**Corvinus University Budapest**
Faculty of Business Administration

# Inventory Models in Reverse Logistics

PhD Dissertation

by

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# Contents

# Preface

Environmental conscious material and inventory management will be studied in this dissertation. It is named as reverse, inverse, or waste disposal logistics in the last decade. In the Hungarian literature there are no uniform definitions for this scientific area. This research field is defined in English speaking countries as reverse logistics. A former name of this idea was inverse logistics, but this name is used mainly in Japan.

The dissertation consists of three chapters. In the first chapter I define reverse logistics and its problems.

In the second chapter I present six deterministic reverse logistics inventory models. These inventory models were constructed in the last three decades. A natural extension of EOQ-type inventory models was examined in the last decade. The first reverse logistic inventory model was built in 1967. The second model was published in 1979. In the eighties no paper was published on this research area. The European Union has aided some research projects in the nineties to support European environmental regulation. The publication of new reverse logistic inventory models is in progress nowadays. Main point of the research is now inventory models with shortage. The dissertation contains all available deterministic EOQ-type models without shortage.

The last chapter investigates shortly the influence of reverse logistics on production planning, and on material requirements planning systems. EOQ-type reverse logistics models can be used, as a basis for the dynamic lot size reverse logistic model. The first publication appeared on this field in 2000. The solution of Wagner-Whitin-type reverse logistic model is not easy, because of the complexity of dynamic programming algorithms. EOQ-type reverse logistic models can serve as heuristics to solve such kind of inventory models. This is a potential application of this research field.

And last I summarize the results of this dissertation.

# 1. Reverse Logistics: A Framework

## 1.1. Introduction

Collection of used products, as paper, bottle, and battery, is a known idea in modern economies. Reuse, remanufacturing and recycling of cars and electronic appliances, and disposal of hazardous waste are very recent research field. The listed activities include a very broad area, and it seems to have different management problems. This chapter summarizes the reverse logistics which offers a theoretical background to solve such kind of business problems.

The reuse is not a new phenomenon in the practice, but a lot of publications are appeared in the international literature in eighties, named reverse logistics. In Hungarian literature there are only a few publications on this research field. The first publication is paper of Rixer (1995). He has called this field as "inverse logistics". Cselényi et al. (1997) has used the expression "recycling logistics", and Mike (2002) has given the name "reverse logistics" which is used in English speaking countries. There are some new publications about reverse logistics in Hungarian, as well. (Richter and Dobos (2003)), Dobos (2004)) Reverse logistics includes not only the material flow from supplier to consumer, but also the material flow of used products from consumer to producer and supplier, in order to reduce the burden of environment.

The aim of this chapter is to present the international (mainly Anglo-Saxon) literature on this field. The environmental regulation becomes rigorous in the European Union and in Hungary. There are recently a lot of environmental regulations about wastes along the life cycle of a product in the European Union. (For example, about used cars.)

The European Union plans to solve environmental problems in the near future by the help of legal regulation. These include the use of renewable environmental resources and energy; avoid wastes, and substitution of non-renewable resources.

The Hungarian Parliament has legislated a law about the waste management in 2000. The aim of this law is to protect the human health and environment, to support the rational use of resources, and to reduce the environmental load, in order to promote sustainable development and economic growth. The law disposes of wastes and activities of its handling. This law does not touch the emissions in the air, and nuclear hazardous wastes. Some of principles are mentioned in this law, as prevention, responsibility of producer, divided responsibility, pollutant pays principle, best available technique, cost efficiency, and so on. The law disposes of responsibility of producers, retailers, consumers, and owners of wastes. Steps of waste management and reuse, and explanation of ideas are included in the law. There are defined collection and transportation of wastes, reuse of wastes and handling. Separate sections present responsibility of handling of communal and hazardous wastes, and organization of waste management. It is to emphasize obligation of publicity and information.

Firms must keep this law, but application of reverse logistic methods can lead to cost savings in long range. Legal registration can not force enterprise to produce an environmental conscious way, but economic earnings can result an environmental friendly production structure of firms.

In this chapter I present shortly the development of reverse logistics, and then I show a conceptual framework, considering the development of this idea in the last decades. After that I look for answer the main questions of reverse logistics: "why-how-what-who". And last I analyze the participants of reverse logistics, examined the main management problems.

**1.2. About development…**

There were economic and historic causes of development of logistics, as it is for the reverse logistics. Retailers have recognized the chance of takeback of products in the United States at the end of eighties, as a tool of market growth. Control of takeback was not directed, because there was no uniform and serious regulation of forms of return policies of used products. The result of this development was that consumers have taken back a number of products. The costs of this process have dramatically increased at the producers and at the retailers, which has reduced the profitability and competitiveness of firms. They have recognized that an effective reverse logistics system is an important integral part of corporate strategy of firms.

The importance of reverse logistics is out of question, but the application of this concept makes more difficult that authors define reverse logistics differently, and the solution of reverse logistics concepts differs from each other at firm level. Because of this difficult applicability, I try to define the idea, and I determine the potential research fields of reverse logistics.

### 1.2.1. Determination of the concept

The reverse logistics was first defined in the eighties. In this time there were published only a few articles in the literature, so the theoretical basis of investigations was unsettled. One of the first publications on this field is the paper of Lambert and Stock (1981). They have defined reverse logistics, as a reverse material flow opposite to supply chain, which is a "bad" process along the material flow of firms. It means that until material flow of traditional supply chain occurs in supplier-producer-wholesaler-retailer-consumer chain, reverse logistics seizes the return material flow of used products, in order to follow this process backward from consumer to supplier.

After the negative definition of Lambert and Stock, Murphy and Poist (1989) have offered a new approach to determine reverse logistics. They have defined reverse logistics, as a material flow of products from consumers to producers in the supply chain. This definition is accepted by Pohlen and Farris (1992), who prefer to apply marketing concepts to reverse logistics. The importance of their paper is that they have named the final consumer, and they have emphasized that the process is reverse in the supply chain. A drawback of this definition is that they have not determined the main activities of reverse logistics, which makes more difficult to limit the framework of reverse logistics.

In the nineties Stock (1992) has given a wide definition, which is a basis for waste management. He stresses the role of logistics, which contains recycling, waste disposal, substitution of hazardous material, reduction of resources, and reuse. This definition of Stock is more accurate than that of earlier. The connection with supply chain activities is missing in this general definition, and the reverse process is not emphasized, as well.

These last approaches are summarized by Kopicky et al. (1993). This definition contains all above-mentioned activities, the reverse movement of materials along the supply chain, opposite to traditional logistics. Kopicky et al. (1993) have introduced information flow in the definition of reverse logistics, which helps on effective practical functioning of reverse logistic systems.

Carter and Ellram (1998) have collected a number of definitions of reverse logistics. I will cite one of the definitions. The more general definition is: "Reverse logistics is such an activity, which helps to continue an environmental effective policy of firms with reuse of necessary materials, remanufacturing, and with reduction of amount of necessary materials". This efficiency touches the personal in production, supply, and consumption process. Carter and Ellram (1998) approach reverse logistics from point of view of environmental protection. Environmental consciousness occurs at three level of activity of firms: governmental regulation, social pressure, and voluntary self restriction.

A next definition contains both traditional and reverse logistics. Council of Logistics Management defines logistics: Logistics is a successful, cost-effective planning, realization, and control of raw material, work-in progress, final products, and connected information from the beginning to consumption, in order to perform consumer's needs.

Rogers and Tibben-Lembke (1999) defines reverse logistics, as: Logistics is a successful, cost-effective planning, realization, and control of raw material, work-in progress, final products, and connected information from consumption to the beginning, in interest of value regain, and handling of wastes.

Reverse Logistics Executive Council (RLEC) has given a more general definition of reverse logistics, which summarizes the above definitions: Reverse logistics is a movement of materials from a typical final consumption in an opposite direction, in order to regain value, or to dispose of wastes. This reverse activity includes tackback of damaged products, renewal and enlargement of inventories through product takeback, remanufacturing of packaging materials, reuse of containers, repair and renovation of products, and handling of obsolete appliances.

European Working Group on Reverse Logistics (REVLOG) has given a similar definition of reverse logistics in 1998. The difference is that the beginning of collection is not only the consumption, but it can be also production, distribution, or use.

The development of concept of reverse logistics was presented. The concept has changed dramatically in the last two decades. Till the first approach has considered reverse logistics, as a bad direction, nowadays the theory of reverse logistics contains marketing, financial, and environmental points of view. In nineteen's reverse logistics has become a well established theory. This complex definition supports the idea that reverse logistics covers all activities along the supply chain.

### 1.3. Factors of reverse logistics: Why? – How? – What? – Who?

After definition of reverse logistics I examine the factors that stand behind this concept. Four questions arise in this context: why, how, what, and who. These questions are answered by Brito and Dekker (2002) most comprehensively.

#### 1.3.1. Why?

This question contains two research fields. First, why send persons used products back, and why accept others used items? I have mentioned the causes of reverse logistics in the second section of these chapter, i.e. economic, legislative, and social consequences. These causes touch the "receiver" of groups. Brito and Dekker (2002) distinguish direct and indirect gains inside of economic advantages. Direct gains are the possibility of profit increase that means a reduction of use of raw materials, decrease of costs of waste disposal, and value added through reuse. Indirect advantages are the "green" image of a firm which is a factor of competitiveness for enterprises. Experiences have supported that environmental conscious functioning of firms results in a stable consumer connection. It is a competitive advantage of firms that increases in profit chances. A strict legislative regulation is a new argument for practical application of reverse logistic processes, which serves as a method for environmental protection. The United States and the European Union are leading in environmental legislation, which forces the firms to keep the law. Thirdly, voluntary social responsibility of firms directs organizations to protect environment. In the practice this voluntary activity increases in competitive advantages of firms.

A second area is to investigate the group of "sender". They have decided to send back a used product to the manufacturer. As by the "receiver", three fields are to be analyzed: return by manufacturers, distributors, and users.

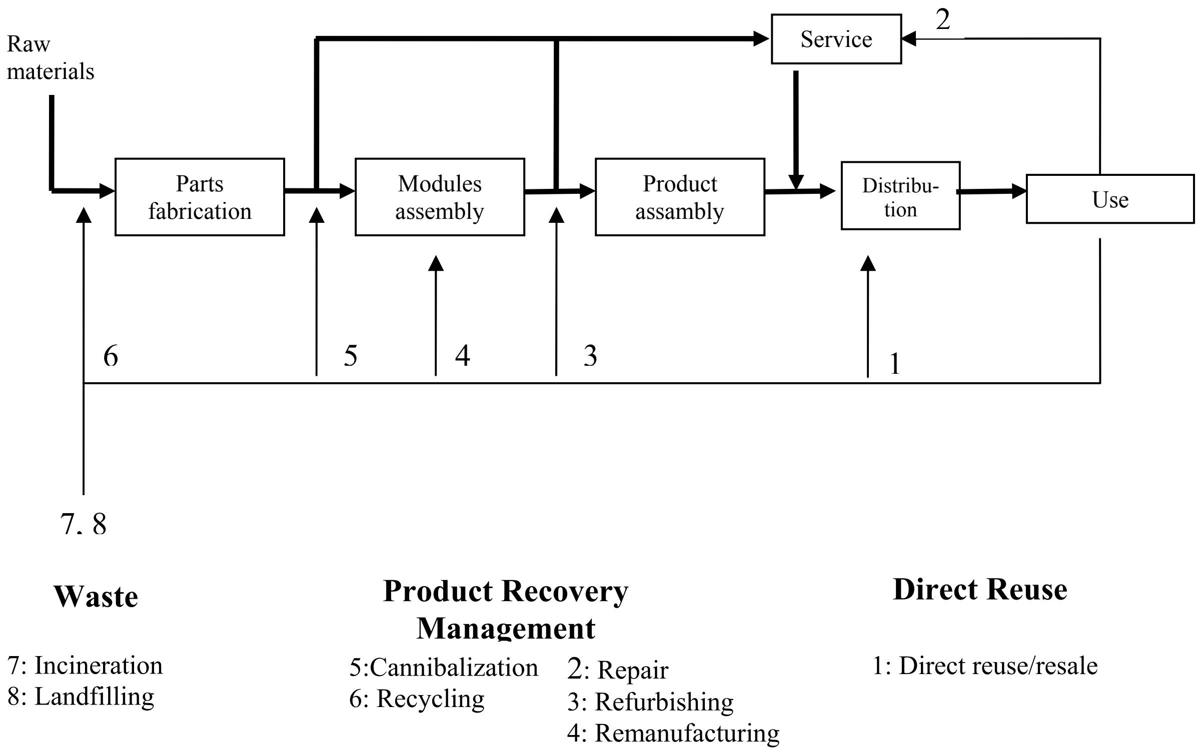



**Figure 1: Integrated Supply Chain, Source: Thierry et. al. (1995)**

Return of used products by manufacturers means a send back in the production process because of raw material surplus, shortage on quality of products and by-products.

Return by distributors means a send back of non sellable, unsold products. These products embody in inventory, defective transport and products, and packaging waste.

Return of users is guarantee, services, and end-of-life products which are at the end of economic and physical span of life of products. A next group of products is the end-of-use products that have no consumer value for their owner, but they can be sold for other consumers. It is very hard to distinguish these last two groups of products, so it is easier to supply some examples. The end-of-life products are, for example, wreck cars, which can be dismantled and its parts or modules can be reused. End-of-use products are rented cars, which can be rented after a known deadline.

### 1.3.2. How?

Now I will investigate the how question: How can be realized a reverse logistics system? I use to answer this question the paper of Thierry et al. (1995). This process consists of eight steps: direct reuse, repair, refurbishing, remanufacturing, cannibalization, recycling, incineration and landfilling. Figure 1 shows the connection between the elements of reverse logistics activities.

*Direct reuse:* the physical and quality property of products is unchangeable in the reverse logistics process.

*Repair:* The product will be transformed, and after this transformation (repair) the product can be used or sold, as a new product. Repair can occur at the user or in a repair shop. Under transformation I understand a change of parts, but other modules or elements of the product are intact.

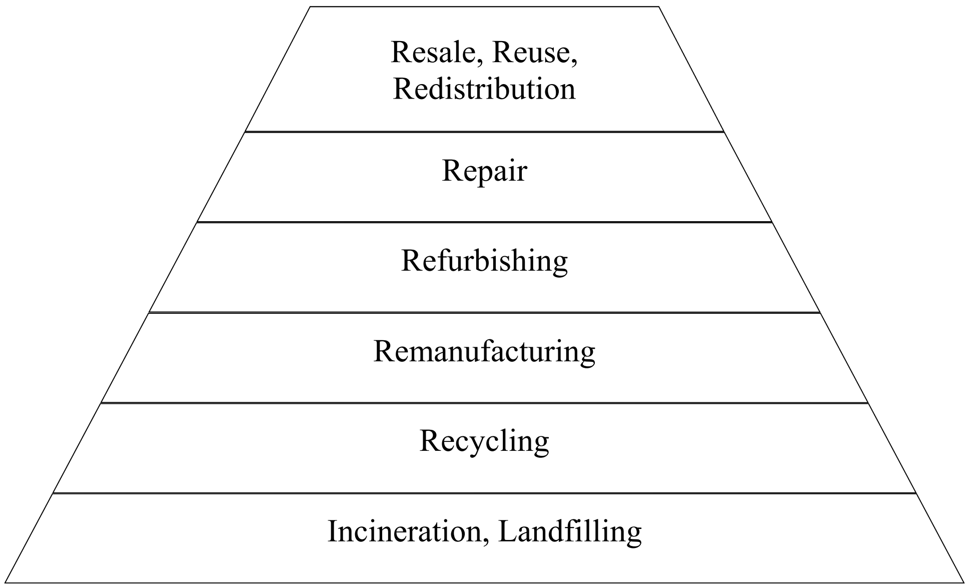



**Figure 2: Hierarchical connection of reverse logistics activities, Source: de Brito – Dekker (2002)**

*Refurbishing:* Refurbished products are dismantled into modules, and then they assembled under less rigorous quality. There are repaired only the defective modules, so the lifespan of products are enlarged.

*Remanufacturing:* From remanufactured products we wait as good quality as a new product. Remanufacturing is more than refurbishing, because all modules and parts are rigorously inspected before use process. Defective modules and parts are totally exchanged in the remanufacturing process.

*Cannibalization:* In this reuse process all of the returned products are dismantled, and there is a rigorous quality inspection. The regained parts and modules are reused in repair, refurbishing, and remanufacturing activities.

*Recycling:* The product loses the original function in this reuse method. The objective of recycling is to recover all usable material. If the quality of recovered materials is appropriate, then they can be used for manufacturing of new products.

*Incineration and landfilling:* These two categories belong to the waste management. Both activities must fill rigorous requirements. Economic advantages can be gained from incineration, if the rising energy is reused.

The above-mentioned fields are summarized in a pyramid in figure 2. This pyramid creates a close connection between reverse logistics and environmental protection. The levels shows which logistic activities promote the protection of environment. Some of the materials and wastes, as products of reverse logistics, can be handled with activities at the bottom of pyramid. The objective of a reverse logistics system is to introduce activities at the top of pyramid. The question is now, if the objective is reuse or reduction of resources, then the pyramid is why not broad at the top. The ideal situation would be an inverse pyramid, but reuse is nowadays not so general.

### 1.3.3. What?

The next question deals with the quality of the returned products in reverse logistics. In this case the assortment of returned products is examined: which factors damage the possibility of reuse, and how will the consumer use the reused products.

Assortment influences the reuse in two ways: homogeneity, and measure of the returned products. Lifespan of products is influenced by perishability, age of elements and amortization of the products, which make more difficult the reuse. A typical example is electronic items, where the technical progress supersedes functioning, but obsolete products.

The way of use of products influences the reuse. It depends on place, intensity, and duration of use, which determine a later remanufacturing. The collected items can be distinguished whether they originate from communal or industrial consumption. (E.g. because of transportation, handling, or quantity.) Here must be mentioned packaging materials, spare parts, or public goods.

### 1.3.4. Who?

The fourth important field is the identification of participants in the reverse logistics. In this context I distinguish the participants of traditional value chain, and of reverse processes, and other participants, e.g. charity organizations. Till some of interested persons organize the reverse process, others deal with the practical realization. It is very important to coordinate the connection between supply chains. One of the coordination mechanisms is a reliable information flow. Necessary information for a successful functioning is summarized in paper of Thierry et al. (1995). On the basis of this article there are four groups:

- Information about product assortment, i.e. about materials, their combination, quality, value, hazard, and possibility of manufacturing (analyzes).
- Information about extent and uncertainty of reverse processes:
    - Warranty – quantity and quality of returned products is uncertain, necessary repair activities are hard to plan.
    - Off-lease and off-rent contracts – they can be estimated very well in quantity and in time, but to estimate the quality is hard.
    - Voluntary buy-back – it depends on the possibility of manufacturer. The advantage of this solution is that it insures inexpensive resources for manufacturing and repair. Waste disposal costs decrease at the consumer, and it makes possible for the manufacturer to sell new products.
- Information about the market of reused products, parts and materials. It is hard to find markets, so competitive advantages are in difference of quality and costs for new and

used products. Reuse can be made by the manufacturer, but other firms can realize the reuse inside and outside supply chain.

- Information about collection of used products and waste disposal. The examination includes organizations involved in the process, obstacles occurred, quantity of returned products, and cost-benefit analyzes.

**1.4. Stakeholders of reverse logistics**

Participants of reverse logistics can be approached in another way. A theoretical background is supplied in paper of Carter and Ellram (1998), in which there are internal and external factors that influence reverse logistics.

In general, there are factors within organizations and between organizations, which are external factors. Internal factors belong interested persons inside of firms, steps for protection of environment, successful applied business ethics standards, and mainly those persons who are responsible for the environment friendly corporate philosophy. Also internal influences have the consumers, supplier, competitors, and government. These four elements are influenced also by the macro environment with social, political, and economic trends that touch reverse logistics indirectly.

The listed sectors have a different effect, and they have several interpretations. Among external factors governmental sector has a most determining influence. It can be accepted from environmental protection point of view, considering that environmental problems initiate most of the questions in the European Union. It must be remarked that law forces enterprises, till other competitors have to consider enterprise competitiveness in the same way. From this point of view a firm must meet the consumer need under keeping the environmental regulation of government. Without keeping governmental instruction an enterprise can not become competitive. There are two views about firm behavior.

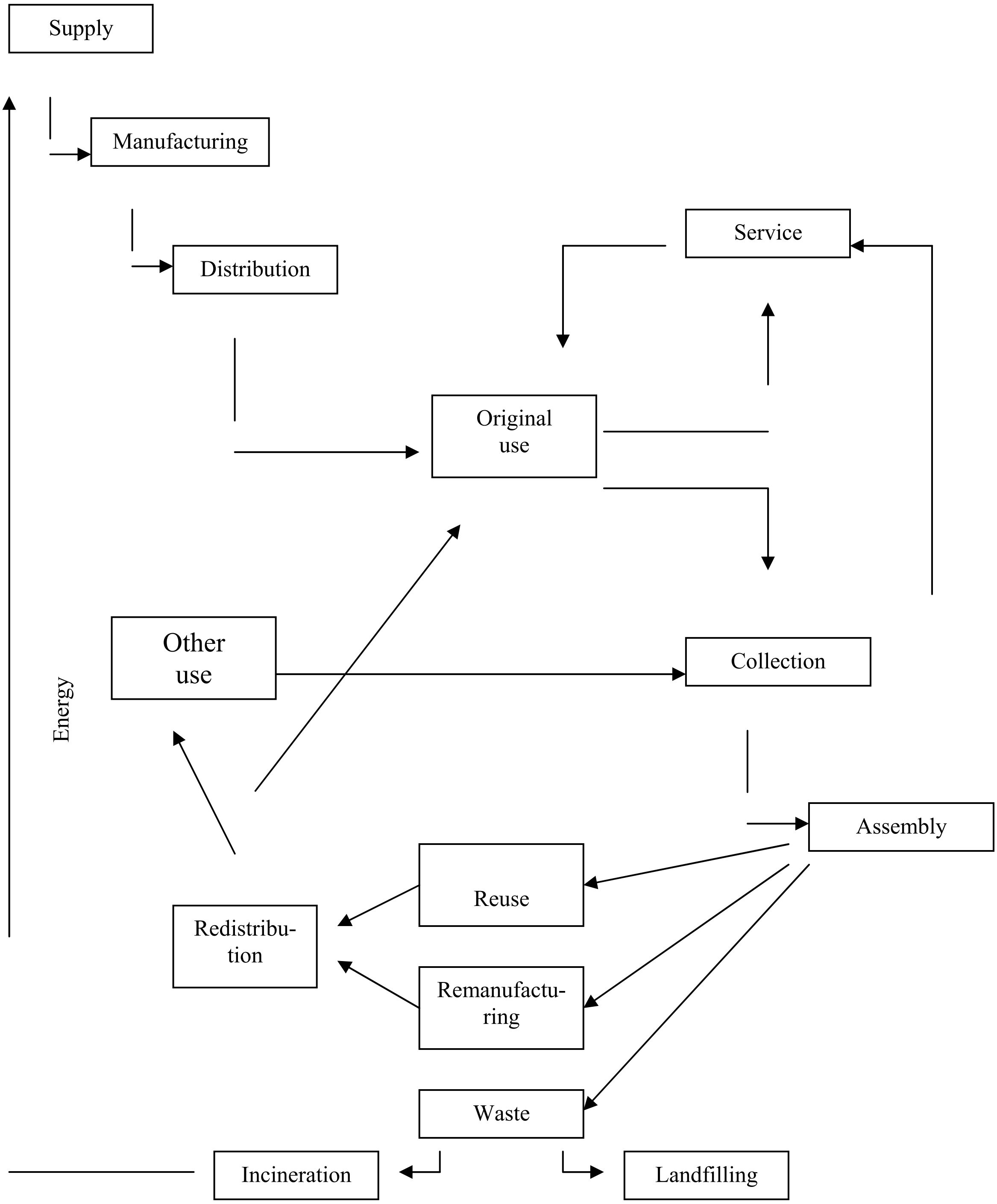



**Figure 3: Connection of reverse logistics processes, Source: Kohut – Nagy (2004)**

Importance of supply side is emphasized with the fact that colleagues of procurement departments purchase more used items, if the permanent good quality of reusable products is secured. Suppliers are responsible for collection and selection of used products generally, but the insurance of expected quality of reusable items necessitates cooperation between supplier and producer firms, which is a crucial logistic activity depending on the attainable information. The quality of returned products has a risk potential for the supplier, so the integration between supplier and producer must be strengthened.

The role of the touched persons is an important internal factor. The owners of the firm can influence the functioning of reverse logistics system. They do not determine the activity of firms directly, but they can hinder it in a long range. Their assistance is a pre-requisite for a successful reverse process.

The role of management is similar to that of owners. Without any assistance of management a reverse logistics system can not be functioned effectively, but the functioning is made by the middle leaders of the firm. They must have good diplomatic and communication skills, and leading ability. They have the work to persuade the touched persons about the necessity of effective reverse logistics system.

Employees belong to the third group of stakeholders, who can help to introduce reverse logistics system through their contribution. Stimulation system can assist the efficiency. The above-mentioned external and internal factors have a synergy effect, i.e. both can make stronger their effect together. The consumer need must be considered, as a general rule. Also the internal and external interest must be considered. Without consideration of these interests a reverse logistics system can not be realized.

Figure 3 summarizes the examined connections about reverse logistics. I emphasize that the processes must be close. The figure presents a paper mill manufacturing process. Some of the important activities are neglected because of the simplicity.

### 1.5. Summary

All of the presented reverse logistics activities can not be found in a firm. There are a number of reasons, why. The available technology, great variety of products, and economic situation of firm influence the enterprise decision about applied reverse logistics system.

I do not investigate, what reverse logistics means for a specific product, and how a successful system could be introduced. These points requires further examinations, e.g. how a final product can be dismantled, which elements and modules of this product can be reused in the manufacturing process. At the same time, it is hard to follow all parts and modules from manufacturers to consumers, then through collection network to reuse fabrication. Nowadays there is no information system to follow the correct material flow along the supply chain. In some cases it is easy to model the reuse process, but in general it is not so. Some of the parts and modules can not reuse, and it is difficult to find an economic sector, where the reuse process can build up effectively. A typical example is computer, from which relatively a few parts can be recovered, and the reuse is economical only in a great extent.

These above-mentioned problems can be eliminated with a cooperation of different industrial sectors, and with a coordinated, reliable information flow between these sectors.

Our starting point was the protection of environment, which is stimulated by legal regulation and by enterprise responsibility. The firms are forced to meet governmental regulation, but a voluntary responsibility is influenced by the available financial sources. In a long range the costs and revenues must be analyzed. Environmental consciousness is not attractive without any economic gains.

The aim of this review is to give an introduction in the theory of reverse logistics systems. This chapter is a starting point to get acquainted with reuse processes, which raise a numbers of questions. This theoretical chapter gives a theoretical background, but the practical application of reverse logistics system needs further empirical investigations. The physical realization faces with technological difficulties, and on the other side the costs must be examined, as well. A successful reverse logistics along the supply chain can contribute to the reduction of loads of environment.

## 2. Economic Order Quantity Models in Reverse Logistics

Reverse logistics is an extension of logistics, which deals with handling and reuse of reusable used products withdrawn from production and consumption process. Such a reuse is e.g. recycling or repair of spare parts. An environmental conscious materials management and/or logistics can be achieved with reuse. It has an advantage from economic point of view, as reduction of environmental load through return of used items in the manufacturing process, but the exploitation of natural resources can be decreased with this reuse that saves the resources from extreme consumption for the future generation.

In this chapter I present three reverse logistic economic order quantity (EOQ). These models are not only shown, but extended, and I show that all of these models lead to the same mathematical structure named meta-model analyzed in the appendix. (Dobos-Richter (2000)) properties of meta-model are presented in the appendix. The following models are presented.

The first reverse logistic (repair/reuse/recycling) model was first investigated by Schrady (1967) in an EOQ context. The paper has examined the cost savings of repair of high cost items at the U.S. Navy Aviation Supply Office in opposite to procurement. The condition of the basic model is that there are only procurement and several repair batches. The question is the lot sizes of procurement and repair.

Model of Nahmias and Rivera (1979) was the second lot sizing model. This model has extended the results of Schrady (1967) with finite repair rate, i.e. the repair process needs time. The repair rate is constant in time. The problem considers waste disposal of a reuse process. In the basic model Nahmias and Rivera (1979) have examined the case of one repair lot size. These investigations were supported by U.S. Air Force Systems Command.

The last model is model of Koh, Hwang, Sohn és Ko (2002). The authors of this paper analyze a model similar to that of Schrady (1967). Till the first two models examine a situation, where the new manufactured/procured and repaired products can arrive in a store, if the inventory level is equal to zero, in this model the recoverable inventory fulfils this property. This inventory strategy was named by Schrady (1967) as “continuous supplement” policy, but modeling of this situation was not published in his paper. The models of Schrady

(1967) and Nahmias and Rivera (1979) have applied an other inventory policy named "substitution". Koh, Hwang, Sohn and Ko (2002) have not expressed the batch sizes, but they investigate two separate cases: number of purchasing batch is one, and repair batch is one. I show a new formulation of the model, which treats these two cases in a general model. Koh et al. (2002), examine an other model. In this model the reuse capacity is not greater than the demand rate. In my investigation I ignore this type of model.

There is a multi product generalization of EOQ-type reverse logistics models published by Mabini, Pintelon and Gelders (1998). They have extended the basic model of Schrady (1967) with capital budget restriction. The examined models have determined the lot sizes, but they have not taken into account that number of lots is integer, and the sensitivity of return process from parameters was not investigated.

After this brief overview I summarize the common conditions of these models.

1. The inventory holding policies are known in the models. It means that in an inventory cycle the inventory status is given and known in time.
2. The demand for new and recovered products is constant and deterministic in time.
3. The return rate of used items is constant and known in time. It is a similar condition to that of last point.
4. The ordering costs of purchasing and setup costs of repair are known.
5. The inventory holding costs of recovered and new products and holding costs of used items waiting for repair are known.
6. There is no shortage in store of recovered and new products and store of returned items.

The first condition defines the inventory holding policy. The variables of these strategies must be determined in a model, i.e. the lot sizes for new and recovered products, number of batches for new and used products, and cycle time. The next four conditions are similar to that of traditional one product EOQ model, i.e. cost structure and demand process. The shortage situation is excluded with the last condition. Consideration of shortage is not a complicated mathematical problem, but the aim of this chapter is to give an introduction in the basic models of EOQ-type reverse logistics models.

I present the models in the following sections.

## 2.1. A Reverse Logistics Model with Procurement and Repair

### 2.1.1. Introduction

A deterministic EOQ-type inventory model for repairable items was first offered by Schrady (1967). This model can be seen as the first reverse logistics model. His model has examined the U.S. Naval Supply Systems Command stock holding problem with repairable items. The repairable items may be scrapped upon a failure, but the products are usually returned from the user to the overhaul and repair point. The repaired items are sent then to the ready-for-issue (RFI) inventory to await demand. Based on the feasibility of repair, the items not sent back are disposed of and they are replaced with new procured products. The returned and not repaired items are held in a second stock point, i.e. the inventory of non-ready-for-issue (NRFI) items are awaiting repair at the overhaul and repair point.

Schrady has offered to inventory holding policy to solve problem: the "continuous supplement" and "substitution" policies. To this last policy he has determined the optimal procurement and repair quantities. It was assumed that there are only one procurement quantity (batch size) and more than one repair quantities.

The aim of the paper is to analyze the introduced substitution policy in a general framework. In this generalization it is allowed a more than one procurement quantity. To solve the problem we use the meta-model. (See appendix.) Schrady has not investigated the integer solution for the repair batch number, it is examined now. We will show that the by Schrady offered solution can be improved in dependence on the recovery (return) rate.

The paper is organized as follows. The next section summarizes the parameters and functioning of the model. In section 3 we construct the inventory holding cost function of the model. Then analyzing the total average costs, we determine the optimal procurement/repair cycle. After eliminating the cycle time we have attained the model in dependence on procurement and repair batch numbers which leads to the meta-model investigated by the author, as well. Section 5 presents the basic model of Schrady with one procurement batch. We will show the optimal integer solution to this model. The following section solves the generalized model with continuous batch numbers.

### 2.1.2. Parameters and functioning of the model

The system contains two inventories. The user's demand can be satisfied from the RFI inventory. The demand of the user is constant in time. The RFI inventory is filled up with procured and repaired items. Shortage is not allowed in this stock point. The procurement and repair quantities are equal. From the user the repairable items are sent back to the overhaul and repair point with a constant rate. The repairable items are stored in the NRFI stock point waiting for repair. After repair products are seen as new and they are sent back to the RFI inventory. The material flow of the model is depicted in Figure 1. We define the variables and parameters as follows:

*The decision variables of the model:*

- $Q_P$ procurement quantity,
- $m$ number of procurements, $m \geq 1$, integer,
- $Q_R$ repair batch size,
- $n$ number of repair batches, $n \geq 1$, integer,
- $T$ procurement/repair cycle time.

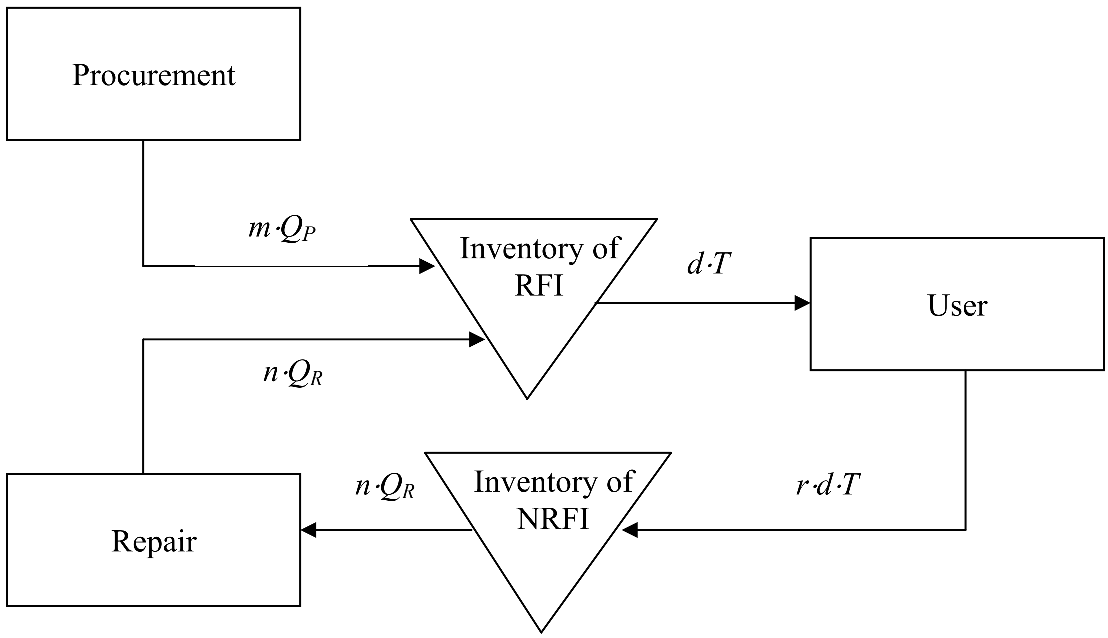
*Parameters of the model:*

- $d$ demand rate, units per unit time,
- $r$ the recovery rate, percent of the demand rate $d$, the scrap rate is $1-r$,
- $A_P$ fixed procurement cost, per order,
- $A_R$ fixed repair batch induction cost, per batch,
- $h_1$ RFI holding cost, per unit per time,
- $h_2$ NRFI holding cost, per unit per time.

The following equalities show relations between the in- and outflows in the stocking points in a procurement/repair cycle.

$$m \cdot Q_P + n \cdot Q_R = d \cdot T$$
$$n \cdot Q_R = r \cdot d \cdot T$$

**Figure 1. Material flow of the model**




The offered "substitution" policy has the next property. The lead times for procurement and repair batches are disregarded, because in deterministic models its influence can be eliminated with a moving away. Let us assume that a procurement/repair cycle begins with induction of a repair cycle. The initial inventory level in NRFI stock point is reduced with a repair batch size. Then the remaining NRFI inventory decreases with a new repair batch, until it reaches the zero inventory level after supply in the RFI inventory. The time history of this policy is shown in Figure 3.

In the next two sections we construct the inventory holding and average inventory cost function of the model.

### 2.1.3. The inventory holding cost function

The holding costs of the model are calculated with the help of the inventory levels in time, as it is presented on Figure 2.

**Lemma 1.**

Let the inventory holding costs for RFI items $H_{RFI}$ and for NRFI items $H_{NRFI}$. Then the cost functions have the next form:

$$H_{RFI} = \frac{h_1}{2 \cdot d} \cdot m \cdot Q_P^2 + \frac{h_1}{2 \cdot d} \cdot n \cdot Q_R^2$$

$$H_{NRFI} = \frac{h_2}{2 \cdot d} \cdot \left( n^2 \cdot \frac{1-r}{r} + n \right) \cdot Q_R^2$$

*Proof.* We will prove the second equation for the NRFI items, the first equation can be calculated in a similar way. Let us divide the area into *n-1* triangles *A*, triangle *C* and *n-1* rectangles $B_1$, $B_2$, ..., $B_{n-1}$. See Figure 3. The length of a repair cycle is $\frac{Q_R}{d}$. The area of a triangle *A* is $\frac{1}{2} \cdot r \cdot Q_R \cdot \frac{Q_R}{d}$. The area of a rectangle $B_i$ is equal to $i \cdot (1-r) \cdot Q_R \cdot \frac{Q_R}{d}$. The maximum inventory level of NRFI items is $n \cdot Q_R - (n-1) \cdot r \cdot Q_R$. The area of triangle *C* is $\frac{1}{2} \cdot [n \cdot Q_R - (n-1) \cdot r \cdot Q_R] \cdot \frac{n \cdot Q_R - (n-1) \cdot r \cdot Q_R}{r \cdot d}$.

Let us now summarize the areas:

$$H_{NFI} = (n-1) \cdot \frac{h_2}{2 \cdot d} \cdot r \cdot Q_R^2 + \frac{h_2}{d} \cdot (1-r) \cdot Q_R^2 \cdot \sum_{i=1}^{n-1} i + \frac{h_2}{2 \cdot r \cdot d} \cdot Q_R^2 \cdot [n - (n-1) \cdot r]^2 .$$

After some elementary calculation we have the equation b).

*Example 1.* Let *d = 1,000*, *r = 0.9*, $h_1$ *= $ 750*, $h_2$ *= $ 100*. Then for this data the inventory holding cost function is

$$H_{RFI} + H_{NRFI} = \frac{1}{10} \cdot m \cdot Q_P^2 + \frac{11}{100} \cdot n \cdot Q_R^2 + \frac{1}{900} \cdot n^2 \cdot Q_R^2$$

**Figure 2. Inventory levels in the RFI and NRFI stock points (*n = 3*, *m = 2*)**

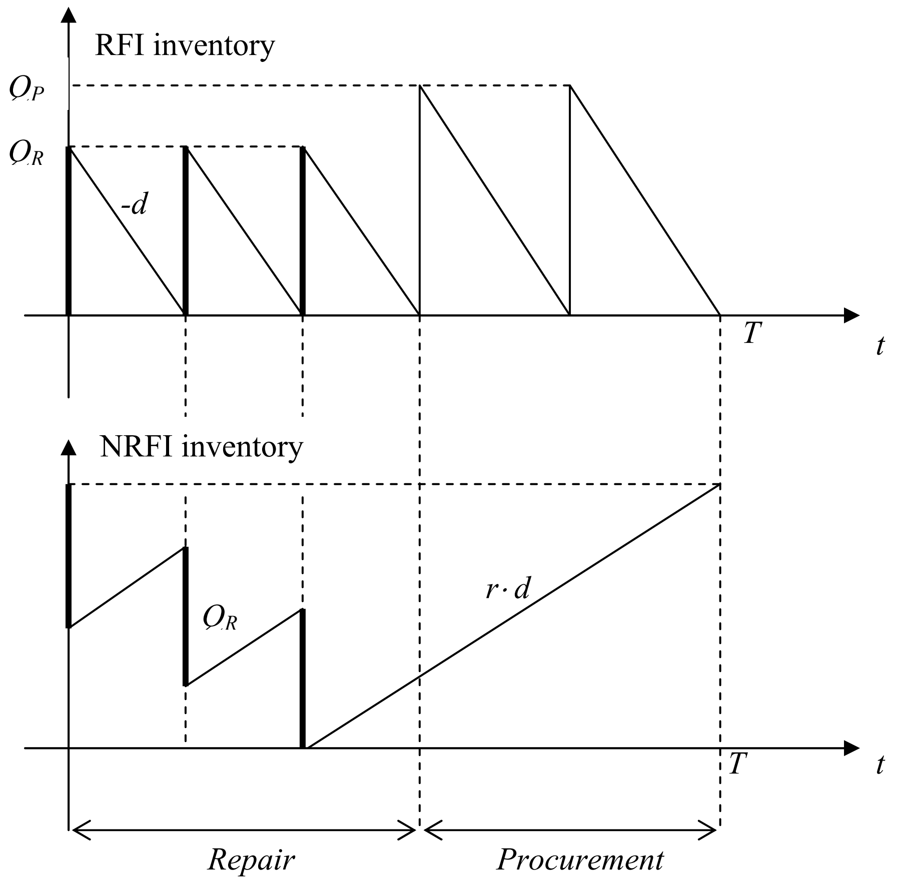
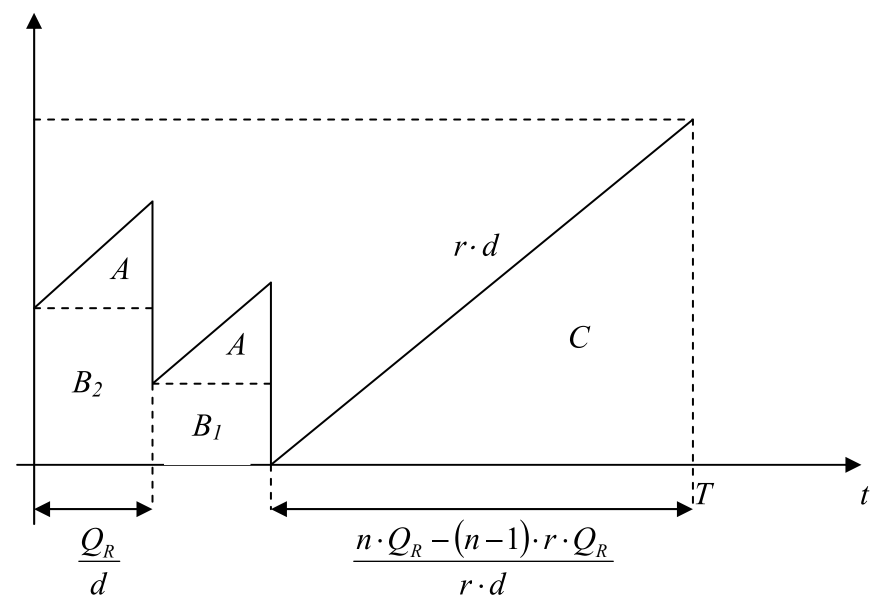



**Figure 3. The calculation of the inventory costs of NRFI items (*m = 3*)**



### 2.1.4. Optimal procurement/repair cycle time

The fixed procurement and repair induction costs

$$F = m \cdot A_P + n \cdot A_R$$

The total average costs are

$$C(T, Q_P, Q_R, n, m, r) = \frac{F + H_{RFI} + H_{NFRI}}{T} =$$

$$= \frac{m \cdot A_P + n \cdot A_R + \frac{h_1}{2 \cdot d} \cdot m \cdot Q_P^2 + \frac{h_1}{2 \cdot d} \cdot n \cdot Q_R^2 + \frac{h_2}{2 \cdot d} \cdot \left( n^2 \cdot \frac{1-r}{r} + n \right) \cdot Q_R^2}{T}$$

Let now use the equations the balance equations

$$Q_P(T, m, r) = \frac{(1-r) \cdot d \cdot T}{m}$$

$$Q_R(T, n, r) = \frac{r \cdot d \cdot T}{n}$$

After substitution the economic order quantities we obtain a simpler cost function:

$$C_1(T, n, m, r) = \frac{m \cdot A_P + n \cdot A_R}{T} + T \cdot \frac{d}{2} \cdot \left[ h_1 \cdot (1-r)^2 \cdot \frac{1}{m} + (h_1 + h_2) \cdot r^2 \cdot \frac{1}{n} + h_2 \cdot r \cdot (1-r) \right]$$

This function is convex in the cycle time then the necessary conditions of optimality are sufficient, as well. The optimal cycle time is

$$T^o(n, m, r) = \sqrt{\frac{2}{d}} \cdot \sqrt{\frac{m \cdot A_P + n \cdot A_R}{h_1 \cdot (1-r)^2 \cdot \frac{1}{m} + (h_1 + h_2) \cdot r^2 \cdot \frac{1}{n} + h_2 \cdot r \cdot (1-r)}}$$

The simplified cost function is

$$C_2(n,m,r)=\sqrt{2\cdot d}\cdot\sqrt{(m\cdot A_P+n\cdot A_R)\cdot\left[h_1\cdot(1-r)^2\cdot\frac{1}{m}+(h_1+h_2)\cdot r^2\cdot\frac{1}{n}+h_2\cdot r\cdot(1-r)\right]}$$

or

$$C_2(n,m,r)=\sqrt{2\cdot d}\cdot\sqrt{A(r)\cdot\frac{m}{n}+B(r)\cdot\frac{n}{m}+C(r)\cdot m+D(r)\cdot n+E(r)}$$

where

$$A(r)=A_P\cdot(h_1+h_2)\cdot r^2,\quad B(r)=A_R\cdot h_1\cdot(1-r)^2,\quad C(r)=A_P\cdot h_2\cdot r\cdot(1-r),$$
$$D(r)=A_R\cdot h_2\cdot r\cdot(1-r),\quad E(r)=A_P\cdot h_1\cdot(1-r)^2+A_R\cdot(h_1+h_2)\cdot r^2$$

*Example 2.* Let $d = 1{,}000$, $r = 0.9$, $h_1 = \$\ 200$, $h_2 = \$\ 20$, $A_P = \$\ 750$, $A_R = \$\ 100$. Then for this data $A(0.9)=133{,}650,\ B(0.9)=200,\ C(0.9)=135,\ D(0.9)=180,\ E(0.9)=193{,}200$

### 2.1.5. The basic model of Schrady

Schrady has investigated the case with only one procurement batch $m = 1$. The cost function of this model is

$$C^S(n,r)=C_2(1,n,r)=\sqrt{2\cdot d}\cdot\sqrt{A(r)\cdot\frac{1}{n}+[B(r)+D(r)]\cdot n+[C(r)+E(r)]}$$

The optimal continuous solution for this case is

**Lemma 2.**

The solution of model of Schrady is

a) if $A_P\cdot(h_1+h_2)\cdot r^2-A_R\cdot h_2\cdot r\cdot(1-r)-A_R\cdot h_1\cdot(1-r)^2>0$, then $n^o(r)=\frac{r}{1-r}\cdot\sqrt{\frac{A_P}{A_R}}\cdot\sqrt{\frac{h_1+h_2}{h_1+h_2\cdot\frac{r}{1-r}}}$ and

$$C^S\left(n^o(r),r\right)=\sqrt{2\cdot d}\cdot\left[(1-r)\cdot\sqrt{A_P\cdot\left(h_1+h_2\cdot\frac{r}{1-r}\right)}+r\cdot\sqrt{A_R\cdot(h_1+h_2)}\right]$$

b) if $A_P\cdot(h_1+h_2)\cdot r^2-A_R\cdot h_2\cdot r\cdot(1-r)-A_R\cdot h_1\cdot(1-r)^2\le 0$,

then $n^o(r)=1$ and

$$C^S\left(n^o(r),r\right)=\sqrt{2\cdot d}\cdot\sqrt{(A_P+A_R)\cdot\left[(h_1+h_2)\cdot r^2+h_1\cdot(1-r)^2+h_2\cdot r\cdot(1-r)\right]}$$

*Proof.* Let us investigate the function $C^S(n,r)$. This function is convex in *n*. The minimal value of the repair batch number is

$$n^o(r)=\sqrt{\frac{A(r)}{B(r)+D(r)}}=\frac{r}{1-r}\cdot\sqrt{\frac{A_P}{A_R}}\cdot\sqrt{\frac{h_1+h_2}{h_1+h_2\cdot\frac{r}{1-r}}}\ .$$

After substitution the optimal value of *n*, we have the condition a) of the lemma. If this number is smaller then one, then the cost function is monotonously increasing for all $n \ge 1$. This fact supports this condition b).

*Remark 1.* The function $F(r)=A_P\cdot(h_1+h_2)\cdot r^2-A_R\cdot h_2\cdot r\cdot(1-r)-A_R\cdot h_1\cdot(1-r)^2$ is quadratic and monotonously increasing between zero and one. Value $F(0)=-A_R\cdot h_1$ is negative and $F(1)=A_P\cdot(h_1+h_2)$ positive, so there exists a recovery rate $r_2$ for which $F(r_2)=0$. Then the optimal batch number is equal to one for all $r\in[0,r_2]$ and it is greater than one for all $r\in(r_2,1]$.

*Remark 2.* The solution for the batch number is not always integer for all $r\in(r_2,1]$. If value $n^o(r)$ is integer then the problem is solved. Let us now assume that $n^o(r)$ is not integer. Let $\underline{n}(r)=int\left(n^o(r)\right)$ denote the maximal integer not greater than $n^o(r)$ and $\overline{n}(r)=int\left(n^o(r)\right)+1$

the minimal integer not smaller than $n^o(r)$. The optimal integer solution can be determined from the following relation

$$n_i^o(r) = \arg\min\left\{C^S(\underline{n}(r)), C^S(\overline{n}(r))\right\}.$$

**Theorem 1.**

The optimal continuous the cycle time and order quantities of model of Schrady are

$$T^o(r) = \begin{cases} \sqrt{\frac{2}{d}} \cdot \sqrt{\frac{A_P + A_R}{h_1 \cdot (1-r)^2 + (h_1 + h_2) \cdot r^2 + h_2 \cdot r \cdot (1-r)}} & r \in [0, r_2] \\ \sqrt{\frac{2}{d}} \cdot \sqrt{\frac{A_P}{h_1 \cdot (1-r)^2 + h_2 \cdot r \cdot (1-r)}} & r \in (r_2, 1] \end{cases}$$

$$Q_P^o(r) = \begin{cases} \sqrt{\frac{2 \cdot d \cdot (A_P + A_R) \cdot (1-r)^2}{h_1 \cdot (1-r)^2 + (h_1 + h_2) \cdot r^2 + h_2 \cdot r \cdot (1-r)}} & r \in [0, r_2] \\ \sqrt{\frac{2 \cdot d \cdot A_P \cdot (1-r)}{h_1 \cdot (1-r) + h_2 \cdot r}} & r \in (r_2, 1] \end{cases}$$

and

$$Q_R^o(r) = \begin{cases} \sqrt{\frac{2 \cdot d \cdot (A_P + A_R) \cdot r^2}{h_1 \cdot (1-r)^2 + (h_1 + h_2) \cdot r^2 + h_2 \cdot r \cdot (1-r)}} & r \in [0, r_2] \\ \sqrt{\frac{2 \cdot d \cdot A_R}{h_1 + h_2}} & r \in (r_2, 1] \end{cases}$$

*Proof.* If $r \in [0, r_2]$, i.e. the optimal repair batch number is one, then after substitution we have the optimal cycle and order quantities. To determine the other case, we use the following relation

$$T^o(n^o(r), r) = \sqrt{\frac{2}{d}} \cdot \frac{n^o(r)}{r} \cdot \sqrt{\frac{A_R}{h_1 + h_2}}.$$

Substituting the optimal repair batch number and cycle time in balance equations, we get the results of the theorem.

Schrady in his paper has not analyzed those cases, for which the optimal batch number is even one. In this formulation we have shown that the solution supplied by Schrady is limited to the case for $r \in (r_2, 1]$. The method proposed in this paper has the same result for the economic order quantities, as obtained by Schrady. The optimal cycle time and economic order quantities for the integer batch number can be calculated with substitution and with some elementary operations.

*Example 3.* Let as in Ex. 2. $d = 1,000$, $r = 0.9$, $h_1 = \$ 200$, $h_2 = \$ 20$, $A_P = \$ 750$, $A_R = \$ 100$. Then for this data the optimal continuous solution and the switching point $r_2$ are $r_2 = 0.2316$ and $n^o = 18.754, m^o = 1, T^o = 0.628\,years, Q_P^o = 62.828, Q_R^o = 30.151, C^S = \$8,357.4$.

**2.1.6. The optimal number of repair and procurement batches**

To minimize the costs in dependence on the batch numbers we apply an auxiliary problem (meta-model). The problem is

$$C_2(m,n,r) = \sqrt{2 \cdot d} \cdot \sqrt{A(r) \cdot \frac{m}{n} + B(r) \cdot \frac{n}{m} + C(r) \cdot m + D(r) \cdot n + E(r)} \to min$$

subject to

$m \geq 1, \quad n \geq 1$. This problem was extensively studied in papers [1-5]. Based on the mentioned papers we examine the continuous solution of this model.

**Theorem 2.**

There are three cases of optimal solutions $(n(r), m(r))$ and the minimum cost expressions $C_3(r)$ in dependence on the return rate for this problem

(i) $A_P \cdot (h_1 + h_2) \cdot r^2 - A_R \cdot h_1 \cdot (1-r)^2 + A_P \cdot h_2 \cdot r \cdot (1-r) < 0$

$$(n(r), m(r)) = \left(1, \sqrt{\frac{A_R}{A_P}} \cdot \frac{1-r}{r} \cdot \sqrt{\frac{h_1}{h_1 + \frac{h_2}{r}}}\right)$$

$$C_3(r) = \sqrt{2 \cdot d} \cdot \left\{(1-r) \cdot \sqrt{A_P \cdot h_1} + \sqrt{A_R \cdot r \cdot [h_1 \cdot r + h_2]}\right\}$$

(ii) $0 \le A_P \cdot (h_1 + h_2) \cdot r^2 - A_R \cdot h_1 \cdot (1-r)^2 + A_P \cdot h_2 \cdot r \cdot (1-r) \le (A_R + A_P) \cdot h_2 \cdot r \cdot (1-r)$

$$(n(r), m(r)) = (1, 1)$$

$$C_3(r) = \sqrt{2 \cdot d} \cdot \sqrt{(A_P + A_R) \cdot \left[(h_1 + h_2) \cdot r^2 + h_1 \cdot (1-r)^2 + h_2 \cdot r \cdot (1-r)\right]}$$

(iii) $A_P \cdot (h_1 + h_2) \cdot r^2 - A_R \cdot h_1 \cdot (1-r)^2 + A_P \cdot h_2 \cdot r \cdot (1-r) > (A_R + A_P) \cdot h_2 \cdot r \cdot (1-r)$

$$(n(r), m(r)) = \left(\sqrt{\frac{A_P}{A_R}} \cdot \frac{r}{1-r} \cdot \sqrt{\frac{h_1 + h_2}{h_1 + h_2 \cdot \frac{r}{1-r}}}, 1\right)$$

$$C_3(r) = \sqrt{2 \cdot d} \cdot \left\{r \cdot \sqrt{A_R \cdot (h_1 + h_2)} + \sqrt{A_P \cdot (1-r) \cdot [h_1 \cdot (1-r) + h_2 \cdot r]}\right\}$$

It is easy to see that the three regions for the optimal solution in dependence on the return rate are not intersected. So we can calculate the values $r_1$ and $r_2$ ($r_1 < r_2$) for which either the procurement batch or the repair batch is equal to one, but the other batch number is greater than one. Between these values both of the batch numbers are equal to one.

*Example 4.* Let as in Ex. 3. $d = 1{,}000$, $h_1 = \$\ 200$, $h_2 = \$\ 20$, $A_P = \$\ 750$, $A_R = \$\ 100$. Then for this data $r_1 = 0.2341$ and $r_2 = 0.2616$. Let now substitute $r = 0.9$ in the optimal solution. Then the optimal values are as calculated in Ex. 3.

The optimal procurement and repair batch sizes and the cycle times of the model are in dependence on the return rate:

**Theorem 3.**

The order quantities and cycle times are

(i) $r \in [0, r_1)$

$$T(r) = \sqrt{\frac{2}{d}} \cdot \sqrt{\frac{A_R}{r \cdot (h_1 \cdot r + h_2)}} \,,$$

$$Q_P(r) = \sqrt{2 \cdot d} \sqrt{\frac{A_P}{h_1}} \,,$$

$$Q_P(r) = \sqrt{2 \cdot d} \sqrt{\frac{A_R \cdot r}{h_1 \cdot r + h_2}} \,.$$

(ii) $r \in [r_1, r_2]$

$$T(r) = \sqrt{\frac{2}{d}} \cdot \sqrt{\frac{A_P + A_R}{h_1 \cdot \left[(1-r)^2 + r^2\right] + h_2 \cdot r}} \,,$$

$$Q_P(r) = \sqrt{2 \cdot d} \sqrt{\frac{(A_P + A_R) \cdot (1-r)^2}{h_1 \cdot \left[(1-r)^2 + r^2\right] + h_2 \cdot r}} \,,$$

$$Q_P(r) = \sqrt{2 \cdot d} \sqrt{\frac{(A_P + A_R) \cdot r^2}{h_1 \cdot \left[(1-r)^2 + r^2\right] + h_2 \cdot r}} \,.$$

(iii) $r \in (r_2, 1]$

$$T(r) = \sqrt{\frac{2}{d}} \cdot \sqrt{\frac{A_P}{h_1 \cdot (1-r)^2 + h_2 \cdot r \cdot (1-r)}} \,,$$

$$Q_P(r) = \sqrt{2 \cdot d} \sqrt{\frac{A_P \cdot (1-r)}{h_1 \cdot (1-r) + h_2 \cdot r}} \,,$$

$$Q_P(r) = \sqrt{2 \cdot d}\sqrt{\frac{A_R}{h_1 + h_2}}\ .$$

The proof is easy; we must substitute the continuous batch numbers in the order quantities and cycle times.

*Example 5.* Let as in Ex. 3. $d = 1{,}000$, $r = 0.05$, $h_1 = \$\ 200$, $h_2 = \$\ 20$, $A_P = \$\ 750$, $A_R = \$\ 100$. Then for this data the minimal cost for the basic model: $C^S(0.05) = 17{,}589.8$ and the minimal cost for the generalized model $C(0.05) = 17{,}002.2$. This means a cost saving of *3.5* percent of the total EOQ related costs.

### 2.1.7. Conclusion

In this paper we have reformulated and solved the model of Schrady. We have shown that for smaller recovery rate it gives a better solution if the procurement batch number is greater than one and on the basis of model of Schrady we can obtain a more effective solution for higher return rate. This result can be interpreted as a generalization of model of Schrady for the case of more than one procurement batch.

## 2.2. A Model with Purchasing and Finite Repair Rate: Substitution Policy

### 2.2.1. Introduction

The model of Nahmias and Rivera (1979) is a natural generalization of model of Schrady (1967). This model takes into account that repair process needs time, i.e. it depends on capacity.

The model and its solution will be presented in three steps. First I show the functioning of the repair-procurement process. After that the cost function will be constructed, and then the optimal decision variables are determined sequentially.

The presented model is an extension of basic model of Nahmias and Rivera (1979). The authors of this article have allowed only one procurement batch size. I allow in this chapter more than one procurement. As it will be shown, the number of repair and procurement batch sizes depends on the return rates.

### 2.2.2. Parameters and functioning of the model

This inventory system contains two stocking points. The demand of the user is satisfied from supply depot. Demand is constant in time in a repair and procurement cycle. Supply depot is filled up from procurement and repair. Shortage is not allowed in this stocking point, so there are always new products. Procurement and repair batch sizes equal. User of spare parts sends back the used products in the repair depot with a constant return rate, till they are waiting for repair. In opposite to the model of Schrady (1967), the capacity of the overhaul department is finite. It is assumed that repair rate is greater than the demand rate. After repair the spare parts are sent back to the supply depot, and they are used as newly purchased products. The length of repair and purchasing lead times are constant, so they do not influence the decision variables. The material flow of the model is shown in figure 1. The used decision variables and parameters are similar to that of used by Schrady (1967). This circumstance makes it easier to compare these models.

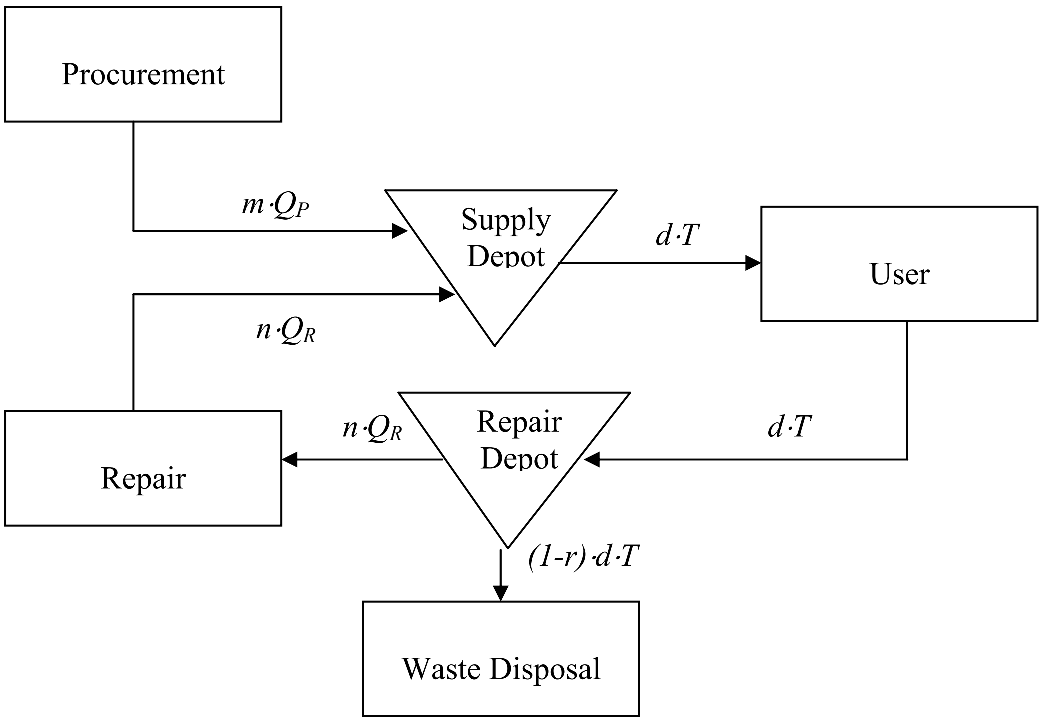
*The decision variables of the model:*

- $Q_P$ procurement quantity,
- $m$ number of procurements, $m \geq 1$, integer,
- $Q_R$ repair batch size,
- $n$ number of repair batches, $n \geq 1$, integer,
- $T$ procurement/repair cycle time.

*Parameters of the model:*

- $d$ demand rate, units per unit time,
- $r$ the recovery rate, percent of the demand rate $d$, the scrap rate is $1-r$,
- $\lambda$ repair rate per unit time, $\lambda > d$,
- $A_P$ fixed procurement cost, per order,
- $A_R$ fixed repair batch induction cost, per batch,
- $h_1$ holding cost in supply depot, per unit per time,
- $h_2$ holding cost in repair depot, per unit per time.

**Figure 1. Material flow of the model**



The following equalities show relations between the in- and outflows in the stocking points in a procurement/repair cycle. These equations make it possible to reduce the number of variables of the model.

$$\begin{aligned} m \cdot Q_P + n \cdot Q_R &= d \cdot T \\ n \cdot Q_R &= r \cdot d \cdot T \end{aligned} \qquad (1)$$

This problem contains waste disposal, but it is not decision variable. The material flow and inventory status are illustrated in figures 1 and 2.

The proposed inventory holding strategy of this model is the substitution policy offered by Schrady (1967). Figure 2 presents the strategy where a cycle is begun with some repair batch sizes and then these lot sizes are followed by some procurement batches. The maximal inventory level is equal to $Q_R\left(1-\frac{d}{\lambda}\right)$, which can be obtained from monographs of inventory controls. Used items are repaired at a rate of $\lambda$ units per time, and $r \cdot d$ units are sent back to repair depot.

**2.2.3. The inventory holding cost function**

Inventory holding costs can be calculated by the help of figure 2. Lemma 1 summarizes this result.

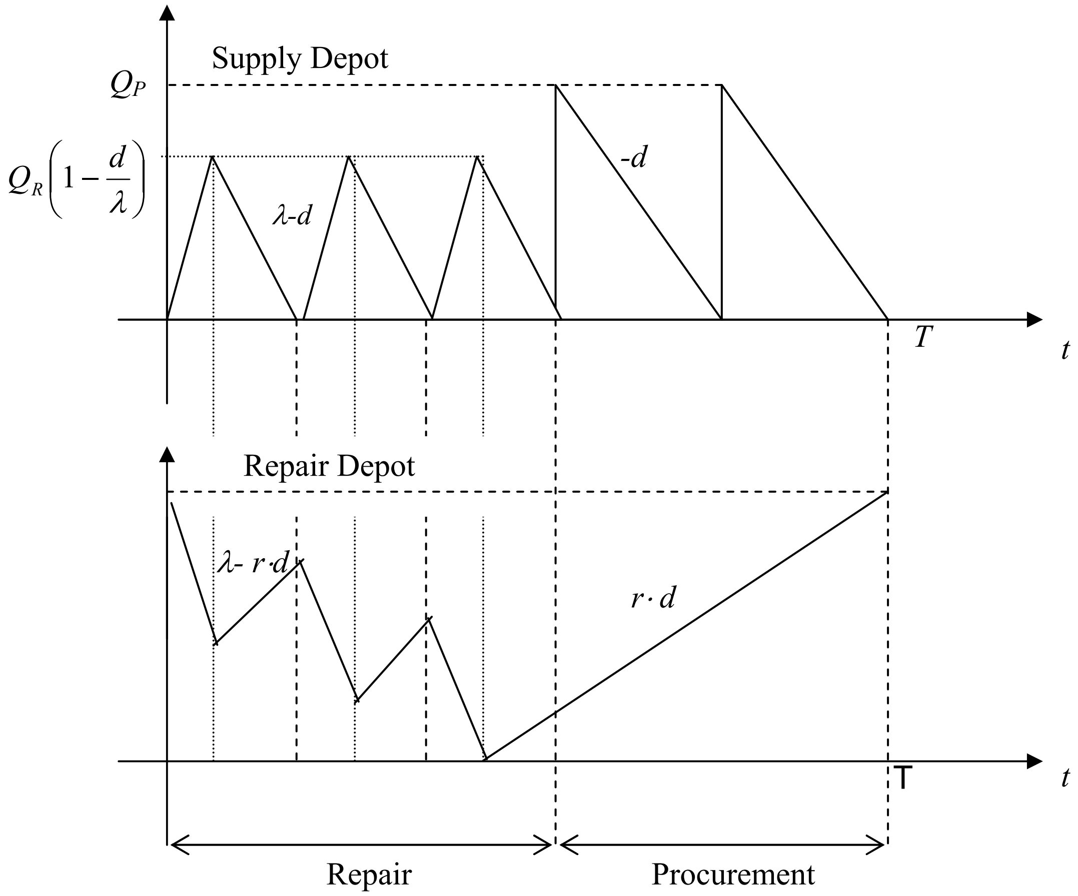
**Lemma 1.**
Let inventory holding cost functions of supply and repair depot be $A_1$ and $A_2$. These two cost functions can be written in the following form:

$$A_1 = \frac{h_1}{2 \cdot d} \cdot m \cdot Q_P^2 + \frac{h_1}{2 \cdot d} \cdot n \cdot Q_R^2 \cdot \left(1-\frac{d}{\lambda}\right),$$

.

$$A_2 = \frac{h_2}{2 \cdot d} \cdot Q_R^2 \cdot \left\{ n^2 \cdot \left(\frac{1}{r}-1\right) + n \right\}$$

**Figure 2. Inventory levels in model of Nahmias és Rivera (*n = 3*, *m = 2*)**




*Proof.* We will prove the second equation for the repair depot; the first equation can be calculated in a similar way. Let us divide the area into *n* triangles *A*, *n-1* triangles *B*, triangle *D* and *n-1* rectangles $C_1$, $C_2$, ..., $C_{n-1}$. See Figure 3. The length of a repair cycle is $\frac{Q_R}{\lambda}$. The area of a triangle *A* is $\frac{1}{2}\cdot\frac{Q_R^2}{\lambda}\cdot\left(1-\frac{r\cdot d}{\lambda}\right)$. The area of a triangle *B* is equal to $\frac{Q_R^2}{2\cdot d}\cdot r\cdot\left(1-\frac{d}{\lambda}\right)^2$.

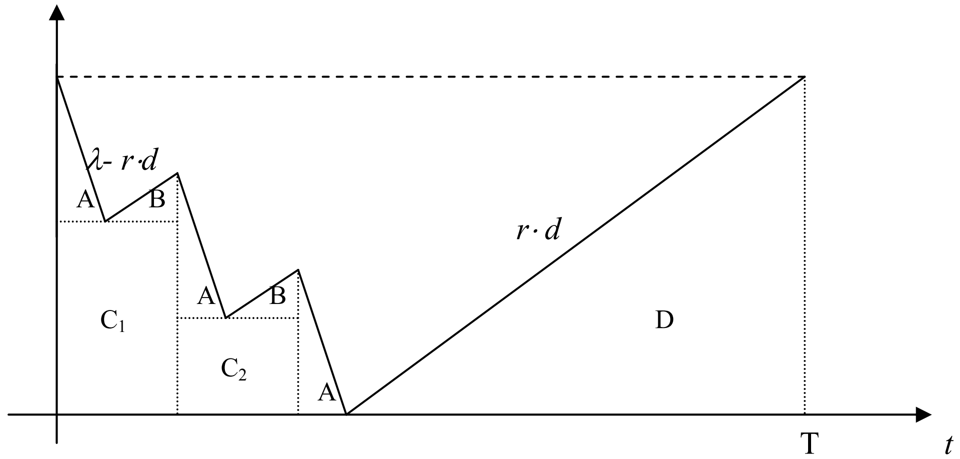
The area of triangle *D* is $\frac{r}{2\cdot d}\cdot\left(Q_R+m\cdot Q_P\right)^2$. The area of a rectangle $C_i$ is equal to $i\cdot(1-r)\cdot\frac{Q_R^2}{d}$.

Let us now summarize the areas:

$$A_2 = n \cdot \frac{h_2}{2} \cdot \frac{Q_R^2}{\lambda} \cdot \left(1 - \frac{r \cdot d}{\lambda}\right) + (n-1) \cdot \frac{h_2}{2} \cdot \frac{Q_R^2}{d} \cdot r \cdot \left(1 - \frac{d}{\lambda}\right)^2 + \frac{h_2}{2} \cdot \frac{r}{d} \cdot (Q_R + m \cdot Q_P)^2 + + h_2 \cdot (1-r) \cdot \frac{Q_R^2}{d} \cdot \sum_{i=1}^{n-1} i.$$

After some elementary calculation we have the second equation.

**Figure 3. The calculation of the inventory costs in repair depot (*m = 3*)**




### 2.2.4. Optimal procurement/repair cycle time

The fixed procurement and repair induction costs are

$$F = m \cdot A_P + n \cdot A_R$$

The total average costs are

$$C(T, Q_P, Q_R, n, m) = \frac{F + A_1 + A_2}{T} =$$

$$= \frac{m \cdot A_P + n \cdot A_R + \frac{h_1}{2 \cdot d} \cdot m \cdot Q_P^2 + \frac{h_1}{2 \cdot d} \cdot n \cdot Q_R^2 \cdot \left(1 - \frac{d}{\lambda}\right) + \frac{h_2}{2 \cdot d} \cdot Q_R^2 \cdot \left\{ n^2 \cdot \left(\frac{1}{r} - 1\right) + n \right\}}{T}.$$

The model leads to the following nonlinear optimization problem:

$$
\left.\begin{aligned}
&C(T, Q_P, Q_R, n, m) \rightarrow \min \\
&m \cdot Q_P + n \cdot Q_R = d \cdot T, \\
&n \cdot Q_R = r \cdot d \cdot T, \\
&T > 0,\ Q_P > 0,\ Q_R > 0,\ n, m \text{ positive integer.}
\end{aligned}\right\} \qquad \text{(P)}
$$

Let us now use the balance equations

$$
Q_P = \frac{(1-r) \cdot d \cdot T}{m}
$$

$$
Q_R = \frac{r \cdot d \cdot T}{n}
$$

After substitution the economic order quantities we obtain a simpler cost function:

$$
C_1(T, n, m) = \frac{m \cdot A_P + n \cdot A_R}{T} + \frac{d}{2} \cdot T \cdot \left[ h_1 \cdot \frac{(1-r)^2}{m} + (h_1 + h_2) \cdot \frac{r^2}{n} \cdot \left(1 - \frac{d}{\lambda}\right) + h_2 \cdot r \cdot (1-r) \right]
$$

This function is convex in the cycle time then the necessary conditions of optimality are sufficient, as well. The optimal cycle time is

$$
T^o = \sqrt{\frac{2}{d}} \cdot \sqrt{\frac{m \cdot A_P + n \cdot A_R}{h_1 \cdot (1-r)^2 \cdot \frac{1}{m} + (h_1 + h_2) \cdot r^2 \cdot \left(1 - \frac{d}{\lambda}\right) \cdot \frac{1}{n} + h_2 \cdot r \cdot (1-r)}} .
$$

The simplified cost function is after substitution

$$
C_2(n, m) = \sqrt{2 \cdot d} \cdot \sqrt{(m \cdot A_P + n \cdot A_R) \cdot \left[ h_1 \cdot \frac{(1-r)^2}{m} + (h_1 + h_2) \cdot \frac{r^2}{n} \cdot \left(1 - \frac{d}{\lambda}\right) + h_2 \cdot r \cdot (1-r) \right]}
$$

or

$$C_2(n,m) = \sqrt{2 \cdot d} \cdot \sqrt{A(r) \cdot \frac{m}{n} + B(r) \cdot \frac{n}{m} + C(r) \cdot m + D(r) \cdot n + E(r)}, \qquad (2)$$

where

$$A(r) = A_P \cdot (h_1 + h_2) \cdot \left(1 - \frac{d}{\lambda}\right) \cdot r^2, \quad B(r) = A_R \cdot h_1 \cdot (1-r)^2, \quad C(r) = A_P \cdot h_2 \cdot (1-r) \cdot r,$$
$$D(r) = A_R \cdot h_2 \cdot (1-r) \cdot r \quad E(r) = A_P \cdot h_1 \cdot (1-r)^2 + A_R \cdot (h_1 + h_2) \cdot \left(1 - \frac{d}{\lambda}\right) \cdot r^2 .$$

Problem (P) is simplified as an integer optimization model $C_2(n,m)$. Model (2) is the meta-model of appendix.

### 2.2.5. The basic model of Nahmias and Rivera

Nahmias and Rivera have investigated the case with only one procurement batch $m = 1$. The cost function of this model is

$$C^{NR}(n) = C_2(1,n) = \sqrt{2 \cdot d} \cdot \sqrt{A(r) \cdot \frac{1}{n} + (B(r) + D(r)) \cdot n + (C(r) + E(r))}.$$

The optimal continuous solution for this case is

**Lemma 2.**

The continuous solution of model of Nahmias and Rivera is

a) if $A_P \cdot (h_1 + h_2) \cdot \left(1 - \frac{d}{\lambda}\right) \cdot r^2 - A_R \cdot h_1 \cdot (1-r)^2 - A_R \cdot h_2 \cdot \left(1 - \frac{d}{\lambda}\right) \cdot (1-r) \cdot r > 0$,

then $n^o = \frac{r}{1-r} \cdot \sqrt{\frac{A_P}{A_R}} \cdot \sqrt{\frac{(h_1 + h_2) \cdot \left(1 - \frac{d}{\lambda}\right)}{h_1 + h_2 \cdot \frac{r}{1-r} \cdot \left(1 - \frac{d}{\lambda}\right)}}$ and

$$C^{NR}(n^o) = \sqrt{2 \cdot d} \cdot \left[(1-r) \cdot \sqrt{A_P \cdot \left(h_1 + h_2 \cdot \frac{r}{1-r} \cdot \left(1 - \frac{d}{\lambda}\right)\right)} + r \cdot \sqrt{A_R \cdot (h_1 + h_2) \cdot \left(1 - \frac{d}{\lambda}\right)}\right],$$

b) if $A_P \cdot (h_1 + h_2) \cdot \left(1 - \frac{d}{\lambda}\right) \cdot r^2 - A_R \cdot h_1 \cdot (1-r)^2 - A_R \cdot h_2 \cdot \left(1 - \frac{d}{\lambda}\right) \cdot (1-r) \cdot r \le 0$,

then $n^o = 1$ and

$$C^{NR}(n^o) = \sqrt{2 \cdot d} \cdot \sqrt{(A_P + A_R) \cdot \left[ (h_1 + h_2) \cdot r^2 \cdot \left(1 - \frac{d}{\lambda}\right) + h_1 \cdot (1-r)^2 + h_2 \cdot r \cdot (1-r) \cdot \left(1 - \frac{d}{\lambda}\right) \right]}.$$

*Proof.* Let us investigate the cost function $C^S(n)$. This function is convex in $n$. The minimal value of the repair batch number is

$$n^o = \sqrt{\frac{A(r)}{B(r) + D(r)}} = \frac{r}{1-r} \cdot \sqrt{\frac{A_P}{A_R}} \cdot \sqrt{\frac{(h_1 + h_2) \cdot \left(1 - \frac{d}{\lambda}\right)}{h_1 + h_2 \cdot \frac{r}{1-r} \cdot \left(1 - \frac{d}{\lambda}\right)}}.$$

After substitution the optimal value of $n$, we have the condition a) of the lemma. If this number is smaller then one, then the cost function is monotonously increasing for all $n \ge 1$. This fact supports this condition b).

*Remark 1.*

Function $F(r) = A_P \cdot (h_1 + h_2) \cdot \left(1 - \frac{d}{\lambda}\right) \cdot r^2 - A_R \cdot h_1 \cdot (1-r)^2 - A_R \cdot h_2 \cdot \left(1 - \frac{d}{\lambda}\right) \cdot (1-r) \cdot r$ is a quadratic function in $r$ and monotonously increasing between zero and one. Value $F(0) = -A_R \cdot h_1$ is negative, and expression $F(1) = A_P \cdot (h_1 + h_2) \cdot \left(1 - \frac{d}{\lambda}\right)$ is positive, so there exists a reuse rate $r_2$, for which $F(r_2) = 0$. Then the optimal batch number is equal to one for all $r \in [0, r_2]$ and it is greater than one for all $r \in (r_2, 1]$.

*Remark 2.*

The solution for the batch number is not always integer for all $r \in (r_2, 1]$. If value $n^o(r)$ is integer then the problem is solved. Let us now assume that $n^o(r)$ is not integer. Let $\underline{n} = \lfloor n^o \rfloor$ denote the maximal integer not greater than $n^o$, and $\underline{n} = \lfloor n^o \rfloor + 1$ the minimal integer not smaller than $n^o$. The optimal integer solution can be determined from the following relation

$$n^i = \arg\min\left\{C^{NR}(\underline{n}), C^{NR}(\overline{n})\right\}.$$

The following theorem summarizes the continuous solution of the basic model, not investigated the integer case.

**Theorem 1.**

The optimal continuous the cycle time and order quantities of model of Nahmias and Rivera are in dependence of reuse rate $r$

$$T^o(r) = \begin{cases} \sqrt{\dfrac{2}{d}} \cdot \sqrt{\dfrac{A_P + A_R}{h_1 \cdot (1-r)^2 + (h_1 + h_2) \cdot r^2 \cdot \left(1 - \dfrac{d}{\lambda}\right) + h_2 \cdot r \cdot (1-r) \cdot \left(1 - \dfrac{d}{\lambda}\right)}} & r \in [0, r_2] \\ \sqrt{\dfrac{2}{d}} \cdot \sqrt{\dfrac{A_P}{h_1 \cdot (1-r)^2 + h_2 \cdot r \cdot (1-r) \cdot \left(1 - \dfrac{d}{\lambda}\right)}} & r \in (r_2, 1] \end{cases},$$

$$Q_P^o(r) = \begin{cases} \sqrt{\dfrac{2 \cdot d \cdot (A_P + A_R) \cdot (1-r)^2}{h_1 \cdot (1-r)^2 + (h_1 + h_2) \cdot r^2 \cdot \left(1 - \dfrac{d}{\lambda}\right) + h_2 \cdot r \cdot (1-r) \cdot \left(1 - \dfrac{d}{\lambda}\right)}} & r \in [0, r_2] \\ \sqrt{\dfrac{2 \cdot d \cdot A_P \cdot (1-r)}{h_1 \cdot (1-r) + h_2 \cdot r \cdot \left(1 - \dfrac{d}{\lambda}\right)}} & r \in (r_2, 1] \end{cases},$$

and

$$Q_R^o(r) = \begin{cases} \sqrt{\dfrac{2 \cdot d \cdot (A_P + A_R) \cdot r^2}{h_1 \cdot (1-r)^2 + (h_1 + h_2) \cdot r^2 \cdot \left(1 - \dfrac{d}{\lambda}\right) + h_2 \cdot r \cdot (1-r) \cdot \left(1 - \dfrac{d}{\lambda}\right)}} & r \in [0, r_2] \\ \sqrt{\dfrac{2 \cdot d \cdot A_R}{h_1 + h_2 \cdot \left(1 - \dfrac{d}{\lambda}\right)}} & r \in (r_2, 1] \end{cases}.$$

*Proof.* If $r \in [0, r_2]$, i.e. the optimal repair batch number is one, then after substitution we have the optimal cycle and order quantities. To determine the other case, we use the following relation

$$T^o(n^o) = \sqrt{\frac{2}{d}} \cdot \frac{n^o}{r} \cdot \sqrt{\frac{A_R}{h_1 + h_2 \cdot \left(1 - \frac{d}{\lambda}\right)}} .$$

Substituting the optimal repair batch number and cycle time in balance equations, we get the results of the theorem.

Nahmias and Rivera in his paper has not analyzed those cases, for which the optimal batch number is even one. In this formulation we have shown that the solution supplied by Nahmias and Rivera is limited to the case for $r \in (r_2, 1]$. The method proposed in this paper has the same result for the economic order quantities, as obtained by Nahmias and Rivera. The optimal cycle time and economic order quantities for the integer batch number can be calculated with substitution and with some elementary operations.

### 2.2.6. The optimal number of repair and procurement batches

To minimize the costs in dependence on the batch numbers we apply an auxiliary problem (meta-model). The problem is

$$C_2(m,n) = \sqrt{2 \cdot d} \cdot \sqrt{A \cdot \frac{m}{n} + B \cdot \frac{n}{m} + C \cdot m + D \cdot n + E} \rightarrow \min$$

subject to

$m \geq 1, \quad n \geq 1$. This problem was extensively studied in papers (Dobos-Richter (2000), Richter (1996a), Richter (1996b), Richter (1997), Richter-Dobos (1999)). Based on the mentioned papers we examine the continuous solution of this model.

**Theorem 2.**

There are three cases of optimal solutions $(n(r), m(r))$ and the minimum cost expressions $C_3(r)$ in dependence on the return rate for this problem

(i) $$A_P \cdot (h_1 + h_2) \cdot \left(1 - \frac{d}{\lambda}\right) \cdot r^2 - A_R \cdot h_1 \cdot (1-r)^2 + A_P \cdot h_2 \cdot \left(1 - \frac{d}{\lambda}\right) \cdot r \cdot (1-r) < 0,$$

$$(n(r), m(r)) = \left(1, \sqrt{\frac{A_R}{A_P\left(1 - \frac{d}{\lambda}\right)}} \cdot \frac{1-r}{r} \cdot \sqrt{\frac{h_1}{h_1 + \frac{h_2}{r}}}\right),$$

$$C_3(r) = \sqrt{2 \cdot d} \cdot \left\{(1-r) \cdot \sqrt{A_P \cdot h_1} + \sqrt{A_R \cdot r \cdot [h_1 \cdot r + h_2]}\right\},$$

(ii)

$$0 \le A_P \cdot (h_1 + h_2) \cdot \left(1 - \frac{d}{\lambda}\right) \cdot r^2 - A_R \cdot h_1 \cdot (1-r)^2 + A_P \cdot h_2 \cdot \left(1 - \frac{d}{\lambda}\right) \cdot r \cdot (1-r) \le (A_R + A_P) \cdot h_2 \cdot r \cdot (1-r),$$

$$(n(r), m(r)) = (1, 1),$$

$$C_3(r) = \sqrt{2 \cdot d} \cdot \sqrt{(A_P + A_R) \cdot \left[(h_1 + h_2) \cdot \left(1 - \frac{d}{\lambda}\right) \cdot r^2 + h_1 \cdot (1-r)^2 + h_2 \cdot \left(1 - \frac{d}{\lambda}\right) \cdot r \cdot (1-r)\right]},$$

(iii)

$$A_P \cdot (h_1 + h_2) \cdot \left(1 - \frac{d}{\lambda}\right) \cdot r^2 - A_R \cdot h_1 \cdot (1-r)^2 + A_P \cdot h_2 \cdot \left(1 - \frac{d}{\lambda}\right) \cdot r \cdot (1-r) > (A_R + A_P) \cdot h_2 \cdot r \cdot (1-r)$$

$$(n(r), m(r)) = \left(\sqrt{\frac{A_P}{A_R}} \cdot \frac{r}{1-r} \cdot \sqrt{\frac{h_1 + h_2}{h_1 + h_2 \cdot \frac{r}{1-r}}}, 1\right),$$

$$C_3(r) = \sqrt{2 \cdot d} \cdot \left\{r \cdot \sqrt{A_R \cdot (h_1 + h_2)} + \sqrt{A_P \cdot \left(1 - \frac{d}{\lambda}\right) \cdot (1-r) \cdot [h_1 \cdot (1-r) + h_2 \cdot r]}\right\}.$$

It is easy to see that the three regions for the optimal solution in dependence on the return rate are not intersected. So we can calculate the values $r_1$ and $r_2$ ($r_1 < r_2$) for which either the procurement batch or the repair batch is equal to one, but the other batch number is greater than one. Between these values both of the batch numbers are equal to one.

### 2.2.7. The integer solution

Let us now apply the results of appendix.

**Theorem 3.**

The optimal integer repair and procurement batch numbers, and cycle times are in dependence of reuse rate $r$

(i) $$A_P \cdot (h_1 + h_2) \cdot \left(1 - \frac{d}{\lambda}\right) \cdot r^2 - A_R \cdot h_1 \cdot (1-r)^2 + A_P \cdot h_2 \cdot \left(1 - \frac{d}{\lambda}\right) \cdot r \cdot (1-r) < 0,$$

$$(n(r), m(r)) = \left(1, \left\lfloor \sqrt{\frac{A_R \cdot h_1 \cdot (1-r)^2}{A_P \cdot \left(1 - \frac{d}{\lambda}\right) \cdot (h_1 \cdot r^2 + h_2 \cdot r)} + \frac{1}{4}} + \frac{1}{2} \right\rfloor \right),$$

(ii)

$$0 \le A_P \cdot (h_1 + h_2) \cdot \left(1 - \frac{d}{\lambda}\right) \cdot r^2 - A_R \cdot h_1 \cdot (1-r)^2 + A_P \cdot h_2 \cdot \left(1 - \frac{d}{\lambda}\right) \cdot r \cdot (1-r) \le (A_R + A_P) \cdot h_2 \cdot r \cdot (1-r),$$

$$(n(r), m(r)) = (1, 1),$$

(iii)

$$A_P \cdot (h_1 + h_2) \cdot \left(1 - \frac{d}{\lambda}\right) \cdot r^2 - A_R \cdot h_1 \cdot (1-r)^2 + A_P \cdot h_2 \cdot \left(1 - \frac{d}{\lambda}\right) \cdot r \cdot (1-r) > (A_R + A_P) \cdot h_2 \cdot r \cdot (1-r)$$

$$(n(r), m(r)) = \left( \left\lfloor \sqrt{\frac{A_P \cdot (h_1 + h_2) \cdot r^2}{A_R \cdot \left(h_1 \cdot (1-r)^2 + h_2 \cdot r \cdot (1-r)\right)} + \frac{1}{4}} + \frac{1}{2} \right\rfloor, 1 \right).$$

Here function $\lfloor \cdot \rfloor$ denotes the maximal integer not greater than the argument. The proof of the theorem is easy, with elementary manipulation we get the results. With these results I have finished the investigation of the model.

### 2.2.8. Conclusion

The model of Nahmias and Rivera is similar to that of model of Schrady. In this chapter I have not presented numerical examples. Nahmias and Rivera (1979) have mentioned some possible extensions in their paper. Such generalizations are e.g. constraints on depot capacity. The costs of waste disposal and return rate as decision variables are proposed by them.

## 2.3. A Model with Purchasing and Finite Repair Rate: Continuous Supplement Policy

### 2.3.1. Introduction

The authors of this model investigate a simple model. The problem is similar to that of Nahmias and Rivera (1979), but they apply an other inventory holding policy, which is initiated by Schrady (1967) and called continuous supplement policy instead of substitution policy. The authors do not express the batch sizes explicitly. The model of Koh et al. (2002) contains a new formulation for continuous supplement policy. They examine the case of capacitated remanufacturing rate that is not greater than production and reuse rate, but I do not analyze that case.

### 2.3.2. Parameters and functioning of the model

The inventory system contains two stocking points. Demand of users is satisfied from depot of usable products. Demand is constant in time during reuse cycle. The depot of usable products is fulfilled from purchase and repair. Shortage is not allowed in this system, so there are always usable products in depot. The procurement and repair batch sizes are equal. Use items return from the consumption process with a known return rate. The capacity of repair department is finite, as it was assumed by Nahmias and Rivers (1979). It is assumed that repair rate is greater than demand rate that is greater than return rate. After repair the spare parts are sent back and they are used as new products. Replenishment and repair lead times are disregarded, because they don not influence the decision variables. The material flow of this model is depicted on figure 1. Let us now define the decision variables and parameters of the model.

*The decision variables of the model:*

- $Q_P$ procurement batch size,
- $m$ number of procurements, $m \geq 1$, integer,
- $Q_R$ repair batch size,
- $n$ number of repair batches, $n \geq 1$, integer,

- $T$ procurement/repair cycle time.

*Parameters of the model:*

- $d$ demand rate, units per unit time,
- $r$ return rate, $d > r$,
- $p$ production rate, $p > d$,
- $C_o$ fixed procurement cost, per order,
- $C_s$ fixed reuse batch induction cost, per batch,
- $C_{h2}$ holding cost of usable products, per unit per time,
- $C_{h1}$ holding cost of reusable items, per unit per time.

The inventory levels are presented in figure 2.

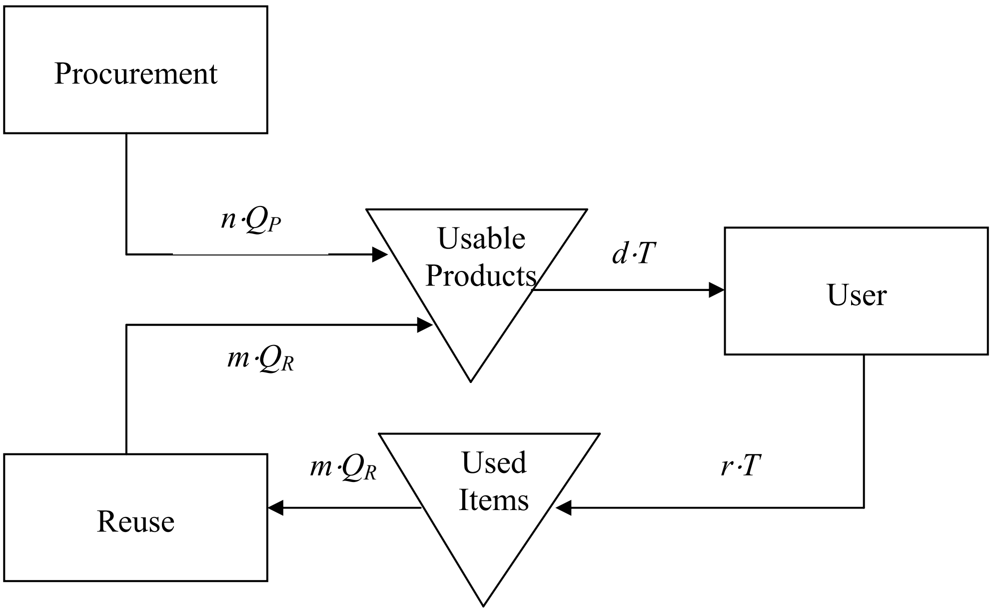


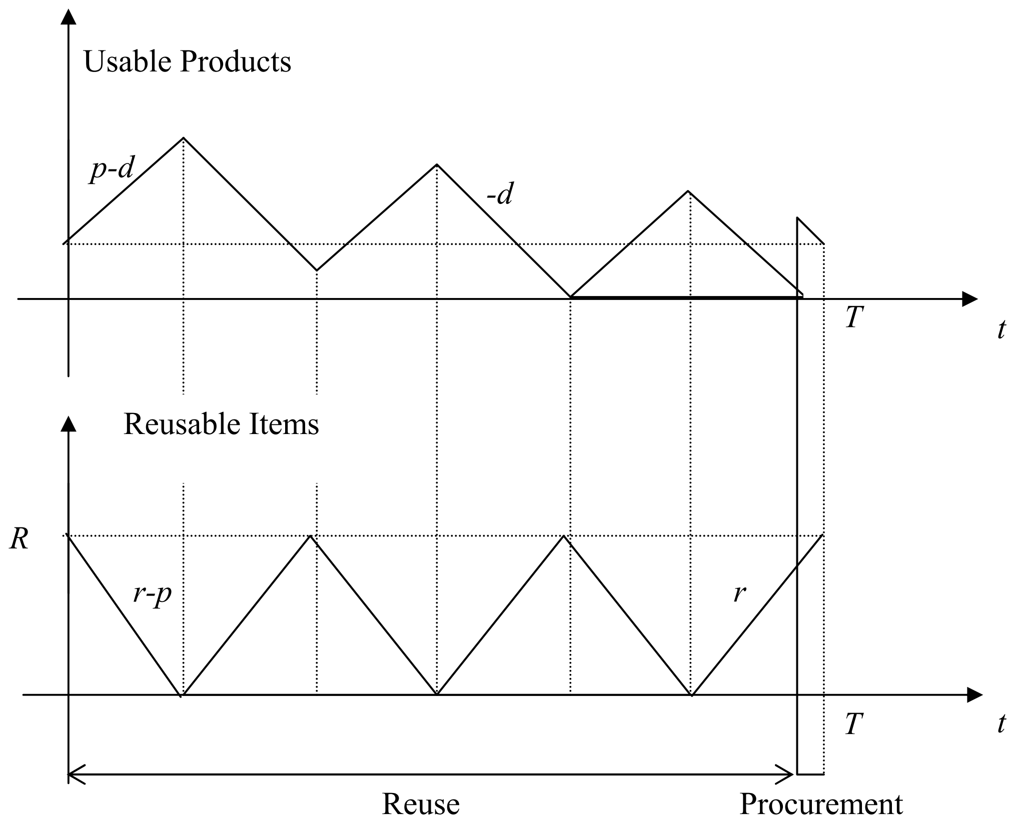

**Figure 1. Material flow in model of Koh, Hwang, Sohn és Ko**

The following equations represent the in- and outflow of materials in stocking points during purchasing-repair cycle. I use these stock-flow identities to reduce the numbers of decision variables.

$$\begin{aligned} m \cdot Q_R + n \cdot Q_P &= d \cdot T \\ m \cdot Q_R &= r \cdot T \end{aligned} \tag{1}$$

The inventory status is different in this model compared to that of Nahmias and Rivera (1979). The authors of this model have assumed that the inventory holding of remanufactured and usable products is more expensive than that of reusable items, i.e. replenishment is economical, if the inventory level is zero. In opposite, Koh et al. (2002) have assumed that inventory level of returned and reusable items must be zero in a reuse cycle. The question arises, when model of Nahmias and Rivera (1979) and when Koh, Hwang, Sohn and Ko (2002) can be applied. In the following this question is answered.

**Figure 2. Inventory status in model of Koh, Hwang, Sohn és Ko**




A difference between two models is that problem of Nahmias and Rivera (1979) is modeled with only one procurement batch. I have generalized this assumption in the last chapter, so the models can be compared. Koh et al. (2002) have separated the examination in two parts: Procurement batch number are equal to one, and repair batch number is one. I do not distinguish between two cases; they are special cases of this general model. An advantage of the presented model is that it can be determined, when and which partial model is applied in dependence of cost parameters and return rates. Of course, these investigations are supported by the meta-model offered in appendix.

I construct the inventory holding cost function in dependence of decision variables in the next section. The solution of the model is led to a nonlinear programming problem.

### 2.3.3. The inventory holding cost function

Calculation of inventory holding costs is made by inventory levels in figure 2. Inventory holding policy is predetermined. The authors calculate these costs independent on optimality of inventory holding strategy. The determination of costs is summarized in lemma 1.

**Lemma 1.**
Let inventory holding cost functions of usable products and reusable items be $S_1$ and $S_2$. These two cost functions can be written in the following form:

$$S_1 = n \cdot C_{h2} \cdot \frac{Q_P^2}{2 \cdot d} + C_{h2} \cdot \frac{Q_R^2}{2} \cdot \left\{ m^2 \cdot \left( \frac{1}{r} - \frac{1}{d} \right) + m \cdot \left( \frac{2}{d} - \frac{1}{p} - \frac{1}{r} \right) \right\}$$

$$S_2 = m \cdot \frac{C_{h1}}{2} \cdot Q_R^2 \cdot \left( \frac{1}{r} - \frac{1}{p} \right)$$

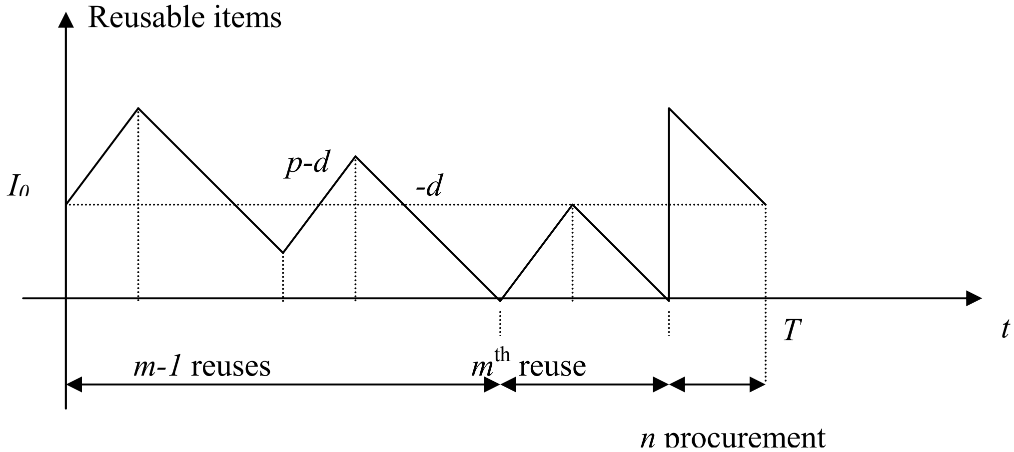
*Proof.* We will prove the second equation for the repair depot; the first equation can be calculated in a similar way. Let us divide the area into *m-1* reuse cycles, the last $m^{\text{th}}$ reuse cycle, and *n* procurement cycles. Inventory holding costs are defined, as integral of a curve. Let $I_0$ denote the initial inventory level in figure 3, which is $(m-1)\cdot\left(\frac{d}{r}-1\right)\cdot Q_R$. With the help of this connection the areas of *m-1* reuse cycles are $\frac{Q_R^2}{2\cdot r}\cdot\left[(m-1)^2\cdot\left(\frac{d}{r}-1\right)+(m-1)\cdot\left(1-\frac{r}{p}\right)\right]$. The area of last reuse cycle is $\frac{Q_R^2}{2}\cdot\left(\frac{1}{d}-\frac{1}{r}\right)$. The areas of *n* procurement cycles can be written as $n\cdot\frac{Q_P^2}{2\cdot d}-\frac{Q_R^2}{2\cdot d}\cdot(m-1)^2\left(\frac{d}{r}-1\right)^2$. See Figure 3.

Let us now summarize the areas:

$$S_1 = \frac{Q_R^2}{2 \cdot r} \cdot \left[ (m-1)^2 \cdot \left( \frac{d}{r} - 1 \right) + (m-1) \cdot \left( 1 - \frac{r}{p} \right) \right] + \frac{Q_R^2}{2} \cdot \left( \frac{1}{d} - \frac{1}{r} \right) + \left[ n \cdot \frac{Q_P^2}{2 \cdot d} - \frac{Q_R^2}{2 \cdot d} \cdot (m-1)^2 \left( \frac{d}{r} - 1 \right)^2 \right].$$

After some elementary calculation we have the second equation.

**Figure 3. The calculation of the inventory costs of reusable items (*m = 3*)**




**2.3.4. Optimal procurement/repair cycle time**

The fixed procurement and repair costs are

$$F = m \cdot C_s + n \cdot C_o .$$

The total average costs are determined in a procurement and repair cycle.

$$C(T, Q_P, Q_R, n, m) = \frac{F + S_1 + S_2}{T} =$$

$$\frac{C_{h2} \cdot \frac{Q_R^2}{2} \cdot \left\{ m^2 \cdot \left( \frac{1}{r} - \frac{1}{d} \right) + m \cdot \left( \frac{2}{d} - \frac{1}{p} - \frac{1}{r} \right) \right\}}{T} + \frac{n \cdot C_o + n \cdot C_{h2} \cdot \frac{Q_P^2}{2 \cdot d}}{T} + \frac{m \cdot C_s + m \cdot C_{h1} \cdot \frac{Q_R^2}{2} \cdot \left( \frac{1}{r} - \frac{1}{p} \right)}{T} \cdot$$

The model leads to the following nonlinear optimization problem:
The model leads to the following nonlinear optimization problem:

$$\left.\begin{array}{l} C(T, Q_P, Q_R, n, m) \to \min \\ \\ m \cdot Q_R + n \cdot Q_P = d \cdot T, \\ m \cdot Q_R = r \cdot T, \\ T > 0,\ Q_P > 0,\ Q_R > 0,\ n, m \text{ pozitív egészértékű.} \end{array}\right\} \qquad \text{(P)}$$

Let us now use the balance equations (1) to simplify the problem. Two continuous variables are substituted in the inventory holding cost function. For the sake of simplicity, let these variables be the batch sizes. Of course, the batch numbers can be chosen, but this choice makes more difficult the investigations.

$$Q_P = \frac{(d - r) \cdot T}{n}$$
$$Q_R = \frac{r \cdot T}{m}$$

After substitution the economic order quantities we obtain a simpler cost function. Let the new cost function denote $C_1(.)$.

$$C_1(T, n, m) = \frac{m \cdot C_s + n \cdot C_o}{T} +$$
$$+ \frac{T}{2} \cdot \left[ \left\{ C_{h2} \cdot r^2 \cdot \left( \frac{2}{d} - \frac{1}{p} - \frac{1}{r} \right) + C_{h1} \cdot r^2 \cdot \left( \frac{1}{r} - \frac{1}{p} \right) \right\} \cdot \frac{1}{m} + \frac{C_{h2} \cdot (d - r)^2}{d} \cdot \frac{1}{n} + C_{h2} \cdot r^2 \cdot \left( \frac{1}{r} - \frac{1}{d} \right) \right]$$

I exclude the cycle time sequentially from this cost function. This function is convex in the cycle time then the necessary conditions of optimality are sufficient, as well. The optimal cycle time is

$$T^o = \sqrt{\frac{2 \cdot (m \cdot C_s + n \cdot C_o)}{\left\{ C_{h2} \cdot r^2 \cdot \left( \frac{2}{d} - \frac{1}{p} - \frac{1}{r} \right) + C_{h1} \cdot r^2 \cdot \left( \frac{1}{r} - \frac{1}{p} \right) \right\} \cdot \frac{1}{m} + \frac{C_{h2} \cdot (d - r)^2}{d} \cdot \frac{1}{n} + C_{h2} \cdot r^2 \cdot \left( \frac{1}{r} - \frac{1}{d} \right)}}.$$

Let this last expression substitute in cost function $C_1(.)$. Then the following cost function $C_2(.)$ is obtained:

$$C_2(n,m)=$$
$$\sqrt{2\cdot(m\cdot C_s+n\cdot C_o)\cdot\left[\left\{C_{h2}\cdot r^2\cdot\left(\frac{2}{d}-\frac{1}{p}-\frac{1}{r}\right)+C_{h1}\cdot r^2\cdot\left(\frac{1}{r}-\frac{1}{p}\right)\right\}\cdot\frac{1}{m}+\frac{C_{h2}\cdot(d-r)^2}{d}\cdot\frac{1}{n}+C_{h2}\cdot r^2\cdot\left(\frac{1}{r}-\frac{1}{d}\right)\right]}$$

or

$$C_2(n,m)=\sqrt{2\cdot d}\cdot\sqrt{A(r)\cdot\frac{m}{n}+B(r)\cdot\frac{n}{m}+C(r)\cdot m+D(r)\cdot n+E(r)},\qquad(2)$$

where

$$A(r)=C_s\cdot C_{h2}\cdot\frac{(d-r)^2}{d},\quad B(r)=C_o\cdot\left[C_{h1}\cdot\left(r-\frac{r^2}{p}\right)+C_{h2}\cdot\left(\frac{2r^2}{d}-\frac{r^2}{p}-r\right)\right],$$
$$C(r)=C_s\cdot C_{h2}\cdot\left(r-\frac{r^2}{d}\right),\quad D(r)=C_o\cdot C_{h2}\cdot\left(r-\frac{r^2}{d}\right),$$
$$E(r)=C_s\cdot\left[C_{h1}\cdot\left(r-\frac{r^2}{p}\right)+C_{h2}\cdot\left(\frac{2r^2}{d}-\frac{r^2}{p}-r\right)\right]+C_o\cdot C_{h2}\cdot\frac{(d-r)^2}{d}$$
.

Problem (P) is simplified as an integer optimization model $C_2(n,m)$. Model (2) is the meta-model of appendix.

### 2.3.5. The integer solution

In the last two chapters I have constructed the continuous solutions for purchasing and repair batch numbers, but now I disregard from them. I apply the expression of appendix direct to produce the discrete solutions.

**Theorem 1.**

The integer solution of the model is

(i) $C_o \cdot \left[ C_{h1} \cdot \left( r - \frac{r^2}{p} \right) + C_{h2} \cdot \left( \frac{2r^2}{d} - \frac{r^2}{p} - r \right) \right] - C_s \cdot C_{h2} \cdot \frac{(d-r)^2}{d} - C_s \cdot C_{h2} \cdot \left( r - \frac{r^2}{d} \right) > 0$,

$$m^o = \left\lfloor \sqrt{\frac{C_o \cdot \left[ C_{h1} \cdot \left( r - \frac{r^2}{p} \right) + C_{h2} \cdot \left( \frac{2r^2}{d} - \frac{r^2}{p} - r \right) \right]}{C_s \cdot C_{h2} \cdot \frac{(d-r)^2}{d} + C_s \cdot C_{h2} \cdot \left( r - \frac{r^2}{d} \right)} + \frac{1}{4}} \right\rfloor + \frac{1}{2}, n^o = 1,$$

(ii)

$$C_o \cdot \left[ C_{h1} \cdot \left( r - \frac{r^2}{p} \right) + C_{h2} \cdot \left( \frac{2r^2}{d} - \frac{r^2}{p} - r \right) \right] - C_s \cdot C_{h2} \cdot \frac{(d-r)^2}{d} - C_s \cdot C_{h2} \cdot \left( r - \frac{r^2}{d} \right) \le 0,$$

$$C_s \cdot C_{h2} \cdot \frac{(d-r)^2}{d} - C_o \cdot \left[ C_{h1} \cdot \left( r - \frac{r^2}{p} \right) + C_{h2} \cdot \left( \frac{2r^2}{d} - \frac{r^2}{p} - r \right) \right] - C_o \cdot C_{h2} \cdot \left( r - \frac{r^2}{d} \right) \le 0,$$

$m^o = 1, n^o = 1$,

(iii) $C_s \cdot C_{h2} \cdot \frac{(d-r)^2}{d} - C_o \cdot \left[ C_{h1} \cdot \left( r - \frac{r^2}{p} \right) + C_{h2} \cdot \left( \frac{2r^2}{d} - \frac{r^2}{p} - r \right) \right] - C_o \cdot C_{h2} \cdot \left( r - \frac{r^2}{d} \right) > 0$

$$m^o = 1, n^o = \left\lfloor \sqrt{\frac{C_s \cdot C_{h2} \cdot \frac{(d-r)^2}{d}}{C_o \cdot \left[ C_{h1} \cdot \left( r - \frac{r^2}{p} \right) + C_{h2} \cdot \left( \frac{2r^2}{d} - \frac{r^2}{p} - r \right) \right] + C_o \cdot C_{h2} \cdot \left( r - \frac{r^2}{d} \right)} + \frac{1}{4}} \right\rfloor + \frac{1}{2}.$$

Here function $\lfloor \cdot \rfloor$ denotes the maximal integer not greater than the argument. The proof of the theorem is easy; with elementary manipulation we get the results. With these results I have finished the investigation of this model.

### 2.3.6. Summary

It is easy to prove that the costs of model of Koh et al. (2002) are higher than that of model of Nahmias and Rivera (1979), if the unit inventory holding costs of usable products are higher than that of reusable items. In other case the model of Koh et al. (2002) functions better, i.e.

with lower costs. This fact was proven by Teunter (2004). After some reformulation of model of Koh et al. (2002) the model becomes problem of a Nahmias and Rivera (1979).

# 3. Inventory Models with Waste Disposal

The objective of the models of last chapter was to minimize the relevant costs, in order to determine the purchasing and repair lot sizes, and the number of their batches. Cost minimization is the primary management goal. A second question arises from management point of view: If the rate of product return is an influenced variable, then how much used products must be recovered in an inventory cycle. The answer will be looked for this question in three models of product recovery management.

The three models consist of two parts: first, the optimal lot sizes and number of batches are determined for reuse/repair and purchasing/manufacturing; secondly, the optimal reuse rate is determined for known unit manufacturing/purchasing, reuse/repair, and waste disposal costs. So I define two steps inventory optimization problems.

The first model was initiated by Richter (1996a). The products (containers in this case) are manufactured in a shop or used containers are repaired in this shop, in order to transport spare parts in them to an other shop. The empty containers are stored at the second shop, and then they are collected and sent back to the first manufacturing-repair shop at the end of the production period. It is decided in the second shop, how many containers are sent back to the first shop for repair and how many containers are disposed off as waste outside. (Waste disposal can mean a secondary market of containers.) The following questions arise in this context: how much percentage of containers are repaired, and lot sizes of manufacturing and repair, if the aim of decision makers is to minimize the relevant costs.

The next model (Teunter (2001)) investigates a remanufacturing situation. Remanufacturing is followed by manufacturing in this model. A known part of sold products is returned from the market, from customers. Waste disposal process begins after ending of remanufacturing, and after that manufacturing process starts. The question is now, how many items must be remanufactured from the returned products, and how many items must be disposed of. This model consists of two parts: minimization of inventory costs, and then determination of reuse rate.

Last, I present a third problem. (Dobos-Richter (2004)) Let us assume that a firm satisfies the demand from production and recycling. I assume that the production and recycling processes go on time, so production and recycling rate is finite. The firm purchases the used products to recycling from market, and the firm can purchase all of its manufactured and used products. The question is the quantity of product bought back from the market so that the firm minimizes the total relevant costs, i.e. the sum of EOQ-type and non EOQ-type costs.

The conditions of the three models are common in inventory holding subsystem. These conditions are found at the end of chapter 2. The conditions must be extended with the assumption that non EOQ-type costs are linear.

Let us now brief summarize the solution of the models. Since the aim of the models is to determine the optimal reuse rate, the results are similar for these problems. The optimal reuse is on the boundary in the optimal solution, i.e. either all of the returned items must be recovered and the rests are replaced from manufacturing, or the demand is met from manufacturing and all returned items are disposed off. I prove this property in these models.

And now I present the models.

## 3.1. A Repair Model with Three Stocking Point

### 3.1.1. Introduction

In the Economic Order Quantity model for one item the optimal quantity to be ordered or to be produced is determined. It is assumed that the sum of fixed cost and holding cost per time unit is minimized and some deterministic stationary demand is satisfied. The advantage of this model consists in its simplicity as well as in the possibility to express explicitly the optimal lot size and the minimum cost. For the inputs $s$ = set-up cost, $h$ = per time unit per unit holding cost, $d$ = time unit demand the practicable optimal lot size $x^* = \sqrt{2ds/h}$ and the minimum cost $K^* = \sqrt{2dsh}$ are discussed in many textbooks of production economics.

It is a reasonable question to ask how these widely used solutions will change if the model is regarded in the framework of reverse logistics or remanufacturing (Fleischmann et al. (1997), Gupta (1995), Kelle and Silver (1989)) and one fraction $\beta$ of products having been produced will be reused and the other fraction $\alpha = 1\text{-}\beta$ will be disposed off. By of one of the authors it was shown in former papers (Richter (1994), Richter (1996a, b, c), Richter (1997), Richter and Dobos (1999)) that just economic pressure (cost minimization) implies a certain level of reuse $\beta$ (or disposal $\alpha$) of products and by this implies a certain ecological attitude, no matter which technical or technological restrictions have to be considered.

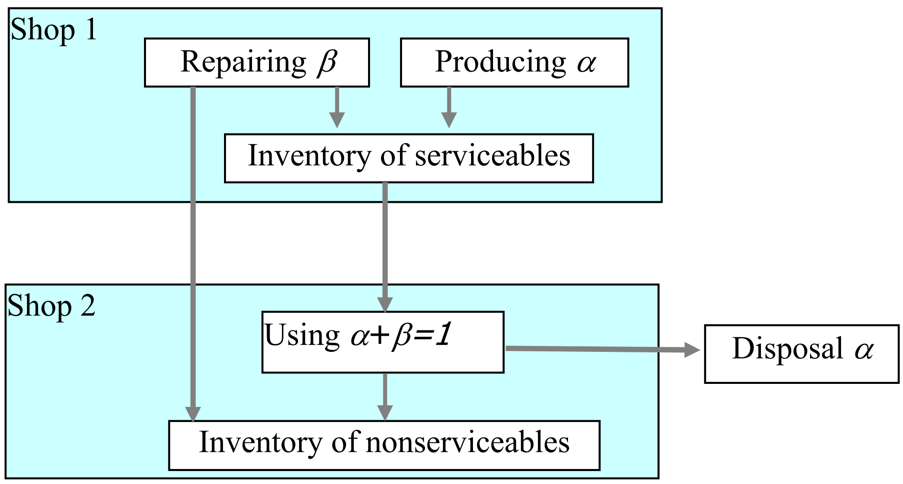



**Figure 1. The 2-shop system**

There is a variety of situations which might be studied and only one of them (Richter and Dobos (1999)) is discussed here: In one shop new products are produced as well as used products are repaired. The equally good regarded new products and repaired products (serviceable items) are then used just a moment in a second shop and after that they are either disposed off at unit cost $e$ or stored as nonserviceable items at per time unit per unit cost $u$ up to the end of a variable collection interval $[0, T]$ (see Fig. 1).

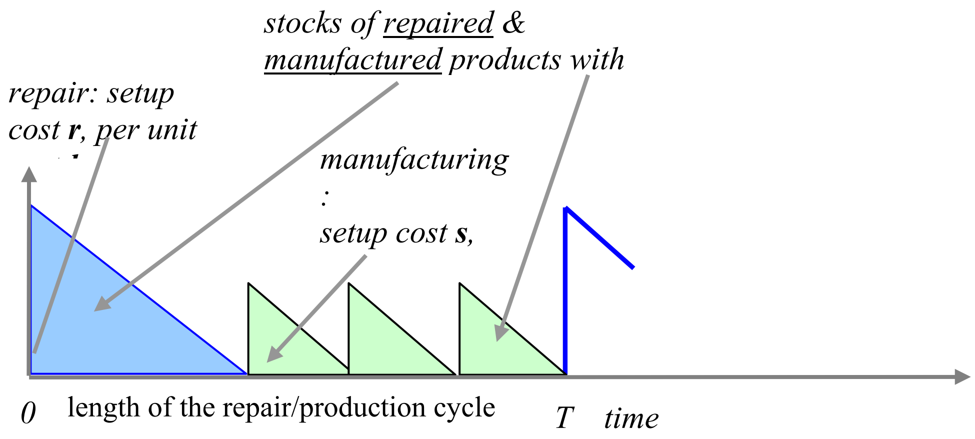



**Figure 2: Cost inputs and inventory stocks for the set-up numbers *m* = *1* and *n* = *3* at the first shop**

In the first shop the process starts with repairing $m$ lots of used products with the size $\beta dT/m$ at unit cost $k$ and at set-up cost $r$, and later $n$ lots of new products with the size $\alpha dT/n$ are produced at unit cost $b$ and at set-up cost $s$. The variable numbers $m$ and $n$ are called set-up numbers. The total lot size $x$ of one repair/production cycle of length $T$ is $m\beta dT/m + n\alpha dT/n = dT$ (see Fig. 2).

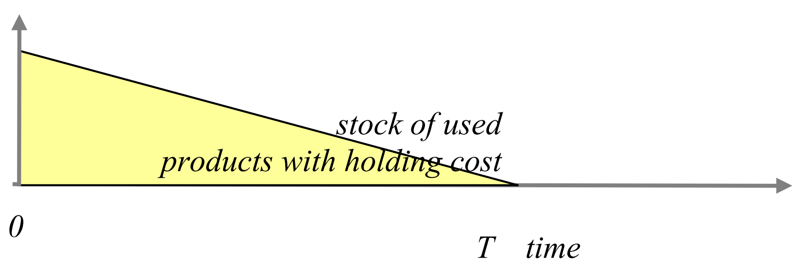



**Figure 3. Cost inputs and inventory stocks for the setup numbers *m* = *1* and *n* = *3* at the second shop**

The products are delivered according to the demand of the second shop and from every $d$ units the fractions $\alpha d$ is disposed off and $\beta d$ is stored (see Fig. 3). The parameters $\alpha, \beta$ are called waste disposal rate and repair rate, respectively.

The problem sketched here covers three levels of complexity and in this way three models which use the functions determined in the previous model:

*Model I:* For given rates and set-up numbers, the cost-minimal total lot size $x(m,n,\alpha) = dT$, i.e. $x(m,n,\alpha) \in \arg\min_{x} G(x,m,n,\alpha)$, can be found by simple calculus. Then the minimal cost is $G(m,n,\alpha) = G(x(m,n,\alpha),m,n,\alpha)$. This model shows the impact of the fixed ecological attitude $\alpha$ and the given set-up numbers on the total lot size and on the length of the repair/production cycle.

*Model II:* For the given rate the optimal set-up numbers $m(\alpha)$ and $n(\alpha)$ can be determined, i.e. $\left(m(\alpha),n(\alpha)\right) \in \arg\min_{m,n} G(m,n,\alpha)$. The minimal cost is then $G(\alpha) = G(m(\alpha),n(\alpha),\alpha)$. This model shows the impact of the fixed ecological attitude $\alpha$ on the value of the set-up numbers and to the lot size.

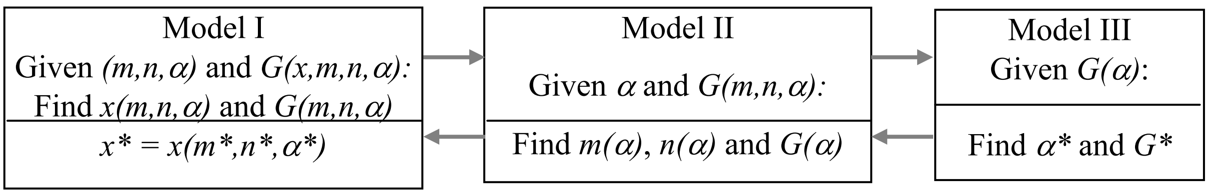



**Figure 4. Three levels of the problem solving**

*Model III:* The optimal $\alpha^* \in \arg\min_{\alpha} G(\alpha)$ can be found. The minimal cost is $G^* = G(\alpha^*)$. This model shows which ecological attitude follows from the overall cost minimization.

The overall minimal cost, i.e. the cost regarded for all three levels is then $G^* = \min_{\alpha} \min_{m,n} \min_{x} G(x,m,n,\alpha)$, or, as presented in Fig. 4.

If the set-up numbers are allowed to be non-integer then a calculus based solution solves also the second level problem (Richter (1994), Richter (1996a, b, c), Richter (1997), Richter and

Dobos (1999))). The function $G(m,n,\alpha)$ has been separated in (Richter (1996a, b, c), Richter (1997), Richter and Dobos (1999))) into $G(m,n,\alpha) = K(m,n,\alpha) + R(\alpha)$ where $K(m,n,\alpha)$ is the so called EOQ-related cost and $R(\alpha) = d(\alpha(e+b)+(1-\alpha)k)$ expresses the EOQ-independent per time unit repair, manufacturing and disposal cost. As long as model I /II are regarded where $\alpha$ is fixed, the cost $R(\alpha)$ does not have impact on the minimum and the main question is the determination of the optimal solutions for $K(m,n,\alpha)$. This remains also true for model III because of the linearity of $R(\alpha)$.

Here, however, it will be asked what happens if the numbers *m* and *n* must be integers. In papers Richter (1994), Richter (1996a, b, c), Richter (1997), Richter and Dobos (1999)) the non-integer problem of minimizing the cost function $G(m,n,\alpha)$, i. e. the non-integer problem of minimizing the cost function $K(m,n,\alpha)$ on $m,n \geq 1$, has been solved by analyzing a special fractional program $S(m,n) \rightarrow min$, subject to $m,n \geq 1$.

Similarly, the integer problem

$$G(m,n,\alpha) \rightarrow min\ [= G(\alpha)], \quad m,n \in \{1,2,...\} \qquad (1)$$

$$K(m,n,\alpha) \rightarrow min\ [= K(\alpha)], \quad m,n \in \{1,2,...\}$$

can be solved if a solution is found for minimizing that auxiliary function $S(m,n)$ on $m,n \in \{1,2,...\}$. In paper Richter and Dobos (1999) the integer problem (1) has been analyzed and it was stated that

- the optimal solution is boundary, i.e. one of the lot numbers equals one under certain conditions,
- some relative error for the non-optimality of boundary solutions can be given,
- the minimum cost functions are partly convex and partly concave under various conditions and
- these results can be applied to the initial models I-III.

In this papers all properties from paper of Richter (1997) will be proved now, will be discussed in greater detail and an example will be provided that shows that the optimal integer solution is not necessarily boundary.

The auxiliary function $S(m,n)$ has been derived in papers of Richter (1994), Richter (1996a, b, c), Richter (1997), Richter and Dobos (1999)) from the following cost function

$$K(m,n,\alpha) = \sqrt{2d(mr+ns)\left(h\frac{\alpha^2}{n}+(h-u)\frac{\beta^2}{m}+u\left(\beta+\beta^2\right)\right)}\ , \qquad (2)$$

which covers the EOQ-related cost from $G(m,n,\alpha)$. Due to monotonicity considerations the function $S(m,n,\alpha) = (mr+ns)(h\alpha^2/n+ (h-u)\beta^2/m+u(\beta+\beta^2))$, (3)

can be analyzed instead of (2). For both functions the relationship $K(m,n,\alpha)= \sqrt{2d\cdot S(m,n,\alpha)}$ holds.

The parameters in the function (3) can be replaced by

$A = rh\alpha^2,\ B = s(h-u)\,\beta^2,\ C = ru(\beta+\beta^2),\ D= su(\beta+\beta^2),\ E = sh\alpha^2+r(h-u)\,\beta^2$ (4)

and, thus, the function

$$S(m,n) = A\frac{m}{n}+B\frac{n}{m}+Cm+Dn+E\ , \qquad (5)$$

appears. If the minimum of the function (5) on $m,n \in \{1,2,...\}$ is determined then also the problem $K(m,n,\alpha) \rightarrow min$ on $m,n \in \{1,2,...\}$ is solved, and, since $G(m,n,\alpha) = K(m,n,\alpha) + R(\alpha)$. Therefore, it will be asked in the next section, which solution can be found for the problem of minimizing the function (5) on the set of positive integers.

### 3.1.2. Application to the repair and waste disposal model

#### 3.1.2.1. The optimal solution

Due to paper of Richter (1997) the relations $A, C, D, B+D, E > 0$ hold and the results of the Lemma 4 of appendix are to be applied.

**Theorem 1 (Richter (1997)):** The continuous optimal solution for minimizing the function (2) is

(i) $\{h > u\} \wedge \{s(h-u)\,\beta^2 \geq r\left(\alpha^2 h+ \beta u(1+\beta)\right)\} \Rightarrow m(\alpha) = \beta\sqrt{\frac{s(h-u)}{r\left(\alpha^2 h+ \beta u(1+\beta)\right)}},\ n(\alpha)= 1,$

(ii) $rh\alpha^2-su\beta(1+\beta) \leq s(h-u)\beta^2 \leq r\left(\alpha^2 h+ \beta u(1+\beta)\right) \Rightarrow (m(\alpha),n(\alpha)) = (1,1),$

(iii) $\alpha^2 rh \geq s\,\beta(\beta h+u) \Rightarrow m(\alpha) =1,\ n(\alpha) = \alpha\sqrt{\frac{rh}{s\beta(\beta h+u)}},$

According to the cases (i) - (iii) the region $(0,1)$ is separated into three subregions and it is clear that the problem of finding an optimal integer solution needs special attention only for $\alpha \in (0, \alpha_1) \cup (\alpha_2,1)$. Therefore the two cases (i) and (iii) will be studied in detail and the results of Theorem 3 from [13] will be extended.

**Lemma 1:** (i) $49A \leq 527C$ holds if $\alpha \leq \alpha_3 = \dfrac{1054u}{49h+1054u}$ and $49h\text{-}527u > 0$ and $\alpha_3 = 2/3$ in the opposite case and (iii) $49B \leq 527D$ holds if $\alpha \geq \alpha_4 = \dfrac{49h-1103u}{49h-576u}$ and $49h > 576u$ and for every $\alpha$ in the opposite case.

Proof: (i) The inequality holds if and only if $49h\alpha^2 \leq 527u\beta(1+\beta)$ or $(49h\text{-}527u)\alpha^2 + 1571u\alpha \leq 1054u$. Since $u > 0$ and $\alpha \geq \alpha^2$ holds the inequality is satisfied at most by $\alpha \leq \alpha_3$.
(iii) The other inequality holds if and only if $49(h\text{-}u)\beta \leq 527u(1+\beta)$ is fulfilled or $(49h\text{-}576u)\beta \leq 527u$. If $49h \leq 576u$ then that inequality holds for every $\beta$ and $\alpha$. In the other case the inequalities $\beta \leq 527u/(49h\text{-}576u)$ and $\alpha \geq \alpha_4$ hold. □

**Lemma 2:** The boundaries of the two sets $\alpha_1$ and $\alpha_2$ fulfill:
(i) $\alpha_1 \leq \dfrac{s(h-u)-2ru}{s(h-u)+r(h-2u)}$ if $s(h\text{-}u) > r(h+u)$ and $\alpha_1 \leq \dfrac{h-u}{2h-u}$ in the other case.
(iii) $\alpha_2 \geq \dfrac{s(h+u)}{s(h+u)+hr}$ if $r > s$ and $\alpha_2 \geq \dfrac{h+u}{2h+u}$ if $r \leq s$

Proof: (i) The relation $B \geq A+C$ holds if $h > u$ and $\beta^2 s(h\text{-}u) \geq r\left(\alpha^2 h + \beta u(1+\beta)\right)$ hold. After some transformation the inequality $\beta^2[s(h\text{-}u) - r(h+u)] + \beta r(2h\text{-}u) \geq rh$ has to be analyzed and the formulas under (i) appears.
(iii) The relation $A \geq B+D$ implies that $rh\alpha^2 \geq s(1\text{-}\alpha)[h(1\text{-}\alpha)+u]$ and that

$$\alpha^2 h(r\text{-}s) + \alpha s(2h+u) \geq s(h+u). \quad (6)$$

If $s=r$ the statement holds obviously. If $r > s$ then

$$\alpha[hr + s(h+u)] = \alpha h(r\text{-}s) + \alpha s(2h+u) \geq \alpha^2 h(r\text{-}s) + \alpha s(2h+u)$$

and every $\alpha$ satisfying (11) satisfies also $\alpha[hr+ s(h+u)] \geq s(h+u)$ (7)
Therefore the lower bound for $\alpha$ obtained from (13) is also a bound for (12).
If $r < s$ then every $\alpha$ satisfying (12) also satisfies $\alpha s(2h+u) \geq s(h+u)$. □

Remark: Due to the Lemmas 1 and 2 boundary optimal integer solutions will be found if

$\alpha_1 \leq \alpha_3$ and $\alpha_4 \leq \alpha_2$, (8)

respectively. In the next Theorem conditions are formulated which secure that the relations (14) hold.

**Theorem 2:** The optimal integer solution of the EOQ repair and waste disposal model is boundary (i) if $\alpha \leq \alpha_1$ and

(ia) $s(h\text{-}u) \leq r(h+u)$ and $49u < 49h \leq 1103u$ or (ib) $49h > 527u$ and $\frac{1152}{49}ur \geq s(h\text{-}u) > r(h+u)$ or (ic) $49u < 49h \leq 527u$ and $2r(h+u) \geq s(h\text{-}u) > r(h+u)$ or

(iii) if $\alpha_2 \leq \alpha$ and

(iiia) $49h \leq 576u$, or (iiib) $49h > 576u$, $r > s$ and $527us(h+u) \geq hr(49h\text{-}1103u)$, or (iiic) $49h > 576u$, $r \leq s$ and $527u^2+1640uh \geq 49h^2$.

Proof: (i) With respect to the previous two lemmas the relation $\alpha_1 \leq \alpha_3$ holds if and only (ia) if $\frac{h-u}{2h-u} \leq \frac{1054u}{49h+1054u}$ or (ib) if $\frac{s(h-u)-2ru}{s(h-u)+r(h-2u)} \leq \frac{1054u}{49h+1054u}$. The analysis of these inequalities produces the mentioned results.

(iiia) If $49h \leq 527u$ then due to Lemma 7 the inequality $\alpha_4 \leq \alpha_2$ holds obviously. If $49h > 527u$ the cases (iiib) and (iiic) are dealt with by analyzing the inequalities

$\alpha_4 = \frac{49h-1103u}{49h-576u} \leq \frac{s(h+u)}{s(h+u)+hr} = \alpha_2$ and $\frac{49h-1103u}{49h-576u} \leq \frac{h+u}{2h+u}$ . □

Remark: Let some simple cases be considered:

*r = s:* Then $49h \leq 576u$ guarantees the boundary property of the optimal integer solution for a wide range of inputs.

*h = u:* Then $\alpha_1 = \alpha_4 = 0$, i.e. the first region is empty and for the second region the boundary property is fulfilled.

By applying the Theorem 2 of the appendix the following property of the boundary optimal integer solution can be found.

**Theorem 3 (Richter and Dobos (1999)):** The boundary optimal solutions for the discrete EOQ repair and waste disposal problem is:

$$(i)\ \alpha \le \alpha_1 \Rightarrow\ m^g(\alpha) = \left\lfloor \sqrt{\frac{B}{A+C} + \frac{1}{4}} + \frac{1}{2} \right\rfloor,\ n^g(\alpha) = 1$$

$$(iii)\ \alpha \ge \alpha_2 \Rightarrow m^g(\alpha) = 1,\ n^g(\alpha) = \left\lfloor \sqrt{\frac{A}{B+D} + \frac{1}{4}} + \frac{1}{2} \right\rfloor \qquad (15)$$

A more detailed expression of the optimal solution will not be given at the moment.

### 3.1.2.2. The boundary solution as an approximate solution

Let now the Theorem 4 of the appendix be applied to the EOQ repair and waste disposal model.

**Theorem 4 (Richter and Dobos (1999)):** The relative error of an optimal boundary solution is $dK_G = \frac{K^b - K^*}{K^*} \le \frac{1}{48}$, where $K^b$ denotes the minimal value for boundary solutions and $K^*$ denotes the global minimum.

Proof: Let $dK_G = \frac{K^b - K^*}{K^*}$ be estimated. Since $K(m,n,\alpha) = \sqrt{2d \cdot S(m,n,\alpha)}$ the relation can be expressed as $dK_G = \frac{\sqrt{S^b} - \sqrt{S^*}}{\sqrt{S^*}} = \frac{S^b - S^*}{(\sqrt{S^b} + \sqrt{S^*}) \cdot \sqrt{S^*}} \le \frac{S^b - S^*}{2S^*} \le \frac{1}{48}$.

Hence, the relative error of optimal boundary solutions is not greater than *2.1 %.*

### 3.1.2.3. Minimum cost for the integer problem

According to the first section the minimum cost $K_g(\alpha) = min\{K(m,n,\alpha): m,n \in \{1,2,...\}\}$ is

$$K_g(\alpha) = \sqrt{2d\left( A\frac{m^g(\alpha)}{n^g(\alpha)} + B\frac{n^g(\alpha)}{m^g(\alpha)} + Cm^g(\alpha) + Dn^g(\alpha) + E \right)}$$

If the set-up numbers are integer this function coincides with $K(\alpha)$. If $\alpha$ is changing there must be switching points where two neighbor set-up numbers are optimal. Below these points are determined and the behavior of the cost function between these switching points is characterized.

**Lemma 3:** If the boundary property holds there are two sets $M$ and $N$ of switching numbers $\alpha$ with the properties

(i) $M = \{\alpha: B = m(m+1)(A+C), m=1,2,...\}$ and

$K_g(\alpha) = K(m^g(\alpha),1,\alpha) = K(m^g(\alpha)+1,1,\alpha)$ for $\alpha \in M$ and

(iii) $N = \{\alpha: A = n(n+1)(B+D), n=1,2,...\}$ and

$K_g(\alpha) = K(1,n^g(\alpha),\alpha) = K(1,n^g(\alpha)+1,\alpha)$ for $\alpha \in N$.

Proof: It follows from $K(m^g(\alpha),1,\alpha) = K(m^g(\alpha)+1,1,\alpha)$ immediately that $B = m^g(\alpha)(m^g(\alpha)+1)(A+C)$. Hence, the structure of $M$ is proved. The second case can be dealt with in the same way. □

Remark: The finite set $M = \{\alpha^{m1}, \alpha^{m2},..., \alpha^{ml}\}$ and $N = \{\alpha^{n1}, \alpha^{n2}, \alpha^{n3},...\}$ separate the regions $(0, \alpha_1)$ and $(\alpha_2,1)$ into such subsets of identical optimal set-up numbers $m^g(\alpha)$ and $n^g(\alpha)$.

Let for instance, $h = u$ and $r = s$. Then the set $N = \{\alpha: \alpha^2 = n(n+1)(2-3\alpha+\alpha^2)\} = \{0.764, 0.883, 0.932, 0.956, 0.97, 0.978...\}$

Let finally the behaviour of the function $K_g(\alpha)$ be studied at the different intervals $[\alpha^{mi}, \alpha^{m,i+1}]$ and $[\alpha^{nj}, \alpha^{n,j+1}]$.

**Lemma 4 (Richter and Dobos (1999)):** The function $K_g(\alpha)$ is

(i) convex on the intervals $[\alpha^{mi}, \alpha^{m,i+1}], i=1,2,...,l,$

(ii) convex on $[\alpha_1, \alpha_2]$ if $4h^2+4hu-u^2 \geq 0,$

(iii$_1$) convex on $[\alpha^{nj}, \alpha^{n,j+1}]$ if $4(h^2 +hu) \geq u^2n(\alpha^{nj})$ and

(iii$_2$) concave in the other cases. In other words, the function is partly piecewise convex and piecewise concave.

Proof:

Let $n = n(\alpha^{nj})$. The careful analysis of $K_g(\alpha) = \sqrt{2d\left(\frac{A}{n(\alpha^{nj})} + Bn(\alpha^{nj}) + C + Dn(\alpha^{nj}) + E\right)}$

shows that the function is convex and concave in the appropriate situations. □

Remark: Lemma 8 (iii) makes clear that for small $\alpha$ and $n(\alpha)$ the function $K_g(\alpha)$ might be convex, although the non-integer continuous function $K(\alpha)$ is concave!

Due to the linearity of $R(\alpha)$ the properties of this function hold also for the total cost function $G_g(\alpha) = K_g(\alpha) + R(\alpha)$, i.e. the total cost is also partly piecewise convex and partly piecewise concave, respectively. As in the continuous case, if $0 < \alpha < 1$, there might exist some optimal (cost minimal) waste disposal rate. If, however, the extreme values $\alpha = 1$ are feasible, then one of them is optimal. (Richter (1997))

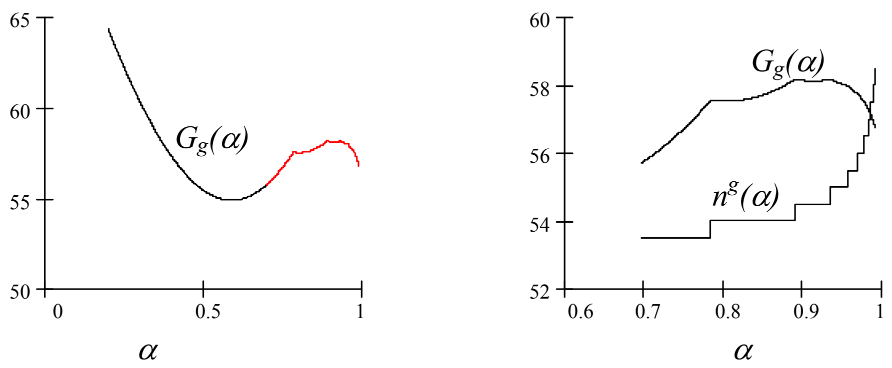



**Figure 5. The minimum cost function $G_g(\alpha)$ and the corresponding set-up numbers $n^g(\alpha)$**

*Example*: Let $s = 200$, $r = 100$, $h = 6$, $u = 3$, $e = 5$, $d = k = b = 1$. Then the optimal integer solution is obviously boundary and the previous Lemmas can be used to express the minimum cost (comp. Fig. 5)

### 3.1.3. Conclusion

The EOQ repair and waste disposal model was analyzed. The variable set-up numbers $n$ and $m$ for production and repair within some collection time interval were supposed to be natural. First, conditions for some auxiliary fractional program were discussed to have optimal integer solutions at the boundary of the feasible region. Secondly, these conditions were used to determine the optimal integer solution and the minimum cost for the repair and waste disposal model for a wide class of model inputs. Thirdly, it was shown that the minimum cost is a partly piecewise convex, partly piecewise concave function of the waste disposal rate and the relative error of optimal boundary solutions is not greater than *2.1 %*. Several problems are subject to further studies as for instance how to determine the optimal integer solution, if the optimal solution is not boundary, how to include non-linear repair cost and finite production/repair rates in the integer model.

## 3.2. A Recoverable Item Inventory System

### 3.2.1. Introduction

Quantitative models for inventory systems with product recovery management provide an actual generalization of classical EOQ models. The classical EOQ model analyzes one product inventory systems. The difficulty of recovery system is that a number of authors have proposed such models. This chapter deals with one of these proposals, we investigate the model of Teunter (1999).

Teunter in his work has stated that in the proposed model there should be either no more than one manufacturing batch and no more than one remanufacturing batch in a cycle. A cycle is a sequence of activities with a fixed number of batches.

The goal of the chapter is to reconsider the Teunter`s model. First the explicit model will be discussed and a solution is given for this model, because the author has neglected to describe the explicit model. After solving the problem, we give a counterexample, where the manufacturing and remanufacturing batches are strictly greater than one. By this counterexample we show that the Teunter's graphical proof, that one of the batch numbers equal to one is not correct. In fact, he proved this property of the batch numbers for the assumption that only relatively prime (coprime) batch numbers of manufacturing and remanufacturing are considered, or in other words, for batch numbers with a greatest common divisors greater than one.

The paper continues the investigations of the proposed model. Now we assume that the planning horizon, as cycle time, is decision variable.

### 3.2.2. The model

Teunter has investigated in his model the following activities:

- remanufacturing,
- disposal and
- manufacturing.

Let a *cycle* be the above-mentioned schedule with fixed batch sizes for manufacturing and remanufacturing. In a planning period there is *only one* cycle. (This can be proved very easily by grouping the remanufacturing and manufacturing lots.)

The goal of the decision maker is to minimize the cost for manufacturing and remanufacturing batch numbers and sizes and for the reuse rate. There are EOQ-oriented setup and holding costs for remanufacturing and manufacturing, linear production and remanufacturing costs, linear disposal cost and holding cost for non-serviceable items.

The notations of the model are the following:

*System parameters:*

- $r$ return rate ($0 \leq r \leq 1$),
- $\lambda$ rate of demand.

*Cost parameters:*

- $K_m$ setup cost for manufacturing,
- $K_r$ setup cost for remanufacturing,
- $h_m$ holding cost for manufactured items,
- $h_r$ holding cost for remanufactured items,
- $h_n$ holding cost for non-serviceable items,
- $c_m$ manufacturing cost,
- $c_r$ remanufacturing cost,
- $c_d$ cost for disposing one non-serviceable item.

Let us assume that $c_m + c_d > c_r$, i.e. the unit manufacturing and disposal costs are strictly greater than the cost of remanufacturing. If the material flow of this model is studied, then it cen be seen that the disposed items must be newly manufactured. For this reason these two costs must be summarized. If remanufacturing is economical, then the unit remanufacturing costs are lower than that of manufacturing and disposal.

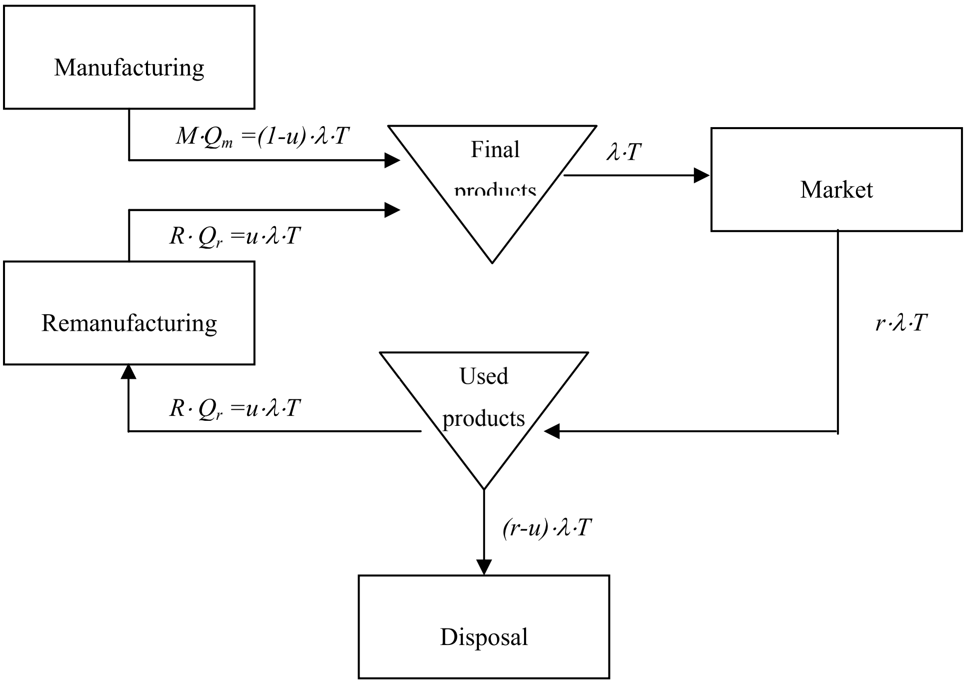
*Decision variables:*

- $Q_m$ batch size for manufacturing,

- $Q_r$ batch size for remanufacturing,
- $M$ number of manufacturing batches, positive integer,
- $R$ number of remanufacturing batches, positive integer,
- $T$ length of the product recovery cycle,
- $u$ reuse rate ($0 \leq u \leq r$).

We assume that all parameters and the decisions variables are nonnegative numbers. We will describe the mathematical model with some application.

**Fig. 1. Material flow in the model**



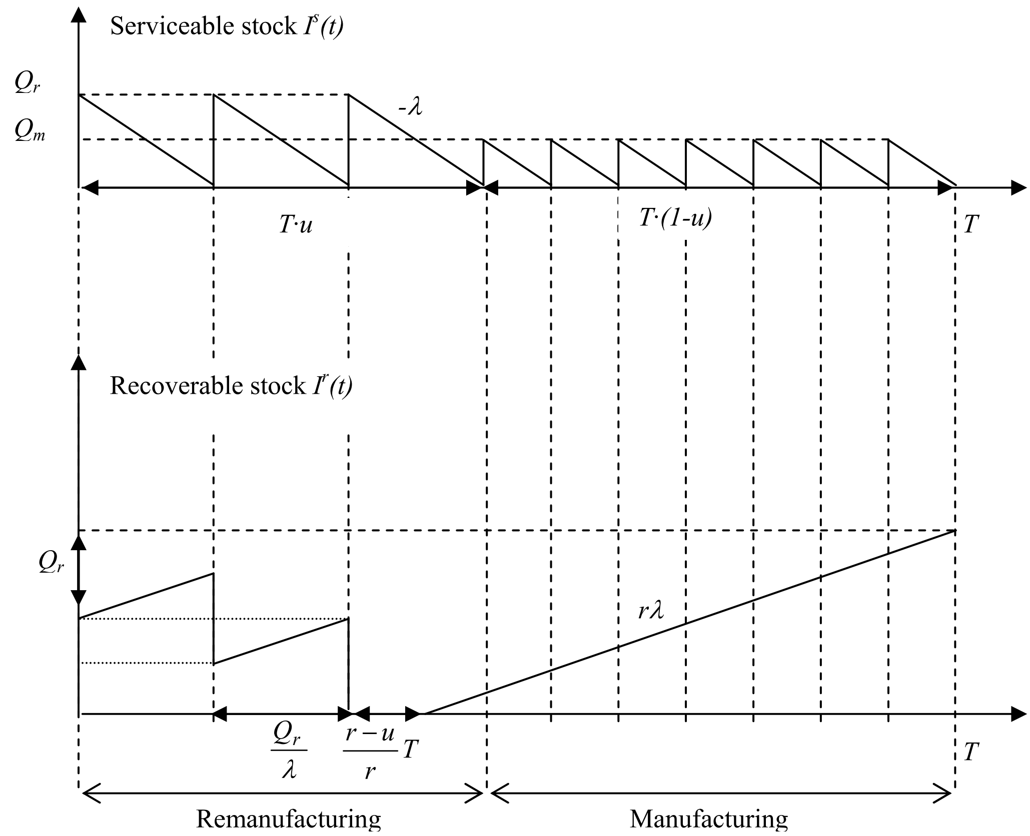

First we examine stock-flow balance of serviceable and recoverable stocks. The equation (1) shows that the sum of manufactured and remanufacted products must cover the demand in a cycle. Equation (2) is the relation between the returned products and the use of these products for remanufacturing and disposal. The material flow of the model is shown in Figure 1.

$$M \cdot Q_m + R \cdot Q_r = \lambda \cdot T \qquad (1)$$

$$R \cdot Q_r + (r-u) \cdot \lambda \cdot T = r \cdot T \qquad (2)$$

From the linear systems (1) and (2) we can write two separate equations for the manufacturing and remanufacturing bathes:

**Fig. 2. Modelling the inventory policy** ($R$=3, $M$=7)




$$M \cdot Q_m = (1-u) \cdot \lambda \cdot T \tag{3}$$

and

$$R \cdot Q_r = u \cdot \lambda \cdot T \tag{4}$$

If the reuse rate is equal to zero, i.e. $u$=0, than the remanufacturing lot size is zero, i.e. $Q_r=0$ in relation (4). It means that all returned parts are disposed, there is no reuse in system and the management problem turns into a simple inventory problem. Another interesting case is, if the return rate equal to reuse rate ($u$=$r$). This case shows an example, when all returned parts are reused and there is no disposal activity. Identity (3) and (4) will be useful to create our cost function.

Now we construct the total cost function. We make it in two steps. In the first step we investigate the inventory holding cost function $H(Q_m,Q_r,T,M,R,u)$ for serviceable and non-serviceable parts.In the second step we describe the linear costs $L(Q_m,Q_r,T,M,R,u)$ of manufacturing, remanufacturing, and disposal.

Let us now calculate the inventory holding costs $H(Q_m,Q_r,T,M,R,u)$. The inventory holding policy is shown in Figure 2. This policy is a predetermined policy and we look for the optimal parameters $(Q_m,Q_r,T,M,R,u)$ of this strategy. Let us assume that the inventory level functions for a known strategy are function $I^s(t)$ for serviceable stock and function $I^r(t)$ for recoverable stock, $0 \le t \le T$. The inventory holding costs are the area below this functions, i.e.

$$H(Q_m,Q_r,M,R,u) = h_r \cdot \int_0^{T\cdot u} I^s(t)dt + h_m \cdot \int_{T\cdot u}^{T} I^s(t)dt + h_n \cdot \int_0^{T} I^r(t)dt .$$

Now we use the property of the inventory policy that the sum of serviceable and recovarable products is a monotone decreasing, linear and continuous function of time in the remanufacturing cycle. So the inventory cycle can be divided into two subcycles.

(1) the demand is satisfied from remanufacturing, and the recoverable stock is positive. The length of this interval is equal to $T \cdot u - \frac{Q_r}{\lambda}$.
(2) The demand is satisfied from the last remanufacturing batch and from manufacturing, and the stock level of recoverable items are monotone nondecreasing. A remanufacturing batch is used in an interval length of $\frac{Q_r}{\lambda}$. The length of this subcycle is $T \cdot (1-u) + \frac{Q_r}{\lambda}$.

So the inventory holding cost function can be expressed with the help of the cycles, as

$$H(Q_m,Q_r,M,R,u) =$$

$$(h_r - h_n) \int_0^{T\cdot u - \frac{q_r}{\lambda}} I^s(t)dt + h_r \int_{T\cdot u - \frac{q_r}{\lambda}}^{T\cdot u} I^s(t)dt + h_m \int_{T\cdot u}^{T} I^s(t)dt + h_n \int_0^{T\cdot u - \frac{q_r}{\lambda}} \left[ I^s(t) + I^r(t) \right] dt + h_n \int_{T\cdot u - \frac{q_r}{\lambda}}^{T} I^r(t)dt \, .$$

We must now calculate the five integrals. The first integral consists of $R$-1 pieces remanufacturing batches. The costs are $(h_r - h_n)\cdot \int_0^{T\cdot u-\frac{q_r}{\lambda}} I^s(t)dt = (R-1)\cdot(h_r - h_n)\cdot \frac{Q_r^2}{2\lambda}$. The second integral is only a remanufacturing batch $h_r \cdot \int_{T\cdot u-\frac{q_r}{\lambda}}^{T\cdot u} I^s(t)dt = h_r \cdot \frac{Q_r^2}{2\lambda}$. The third value is the cost of inventory holding of manufactured products, which consists of $M$ batches $h_m \cdot \int_{T\cdot u}^{T} I^s(t)dt = M \cdot h_m \cdot \frac{Q_m^2}{2\lambda}$. The computation of the fourth integral is a little bit complicated. We have pointed out that the sum of the inventory levels $I^s(t)+I^r(t)$ is a monotone decreasing linear function. The tangent of this linear function is $(1-r)\cdot\lambda$. In point of time $T\cdot u - \frac{Q_r}{\lambda}$ this function has a value of $Q_r$. With this assumption the value of the integral is $h_n \cdot \int_0^{T\cdot u-\frac{q_r}{\lambda}} \left[I^s(t)+I^r(t)\right]dt = h_n \cdot \left[Q_r + (1-r)\cdot\frac{\lambda}{2}\cdot\left(T\cdot u - \frac{Q_r}{\lambda}\right)\right]\cdot\left(T\cdot u - \frac{Q_r}{\lambda}\right)$. The fifth, last integral is $h_n \cdot \int_{T\cdot u-\frac{q_r}{\lambda}}^{T} I^r(t)dt = h_n \cdot r \cdot \frac{\lambda}{2}\cdot\left(\frac{1-r}{r}\cdot T\cdot u + \frac{Q_r}{\lambda}\right)^2$.

Summing up the integrals and with elementary calculations, we have the following expression for the inventory holding costs:

$$H(Q_m, Q_r, T, M, R, u) = R\cdot h_r \cdot \frac{Q_r^2}{2\lambda} + M\cdot h_m \cdot \frac{Q_m^2}{2\lambda} + h_n \cdot \frac{Q_r^2}{2\lambda}\cdot\left\{R^2 \cdot \frac{1-r}{r} + R\right\}.$$

In this expression we have applied from equation (4) that $T\cdot u = \frac{R\cdot Q_r}{\lambda}$.

The linear costs of manufacturing, remanufacturing, and disposal can be calculated very easily:

$$L(Q_m,Q_r,T,M,R,u) = c_m\cdot M\cdot Q_m + c_r\cdot R\cdot Q_r + c_d\cdot\lambda\cdot T\cdot(r-u) = T\cdot\lambda\cdot\left[u\cdot(c_r - c_m - c_d) + (c_m + c_d\cdot r)\right]$$

Now we can formulate the average cost function $C_a(Q_{m,},Q_r,T,M,R,u)$ summing up the total inventory holding costs (setup and inventory holding costs) and the linear manufacturing, remanufacturing, and disposal costs, and divided with the length of the cycle:

$$C_a(Q_m,Q_r,T,M,R,u)=$$
$$=\frac{R\cdot K_r+R\cdot h_r\cdot\frac{Q_r^2}{2\lambda}+M\cdot K_m+M\cdot h_m\cdot\frac{Q_m^2}{2\lambda}+h_n\cdot\frac{Q_r^2}{2\lambda}\cdot\left\{R^2\cdot\frac{1-r}{r}+R\right\}}{T}+$$
$$+u\cdot\lambda\cdot(c_r-c_m-c_d)+\lambda\cdot(c_m+c_d\cdot r)$$

This way we have constructed a non-linear mixed-integer mathematical programming problem:

$$\left.\begin{array}{l} C_a(Q_m,Q_rT,M,R,u)\rightarrow \min \\ \text{such that} \\ M\cdot Q_m=(1-u)\cdot\lambda\cdot T \\ R\cdot Q_r=u\cdot\lambda\cdot T \\ 0\le u\le r \\ Q_m\ge 0, Q_r\ge 0, T>0, M,R \text{ positive integers} \end{array}\right\} \quad \text{(P)}$$

Before solving problem (P) we will show that

In the next section we will solve the problem for the relevant variables.

### 3.2.3. Solutions of the model

In this section we solve proble (P) in two different way. The difference is the order of eliminating the continuous variables from the cost function using equations (3) and (4). The first method offers to eliminate the integer variables *R* and *M*, in order to express the manufacturing and remanufacturing lot sizes. Second method suggests elimination the lot sizes in order to investigate an integer programming problem. Let us follow this two ways.

### 3.2.3.1. Elimination of lot sizes $Q_r$ and $Q_m$

We can follow an other way to solve the problem. Let us substitute the manufacturing and remanufacturing lot sizes $Q_m = \frac{(1-u)\cdot\lambda\cdot T}{M}$ and $Q_r = \frac{u\cdot\lambda\cdot T}{R}$ in the cost function from equations (3) and (4). After substitution the the problem has the next form:

$$\left.\begin{aligned}
&\frac{R\cdot K_r + M\cdot K_m}{T} + T\cdot\left[\frac{h_r+h_n}{2}\cdot u^2\cdot\lambda\cdot\frac{1}{R} + \frac{h_m}{2}\cdot(1-u)^2\cdot\lambda\cdot\frac{1}{M} + \frac{h_n}{2}\cdot u^2\cdot\lambda\cdot\left(\frac{1}{r}-1\right)\right]\\
&+u\cdot\lambda\cdot\left(c_r - c_m - c_d\right) + \lambda\cdot\left(c_m + c_d\cdot r\right) \rightarrow \min\\
&\text{such that}\\
&0\le u\le r\\
&T>0,\\
&M, R \text{ positive integers}
\end{aligned}\right\}\qquad (P^R)$$

The cost function is now convex in the length of the cycle, so the optimal length can be calculated as follows:

$$T^o(M,R,u) = \sqrt{\frac{R\cdot K_r + M\cdot K_m}{\frac{h_r+h_n}{2}\cdot u^2\cdot\lambda\cdot\frac{1}{R} + \frac{h_m}{2}\cdot(1-u)^2\cdot\lambda\cdot\frac{1}{M} + \frac{h_n}{2}\cdot u^2\cdot\lambda\cdot\left(\frac{1}{r}-1\right)}}\,.$$

After substitution the optimal length in the cost function, we have the following cost function $C_I(R,M,u)$:

$$\begin{aligned}
&C_I(M,R,u) =\\
&\sqrt{\left(R\cdot K_r + M\cdot K_m\right)\left[\frac{h_r+h_n}{2}\cdot u^2\cdot\lambda\cdot\frac{1}{R} + \frac{h_m}{2}\cdot(1-u)^2\cdot\lambda\cdot\frac{1}{M} + \frac{h_n}{2}\cdot u^2\cdot\lambda\cdot\left(\frac{1}{r}-1\right)\right]} +\\
&+u\cdot\lambda\cdot\left(c_r - c_m - c_d\right) + \lambda\cdot\left(c_m + c_d\cdot r\right)
\end{aligned}$$

The function $C_I(R,M,u)$ can be written in the next form

$$C_I(M,R,u)=\sqrt{2\lambda\left(A(u)\frac{R}{M}+B(u)\frac{M}{R}+C(u)R+D(u)M+E(u)\right)}+\lambda u(c_r-c_m-c_d)+\lambda(c_m+c_d r) \quad (5)$$

where

$$A(u)=K_r\cdot h_m\cdot(1-u)^2,\ B(u)=K_m\cdot(h_r+h_n)\cdot u^2,\ C(u)=K_r\cdot h_n\cdot\left(\frac{1}{r}-1\right)\cdot u^2,$$
$$D(u)=K_m\cdot h_n\cdot\left(\frac{1}{r}-1\right)\cdot u^2,\ E(u)=K_r\cdot(h_r+h_n)\cdot u^2+K_m\cdot h_m\cdot(1-u)^2 ,$$

and variables *R* and *M* are positive integers, and $0\le u\le r$.

The manufacturing and remanufacturing lots are in this case

$$Q_m(M,R,u)=\frac{(1-u)\cdot\lambda}{M}\cdot\sqrt{\frac{R\cdot K_r+M\cdot K_m}{\frac{h_r+h_n}{2}\cdot u^2\cdot\lambda\cdot\frac{1}{R}+\frac{h_m}{2}\cdot(1-u)^2\cdot\lambda\cdot\frac{1}{M}+\frac{h_n}{2}\cdot u^2\cdot\lambda\cdot\left(\frac{1}{r}-1\right)}}$$

and

$$Q_r(M,R,u)=\frac{u\cdot\lambda}{R}\cdot\sqrt{\frac{R\cdot K_r+M\cdot K_m}{\frac{h_r+h_n}{2}\cdot u^2\cdot\lambda\cdot\frac{1}{R}+\frac{h_m}{2}\cdot(1-u)^2\cdot\lambda\cdot\frac{1}{M}+\frac{h_n}{2}\cdot u^2\cdot\lambda\cdot\left(\frac{1}{r}-1\right)}}.$$

Function $C_I(R,M,u)$ is quasiconvex in *R* and *M* and convex in *u*. This property guarantees the existence of optimal solution, as it is prooved by Dobos and Richter (2000). Let us now introduce an auxilliary function $S(R,M,u)$, as follows

$$S(R,M,u)=A(u)\cdot\frac{R}{M}+B(u)\cdot\frac{M}{R}+C(u)\cdot R+D(u)\cdot M+E(u).$$

We look for an optimal solution of this function for remanufacturing and manufacturing batches *R* and *M*. This function is the expression under the square in (5). Due to monotonicity considerations the function $S(R,M,u)$ can be analysed for solving batch sizes $R$ and $M$, where all coefficients $A(u)$, $B(u)$, $C(u)$, $D(u)$ and $E(u)$ are positive.

### 3.2.3.2. The continuous solution of the problem

Due to the relations $A(u)$, $B(u)$, $C(u)$, $D(u)$, $E(u) > 0$ hold and the results of the Theorem 1 of the appendix are to be applied.

**Theorem 1:** The continuous optimal solution for minimizing the function $C_I(R,M,u)$ for $R$ and $M$ is

(i) $$u \le \max\left\{ \frac{\sqrt{K_r \cdot h_m}}{\sqrt{K_r \cdot h_m} + \sqrt{K_m \cdot \left(h_r + h_n \cdot \frac{1}{r}\right)}},\quad r \right\},$$

$$R(u) = 1,\ M(u) = \frac{1-u}{u} \cdot \sqrt{\frac{K_r \cdot h_m}{K_m \cdot \left(h_r + h_n \cdot \frac{1}{r}\right)}},$$

$$C(u) = \sqrt{2 \cdot \lambda} \cdot \left[ (1-u) \cdot \sqrt{K_m \cdot h_m} + u \cdot \sqrt{K_r \cdot \left[h_r + h_n \cdot \frac{1}{r}\right]} \right] + $$
$$+ u \cdot \lambda \cdot (c_r - c_m - c_d) + \lambda \cdot (c_m + c_d \cdot r)$$

(ii) $$\frac{\sqrt{K_r \cdot h_m}}{\sqrt{K_r \cdot h_m} + \sqrt{K_m \cdot \left(h_r + h_n \cdot \frac{1}{r}\right)}} \le u \le \frac{\sqrt{K_r \cdot h_m}}{\sqrt{K_r \cdot h_m} + \sqrt{K_m \cdot (h_r + h_n) - K_r \cdot h_n \cdot \left(\frac{1}{r} - 1\right)}},$$

$$R(u) = 1,\ M(u) = 1,$$

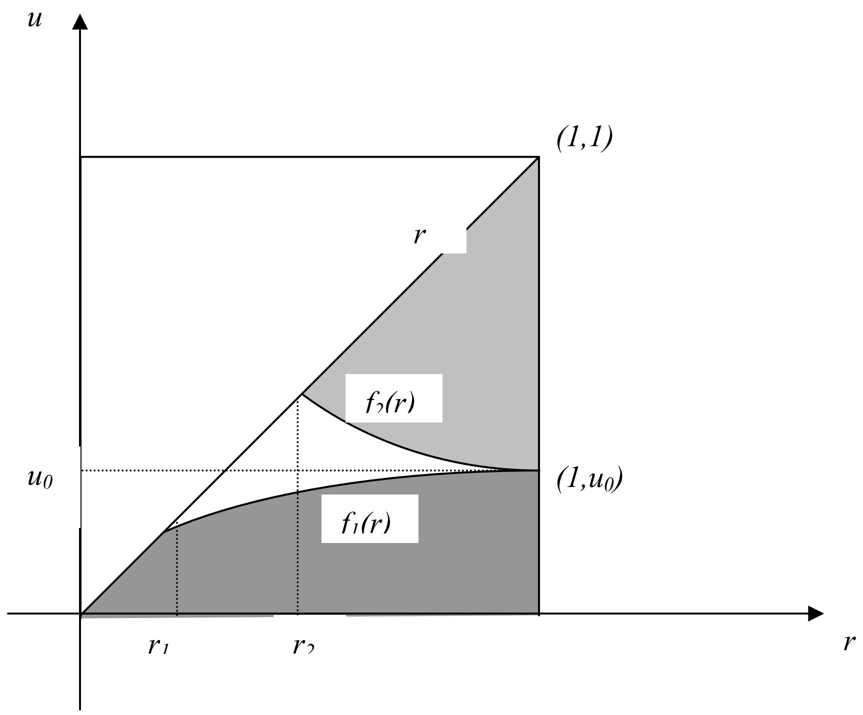
$$C(u) = \sqrt{2 \cdot \lambda \cdot (K_r + K_m) \cdot \left[ h_m \cdot (1-u)^2 + (h_r + h_n) \cdot u^2 + h_n \cdot \left(\frac{1}{r} - 1\right) \cdot u^2 \right]} + $$
$$+ u \cdot \lambda \cdot (c_r - c_m - c_d) + \lambda \cdot (c_m + c_d \cdot r)$$

(iii) $$\frac{\sqrt{K_r \cdot h_m}}{\sqrt{K_r \cdot h_m} + \sqrt{K_m \cdot (h_r + h_n) - K_r \cdot h_n \cdot \left(\frac{1}{r} - 1\right)}} \le u \le r,$$

$$R(u) = u \cdot \sqrt{\frac{K_m \cdot (h_r + h_n)}{K_r \cdot h_m \cdot (1-u)^2 + K_r \cdot h_n \cdot \frac{1-r}{r} \cdot u^2}}, \; M(u) = 1,$$

$$C(u) = \sqrt{2 \cdot \lambda} \cdot \left[ u \cdot \sqrt{K_r \cdot (h_r + h_n)} + \sqrt{K_m \cdot \left[ h_m \cdot (1-u)^2 + h_n \cdot \left( \frac{1}{r} - 1 \right) \cdot u^2 \right]} \right] + \\ + u \cdot \lambda \cdot (c_r - c_m - c_d) + \lambda \cdot (c_m + c_d \cdot r)$$

**Figure 3. Set of *u* in dependence on returne rate *r***




Cost function *C(u)* can be written in the following form:

$$C(u) = \begin{cases} \sqrt{2 \cdot \lambda} \cdot \left[ (1-u) \cdot \sqrt{K_m \cdot h_m} + u \cdot \sqrt{K_r \cdot \left( h_r + h_n \cdot \frac{1}{r} \right)} \right] + \\ + u \cdot \lambda \cdot (c_r - c_m - c_d) + \lambda \cdot (c_m + c_d \cdot r) & u \le f_1(r) \\ \sqrt{2 \cdot \lambda \cdot (K_r + K_m) \cdot \left[ h_m \cdot (1-u)^2 + (h_r + h_n) \cdot u^2 + h_n \cdot \left( \frac{1}{r} - 1 \right) \cdot u^2 \right]} + \\ + u \cdot \lambda \cdot (c_r - c_m - c_d) + \lambda \cdot (c_m + c_d \cdot r) & f_1(r) \le u \le f_2(r) \\ \sqrt{2 \cdot \lambda} \cdot \left[ u \cdot \sqrt{K_r \cdot (h_r + h_n)} + \sqrt{K_m \cdot \left[ h_m \cdot (1-u)^2 + h_n \cdot \left( \frac{1}{r} - 1 \right) \cdot u^2 \right]} \right] + \\ + u \cdot \lambda \cdot (c_r - c_m - c_d) + \lambda \cdot (c_m + c_d \cdot r) & f_2(r) \le u \le r \end{cases}$$

where

$$f_1(r) = \max\left\{ \frac{\sqrt{K_r \cdot h_m}}{\sqrt{K_r \cdot h_m} + \sqrt{K_m \cdot \left( h_r + h_n \cdot \frac{1}{r} \right)}}, \quad r \right\},$$

and

$$f_2(r) = \frac{\sqrt{K_r \cdot h_m}}{\sqrt{K_r \cdot h_m} + \sqrt{K_m \cdot (h_r + h_n) - K_r \cdot h_n \cdot \left( \frac{1}{r} - 1 \right)}}.$$

Let us now represent the function $f_1(r)$ and $f_2(r)$. This figure shows the set of possible disposal rate $u$ for given return rates $r$. The functions $f_1(r)$ and $f_2(r)$ contain the points where the lot numbers are equal to one. If the return rate is smaller than $r_1$, then the cost function $C(u)$ consists of function described in point (i) of theorem, and the number of lots is one for remanufacturing. In this case the cost function is linear. If the return rate is between $r_1$ and $r_2$, then the cases (i) and (ii) of theorem occur. And if the return rate is over $r_2$, then all three cases occur. Points $r_1$ and $r_2$ can be calculated as solution of equations $f_1(r) = r$, and $f_2(r) = r$. The function $C(u)$ is convex for all return rate $u$.

**Lemma 1:** Cost function $C(u)$ is convex in $u$, and twice continuously differentiable.

The lemma can be proven very easily with calculus. The reuse rate will be determined in the following section.

### 3.2.3.3. The determination of optimal reuse rate

Now the cost optimal reuse rate will be calculated, and the dependence of the optimal solution on return rate is examined. Let us assume that

$$h_m \cdot (1 - r_2) > h_r \cdot r_2 + h_n.$$

This assumption shows that the inventory holding costs of newly manufactured products are higher than that of used and returned, and then remanufactured products. Return rate $r_2$ is a switching point. This is the highest return rate where the manufacturing and remanufacturing rates are equal to one. The inequality points out that ramanufacturing is more economical that manufacturing.

The problem is now

$$\min_{0 \le u \le r} C(u) .$$

I solve the problem for three cases in dependence on return rate.

**(i) $0 \le r \le r_1$**

The cost function $C(u)$ has the next form

$$C(u) = u \cdot \sqrt{\lambda} \cdot \left[ \sqrt{2 \cdot K_r \cdot \left( h_r + h_n \cdot \frac{1}{r} \right)} - \sqrt{2 \cdot K_m \cdot h_m} + \sqrt{\lambda} \cdot (c_r - c_m - c_d) \right] + \\ + \sqrt{2 \cdot \lambda \cdot K_m \cdot h_m} + \lambda \cdot (c_m + c_d \cdot r) .$$

It is easy to see that the optimal reuse rate is:

$$u^o = \begin{cases} 0 & r < \dfrac{2 \cdot K_r \cdot h_n}{\left[ \sqrt{\lambda} \cdot (c_m + c_d - c_r) + \sqrt{2 \cdot K_m \cdot h_m} \right]^2 - 2 \cdot K_r \cdot h_r} \\ r & \dfrac{2 \cdot K_r \cdot h_n}{\left[ \sqrt{\lambda} \cdot (c_m + c_d - c_r) + \sqrt{2 \cdot K_m \cdot h_m} \right]^2 - 2 \cdot K_r \cdot h_r} \le r \le r_1 \end{cases} .$$

It shows that it is optimal to manufacture without remanufacturing and to dispose of all returned units, if the return rate is very low.

**(ii) $r_1 < r \le r_2$**

In this case the cost function $C(u)$ has two parts.

$$C(u)=\begin{cases}\sqrt{2\cdot\lambda}\cdot\left[(1-u)\cdot\sqrt{K_m\cdot h_m}+u\cdot\sqrt{K_r\cdot\left(h_r+h_n\cdot\frac{1}{r}\right)}\right]+ \\ +u\cdot\lambda\cdot(c_r-c_m-c_d)+\lambda\cdot(c_m+c_d\cdot r) & 0\le u\le f_1(r) \\ \sqrt{2\cdot\lambda\cdot(K_r+K_m)\cdot\left[h_m\cdot(1-u)^2+(h_r+h_n)\cdot u^2+h_n\cdot\left(\frac{1}{r}-1\right)\cdot u^2\right]}+ \\ +u\cdot\lambda\cdot(c_r-c_m-c_d)+\lambda\cdot(c_m+c_d\cdot r) & f_1(r)\le u\le r\end{cases}.$$

Since cost function $C(u)$ is a convex function, it is enough to examine the function on point $r_2$, whether the function in this point is decreasin or increasing. Let us now reformulate the condition in the follwing form:

$$r\le r_2<\frac{h_m-h_n}{h_m+h_r}.$$

It means that the cost function is monotonuously decreasing, so $u^o = r$. For this case it is optimal all returned items to remanufacture withaout any waste disposal.

**(iii) $r_2 < r \le 1$**

Now the cost function is

$$C(u)=\begin{cases}\sqrt{2\cdot\lambda}\cdot\left[(1-u)\cdot\sqrt{K_m\cdot h_m}+u\cdot\sqrt{K_r\cdot\left(h_r+h_n\cdot\frac{1}{r}\right)}\right]+ \\ +u\cdot\lambda\cdot(c_r-c_m-c_d)+\lambda\cdot(c_m+c_d\cdot r) & u\le f_1(r) \\ \sqrt{2\cdot\lambda\cdot(K_r+K_m)\cdot\left[h_m\cdot(1-u)^2+(h_r+h_n)\cdot u^2+h_n\cdot\left(\frac{1}{r}-1\right)\cdot u^2\right]}+ \\ +u\cdot\lambda\cdot(c_r-c_m-c_d)+\lambda\cdot(c_m+c_d\cdot r) & f_1(r)\le u\le f_2(r) \\ \sqrt{2\cdot\lambda}\cdot\left[u\cdot\sqrt{K_r\cdot(h_r+h_n)}+\sqrt{K_m\cdot\left[h_m\cdot(1-u)^2+h_n\cdot\left(\frac{1}{r}-1\right)\cdot u^2\right]}\right]+ \\ +u\cdot\lambda\cdot(c_r-c_m-c_d)+\lambda\cdot(c_m+c_d\cdot r) & f_2(r)\le u\le r\end{cases}$$

Let us differentiate this function in point $r$, and examine the differential function. The expression is

$$C'(r) = \sqrt{2 \cdot \lambda \cdot K_r \cdot (h_r + h_n)} - \sqrt{2 \cdot \lambda \cdot K_m} \cdot \frac{\sqrt{1-r} \cdot (h_m - h_n)}{\sqrt{[h_m \cdot (1-r) + h_n \cdot r]}} + \lambda \cdot (c_r - c_m - c_d).$$

This derivated function in point $r$ is decreasing. Since $C'(r_2) > C'(r)$ it follows that $C'(r) < 0$, so the function is monotonuously decreasing in $u$. The optimal reuse rate is $u^o = r$.

**3.2.4. Conclusion**

An inventory model was investigated in thsi chapter. The question was for which lot size and reuse rate are the costs lowest. If the sum of unit manufacturing and disposal costs is higher than remanufacturing costs, and the inventory holding costs of newly manufactured products are higher than that of remanufactured and reused items, then there are two cases. If the return rate is very low, then it is optimal to manufacture and to dispose of all returning items. After a given return rate it is optimal all returned items to remanufacture without disposal. In dependence of this return rate $u$ all manufacturing and remanufacturing lot sizes can be determined.

## 3.3. A production-recycling model with buybacking

### 3.3.1. Introduction

Classical logistic systems manage the material and related informational flow from raw material until the final products are delivered to the customer. This means a forward flow. Reverse logistics manages backward process, i.e. the used and reusable parts and products return from the customers to the producers. Environmental consciousness forces companies to initiate such product recovery systems. These way natural resources can be saved for the future generations, so the firms can contribute to the sustainable development efforts. This work analyzes a situation where the returned items are recycled and the firm saves with the recycling the mining of other natural resources.

In this paper a model of the EOQ type is developed and analyzed, in which a producer serves a stationary product demand occurring at the rate $D > 0$. This demand is served by producing or procuring new items as well as by recycling some part $0 \leq \delta \leq 1$ of the used products coming back to the producer at a constant return rate $d = \alpha D, 0 \leq \alpha \leq 1$. It is assumed that the producer is in the situation to buy back all used product to recycle and/or to dispose off them. The parameters $\delta$ and $\alpha$ are called *marginal use rate* and *marginal buyback (return) rate*, respectively. The remaining part of the non-serviceable products $(1-\delta)d$ will be disposed off. $(1-\delta)$ is called *marginal disposal rate.*

First, an analysis of the situation is provided. The inventory stocks for *serviceable* products from the *production and recycling processes* (PRP) and for the *non-serviceable* items is determined. On the basis of these results the lot sizes and cycle times for the PRP can be found which minimize the per time unit total set-up and holding cost. This results in the explicit determination of a function $C_I(\alpha,\delta)$ which expresses these minimal costs as function of the marginal use and buyback rates.

Secondly, if linear waste disposal, production, recycling and buyback costs are introduced, the problem appears at which $\delta$ and $\alpha$ the total set-up, holding and linear costs $C_I(\alpha,\delta)+C_N(\alpha,\delta)$ is minimal. In this formulation the producer makes decision about how much used items buy back to recycle.

In this paper we examine a production/recycling system with predetermined production-inventory policy and assume that there is no difference between newly produced and recycled items, i.e. we apply the "as-good-as-new" principle. The paper is organized as follows. The next section introduces the used parameters and decision variables as well as the functioning of the production and recycling processes. In the section 3 the cost function of the inventory holding will be constructed. In the following two sections we determine the cost minimal cycle time and batch numbers for production and recycling in dependence on the buyback and use rates. Section 6 provides the optimal buyback and use rates for the inventory holding cost, while section 7 shows the optimal policy of the total (EOQ and non-EOQ related) cost model. In the last section we summarize the obtained results and show some directions of generalization.

### 3.3.2. Parameters and functioning of the system

To model the production-recycling we use the following parameters and decision variables. The material flow of the modeled situation is shown in Fig. 1 with the introduced parameters and decision variables.

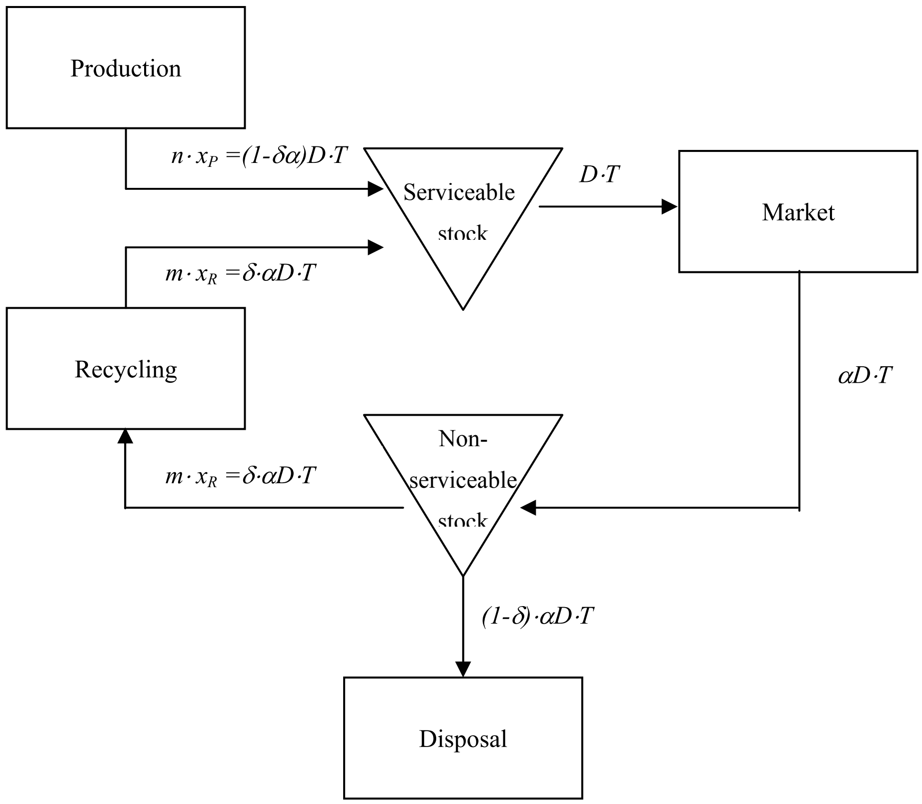
*Lot-size related parameters of the model*:

- $D$ demand rate,
- $P=\frac{1}{\beta}D$ production rate ($\beta < 1$),
- $d=\alpha D$ buyback rate ($0 \leq \alpha \leq 1$),
- $R=\frac{1}{\gamma}D$ recycling rate ($\gamma < 1$),
- $S_R$ setup costs of recycling,
- $S_P$ setup costs of production,
- $h_s$ holding cost of serviceable items,
- $h_n$ holding cost of non-serviceable items.

*Lot-size independent cost parameters:*

- $C_w$ waste disposal cost for $(1-\delta)\cdot\alpha D\cdot T$,
- $C_P$ linear production cost for $(1-\delta\alpha)D\cdot T$,
- $C_R$ linear recycling cost for $\delta\cdot\alpha D\cdot T$,
- $C_B$ buyback cost for $\alpha\cdot D\cdot T$.

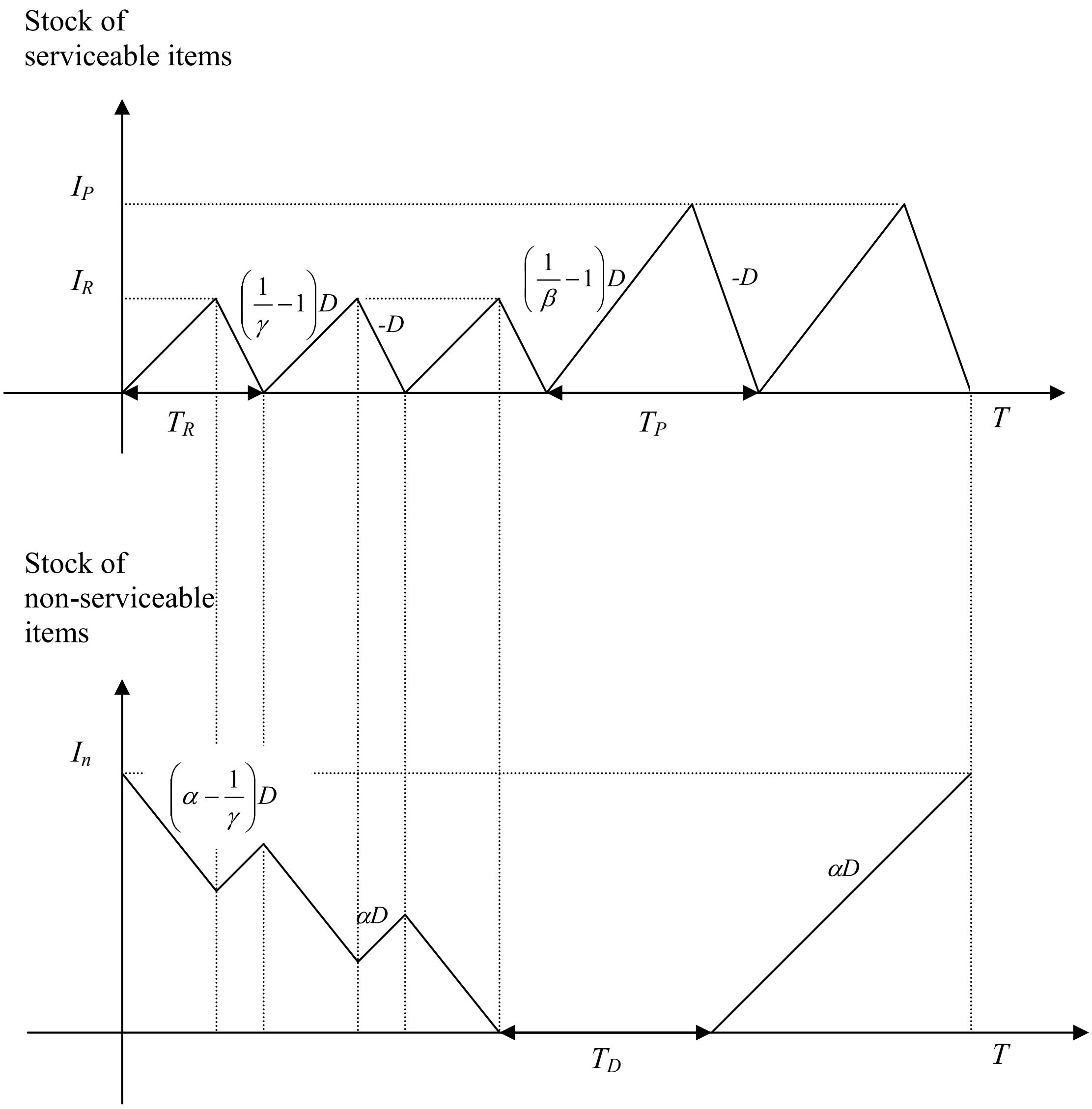
**Figure 1. The material flow in the model in a production and recycling cycle**




*Decision variables of the model*:

- $\delta$ marginal use rate,
- $\alpha$ marginal buyback rate,
- $m$ number of recycling lots, positive integer,
- $T_R$ time interval of recycling,
- $x_R$ recycling lot size, $x_R = D\cdot T_R$
- $n$ number of production lots, positive integer,
- $T_P$ time interval of production,
- $x_P$ recycling lot size, $x_P = D\cdot T_P$
- $T$ length of production and recycling cycles.

**Figure 2. Inventory status in the model (*m* = 3, *n* = 2)**




The demand is satisfied by recycling the non-serviceable products during time period $T_R$ , and stored until the end of this cycle as well as the used products arrive at the rate $d = \alpha D < D$ in non-serviceable stocking point (compare Fig. 2). Due to the given recycling rate $R > D = \gamma R$ the process of recycling lasts for some $\gamma \cdot T_R$ time units. When the recycling process is stopped the demand can be served by the accumulated stock of recycled products. Parameter of this figure $T_R$ denotes the length of the *recycling cycle*.

After recycling the producer serves a demand of one product, which appears at a constant rate $D > 0$ The producer has to determine how much of new items to produce at a rate $P$, $D = \beta P < P$. Depending on this information he can found out how long he has to store the excess production. The time interval in which production and carrying new production is accomplished is called the *production cycle* and it is be denoted by $T_P$. The time interval $T = m \cdot T_R + n \cdot T_P$ gives the length of the *production and recycling cycles*.

The process of storing and disposing off non-serviceable goods can be organized in the following way: the $(1-\delta)dT$ units which have to be disposed during some interval $T$ are disposed during the time *disposal interval* $T_D = (1-\delta)T$ just when they arrive. Hence some stock of non-serviceable items is set up during the *collection interval* $T_{RC} = T - T_D = \delta T$.

At the end of the production cycle the inventory stock of non-serviceable products attains its peak $I_n = [(1-\alpha)m+\alpha(1-\gamma)]\cdot DT_R$ which is the initial inventory level at the beginning of the production and recycling cycle. At the end of a recycling period the inventory stock of serviceable recycled products attains its peak $I_R = (1-\gamma)\cdot DT_R$. The peak of the inventory stock of newly produced items is $I_P = (1-\beta)\cdot DT_P$.

### 3.3.3. Determination of the inventory cost

Let $h_s$ denote the inventory cost for serviceable items per unit and time unit, and let $h_n$ denote the same cost for non-serviceable items. If the length of the production and recycling cycle $T$ is given the average inventory cost $H_P$, $H_R$, $H_n$ for the newly produced items, recycled items and for the non-serviceable items, correspondingly, are as shown in Lemma 1. Let us now assume that the return rate $\alpha$ and the use rate $\delta$ are positive, i.e. there is recycling and the buyback and use rates are not equal to one, i.e. there is production, as well.

**Lemma 1**: The average inventory costs are in this model:

$$H_R = \frac{1}{2} m \cdot I_R T_R \cdot h_s = \frac{1}{2} \cdot DT^2 \cdot h_s (1-\gamma) \alpha^2 \delta^2 \cdot \frac{1}{m} \qquad (1)$$

$$H_P = \frac{1}{2} n \cdot I_P T_P \cdot h_s = \frac{1}{2} \cdot DT^2 \cdot h_s (1-\beta)(1-\alpha\delta)^2 \cdot \frac{1}{n} \qquad (2)$$

$$H_n = \frac{1}{2} \cdot DT^2 \cdot h_n (1-\gamma)\alpha^2 \delta^2 \cdot \frac{1}{m} + \frac{1}{2} \cdot DT^2 \cdot h_n \alpha (1-\alpha) \delta^2 \qquad (3)$$

*Proof.* We will prove equality (3), the other cases can be proved in the same way. The inventory holding costs of non-serviceable items can be computed with dividing the area into *m* triangles *A*, *(m-1)* triangles *B*, triangle *C* and rectangles $D_1$, $D_2$, …, $D_{m-1}$. (See Figure 3.) The area of triangle *A* is

$$T_A = \frac{1}{2} \cdot \gamma T_R \cdot \left(\frac{1}{\gamma} - 1\right) D \cdot \gamma T_R = \frac{1}{2} \cdot \gamma (1-\alpha\gamma) \cdot DT_R^2 .$$

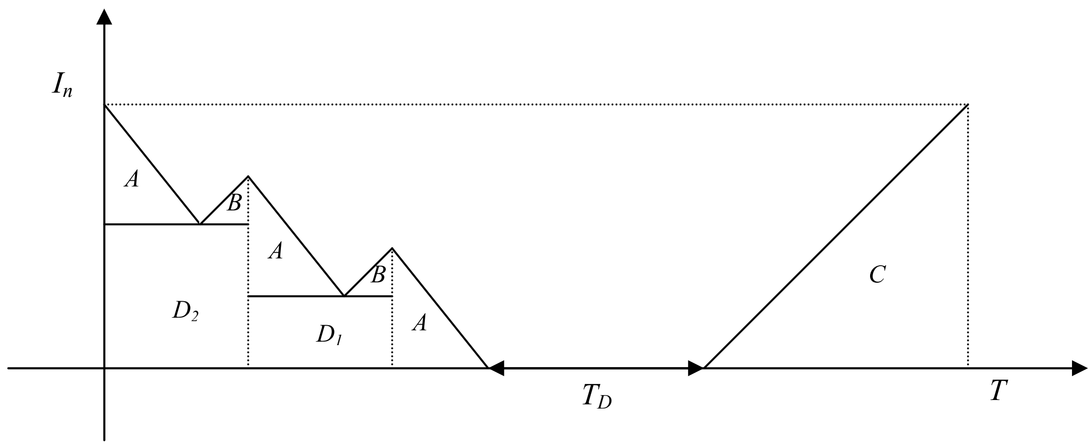
The area of triangle *B* is equal to

$$T_B = \frac{1}{2} \cdot (1-\gamma) T_R \cdot \alpha D \cdot (1-\gamma) T_R = \frac{1}{2} \cdot \alpha (1-\gamma)^2 \cdot DT_R^2 .$$

The area of triangle *C* is

$$T_C = \frac{1}{2} \cdot [(1-\alpha)m + \alpha(1-\gamma)] DT_R \cdot \frac{(1-\alpha)m + \alpha(1-\gamma)}{\alpha D} \cdot DT_R = \frac{1}{2} \cdot \frac{1}{\alpha} [(1-\alpha)m + \alpha(1-\gamma)]^2 \cdot T_R^2 .$$

**Figure 3. Inventory status for the non-serviceable stock**




The area of rectangle $D_i$ is

$$T_{D_i} = i \cdot (1-\alpha) D T_R^2 \, .$$

The total costs are now

$$H_n = m \cdot T_A + (m-1) \cdot T_B + T_C + \sum_{i=1}^{m-1} T_{D_i} \, .$$

After some simple calculation we get the result of (3).

**Lemma 2**: The total inventory cost per time unit is

$$H_T = \frac{H_P + H_R + H_n}{T} = \frac{1}{2} TD \cdot V(m,n,\alpha,\delta) \qquad (4)$$

with

$$V(m,n,\alpha,\delta) = (h_s + h_n)(1-\gamma)\alpha^2\delta^2 \cdot \frac{1}{m} + h_s(1-\beta)(1-\alpha\delta)^2 \cdot \frac{1}{n} + h_n\alpha(1-\alpha)\delta^2 \qquad (5)$$

*Proof.* Formulas (4) and (5) are obtained, if the cost and time parameters on the left-hand side of (4) are substituted by the expressions (1) – (3).

*Example 1*: Let $D = 1{,}000$, $h_s = 850$, $h_n = 80$, $\beta = 2/3$, $\gamma = 2/3$, $m = 1$, $n = 2$, $\alpha = 1/2$ and $\delta = 2/3$. For this data $V(2,1,1/2,2/3) = 0.167h_s + 0.130h_n = 106.296$ and $H_T = \frac{1}{2} \cdot T \cdot 1{,}000 \cdot 106.296$ $53{,}148.1T$ hold.

The function $V(m,n,\alpha,\delta)$ expresses the total inventory cost per time unit and per demand unit.

### 3.3.4. Total cost minimization for the cycle time

Let the setup cost $S$ per production and recycling cycle as the sum of setup costs $S_P$ and $S_R$ for the production and the recycling, respectively, be given. Then the setup cost per time unit is

$S_T(m,n) = S_R \cdot m + S_P \cdot n$.

The average inventory costs of the model $C_A(T,m,n,\alpha,\delta)$ can be written in the following form

$$C_A(T,m,n,\alpha,\delta) = \frac{S_T(m,n)}{T} + \frac{1}{2} TD \cdot V(m,n,\alpha,\delta) \to \min \quad (6)$$

Because of the convexity of the cost function in the production and recycling cycle time the cost optimal cycle time is $T^o(m,n,\alpha,\delta) = \sqrt{\frac{2S_T(m,n)}{D \cdot V(m,n,\alpha,\delta)}}$ (7)

and the minimal total setup and inventory cost per time unit is

$$\widetilde{C}_A(m,n,\alpha,\delta) = \sqrt{2D \cdot S_T(m,n) \cdot V(m,n,\alpha,\delta)}\,. \quad (8)$$

The optimal recycling and production cycle times are

$$T_R^o(m,n,\alpha,\delta) = \frac{\alpha\delta}{m} \sqrt{\frac{2S_T(m,n)}{D \cdot V(m,n,\alpha,\delta)}}, \quad (9)$$

$$T_P^o(m,n,\alpha,\delta) = \frac{1-\alpha\delta}{n} \sqrt{\frac{2S_T(m,n)}{D \cdot V(m,n,\alpha,\delta)}}. \quad (10)$$

The optimal lot sizes are

$$x_R^o(m,n,\alpha,\delta) = \frac{\alpha\delta}{m} \sqrt{\frac{2DS_T(m,n)}{V(m,n,\alpha,\delta)}}, \quad (11)$$

$$x_P^o(m,n,\alpha,\delta) = \frac{1-\alpha\delta}{n} \sqrt{\frac{2DS_T(m,n)}{V(m,n,\alpha,\delta)}}. \quad (12)$$

*Example 2*: Let as in examples 1 $D = 1{,}000$, $h_s = 850$, $h_n = 80$, $\beta = 2/3$, $\gamma = 2/3$, $m = 1$, $n = 2$, $\alpha = 1/2$ and $\delta = 2/3$. It is known from Example 1 that $V(m,n,\alpha,\delta) = 106.296$ and $H_T =$

*53,148.1T* hold. Setting $S_P$ = *1,960* and $S_R$ = *440* the total cost per time unit is according to formula (6) $C_A\left(T,1,2,\frac{1}{2},\frac{2}{3}\right)=\frac{4{,}360}{T}+53{,}148.1T$. The optimal length of the production cycle and recycling cycle is $T\left(1,2,\frac{1}{2},\frac{2}{3}\right)=0.286$ year or 104 days. The minimal cost per time unit is

$$\widetilde{C}_A\left(1,2,\frac{1}{2},\frac{2}{3}\right)=2\sqrt{4{,}360\cdot 53{,}148.1}=30{,}445.1\,.$$

**3.3.5. The optimal number of lots for production and recycling**

Now we will minimize the cost function $\widetilde{C}_A(m,n,\alpha,\delta)$ in order to determine the optimal number of lots. After some calculation this cost function can be written in the following the form

$$\widetilde{C}_A(m,n,\alpha,\delta)=\sqrt{2D\cdot\left[A(\alpha,\delta)\cdot\frac{m}{n}+B(\alpha,\delta)\cdot\frac{n}{m}+C(\alpha,\delta)\cdot m+D(\alpha,\delta)\cdot n+E(\alpha,\delta)\right]} \qquad (13)$$

where

$$\begin{aligned}
&A(\alpha,\delta)=S_R h_s(1-\beta)(1-\alpha\delta)^2, \quad B(\alpha,\delta)=S_P(h_s+h_n)(1-\gamma)\alpha^2\delta^2,\\
&C(\alpha,\delta)=S_R h_n\alpha(1-\alpha)\delta^2, \quad D(\alpha,\delta)=S_P h_n\alpha(1-\alpha)\delta^2,\\
&E(\alpha,\delta)=S_R(h_s+h_n)(1-\gamma)\alpha^2\delta^2+S_P h_s(1-\beta)(1-\alpha\delta)^2
\end{aligned}$$

To solve this problem we can introduce a relaxed auxiliary problem (meta-model) (Richter (1996), Dobos and Richter (2000)):

$$S(m,n)=A\frac{m}{n}+B\frac{n}{m}+Cm+Dn+E\rightarrow\min, \qquad m\geq 1,\ n\geq 1.$$

Applying the results of Dobos and Richter (2000), the optimal continuous solution for the lots number *(m,n)* is

**Lemma 3**: There are three cases of optimal continuous solutions *(m(α,δ),n(α,δ))* and minimum cost expressions $C_I(\alpha,\delta)$ for the function (13):

(i) $A(\alpha,\delta) \geq B(\alpha,\delta)+D(\alpha,\delta),\ B(\alpha,\delta) \leq A(\alpha,\delta)+C(\alpha,\delta)$

$$\left(m^o(\alpha,\delta),n^o(\alpha,\delta)\right)=\left(1,\ \frac{1-\alpha\delta}{\delta}\sqrt{\frac{S_R}{S_P}}\sqrt{\frac{h_s(1-\beta)}{(h_s+h_n)(1-\gamma)\alpha^2+h_n\alpha(1-\alpha)}}\right)$$

$$C_I(\alpha,\delta)=\sqrt{2D}\left\{(1-\alpha\delta)\cdot\sqrt{S_P h_s(1-\beta)}+\delta\cdot\sqrt{S_R\left[(h_s+h_n)(1-\gamma)\alpha^2+h_n\alpha(1-\alpha)\right]}\right\}$$

(ii) $A(\alpha,\delta) \leq B(\alpha,\delta)+D(\alpha,\delta),\ B(\alpha,\delta) \leq A(\alpha,\delta)+C(\alpha,\delta)$

$$\left(m^o(\alpha,\delta),n^o(\alpha,\delta)\right)=(1,1)$$

$$C_I(\alpha,\delta)=\sqrt{2D(S_R+S_P)\left[h_s(1-\beta)(1-\alpha\delta)^2+(h_s+h_n)(1-\gamma)\alpha^2\delta^2+h_n\alpha(1-\alpha)\delta^2\right]}$$

(iii) $A(\alpha,\delta) \leq B(\alpha,\delta)+D(\alpha,\delta),\ B(\alpha,\delta) \geq A(\alpha,\delta)+C(\alpha,\delta)$

$$\left(m^o(\alpha,\delta),n^o(\alpha,\delta)\right)=\left(\alpha\delta\cdot\sqrt{\frac{S_P}{S_R}}\cdot\sqrt{\frac{(h_s+h_n)(1-\gamma)}{h_s(1-\beta)(1-\alpha\delta)^2+h_n\alpha(1-\alpha)\delta^2}},\ 1\right)$$

$$C_I(\alpha,\delta)=\sqrt{2D}\left\{\alpha\delta\cdot\sqrt{S_R(h_s+h_n)(1-\gamma)}+\sqrt{S_P\left[h_s(1-\beta)(1-\alpha\delta)^2+h_n\alpha(1-\alpha)\delta^2\right]}\right\}$$

Note that the expressions for the found optimal lot (batch) numbers are not necessarily integer! Nevertheless we shall see in the next section that this (immediately practically not very useful) result will help us to prove that the mixed strategies are dominated by pure ones.

Now we introduce the following functions

$$\delta_1(\alpha)=\frac{\sqrt{S_R\cdot h_s(1-\beta)}}{\alpha\sqrt{S_R\cdot h_s(1-\beta)}+\sqrt{S_P\left[(h_s+h_n)(1-\gamma)\alpha^2+h_n\alpha(1-\alpha)\right]}}$$

and

$$\delta_2(\alpha) = \frac{\sqrt{S_R \cdot h_s(1-\beta)}}{\alpha\sqrt{S_R \cdot h_s(1-\beta)} + \sqrt{S_P \cdot (h_s + h_n)(1-\gamma)\alpha^2 - S_R \cdot h_n \alpha(1-\alpha)}}.$$

Functions $\delta_1(\alpha)$ and $\delta_2(\alpha)$ are such switching points for which the optimal number of lots *(m,n)* is equal to one. This is shown in Figure 5. Function $\delta_1(\alpha)$ separates the cases (i) and (ii) and $\delta_2(\alpha)$ the cases (ii) and (iii). To calculate the functions, we have used the conditions in cases of equality. It is easy to see that $\delta_1(\alpha) \le \delta_2(\alpha)$. The proof is left to the reader.

Let us now define the possible sets for $(\alpha,\delta)$ with the help of functions $\delta_1(\alpha)$ and $\delta_2(\alpha)$:

$$I = \{(\alpha,\delta) \mid \delta \le \delta_1(\alpha), 0 \le \alpha \le 1, 0 \le \delta \le 1\},$$

$$J = \{(\alpha,\delta) \mid \delta_1(\alpha) \le \delta \le \delta_2(\alpha), 0 \le \alpha \le 1, 0 \le \delta \le 1\},$$

$$K = \{(\alpha,\delta) \mid \delta \ge \delta_2(\alpha), 0 \le \alpha \le 1, 0 \le \delta \le 1\}.$$

The set $I$ is represented by the borders of the possible values of $(\alpha,\delta)$, the function $\delta_1(\alpha)$ and the points $(\alpha_1,1)$ and $(1,\delta_0)$, where value $\alpha_1$ is the solution of the following equality for $\alpha$

$$1 = \frac{\sqrt{S_R \cdot h_s(1-\beta)}}{\alpha\sqrt{S_R \cdot h_s(1-\beta)} + \sqrt{S_P\left[(h_s + h_n)(1-\gamma)\alpha^2 + h_n\alpha(1-\alpha)\right]}} = \delta_1(\alpha)$$

and

$$\delta_0 = \frac{\sqrt{S_R \cdot h_s(1-\beta)}}{\sqrt{S_R \cdot h_s(1-\beta)} + \sqrt{S_P \cdot (h_s + h_n)(1-\gamma)}}.$$

And the set $K$ is represented by the borders of the possible values of $(\alpha,\delta)$, the function $\delta_2(\alpha)$ and the points $(\alpha_2,1)$ and $(1,\delta_0)$, where value $\alpha_1$ is the solution of the following equality for $\alpha$

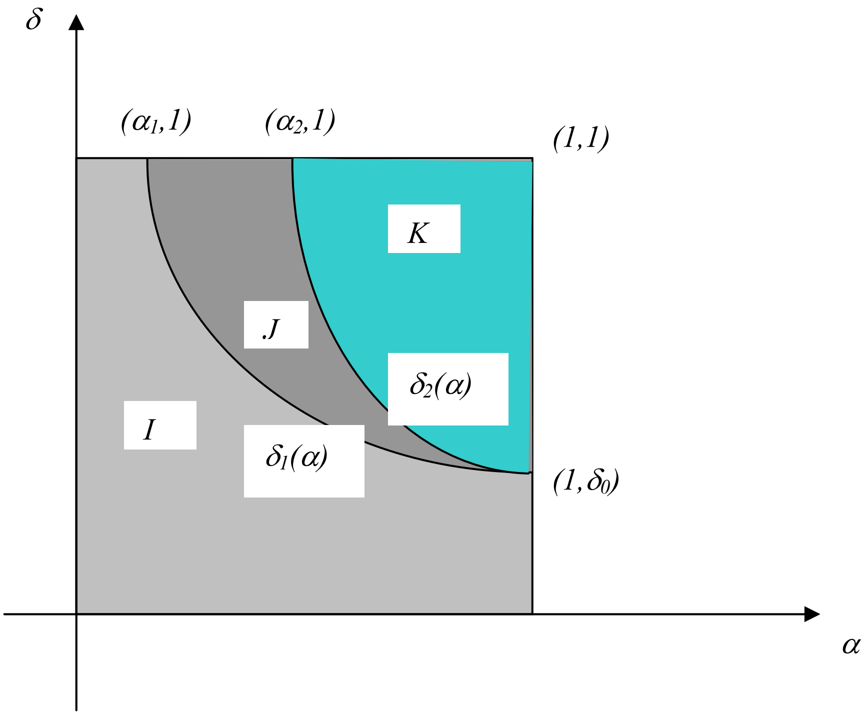
$$1=\frac{\sqrt{S_R\cdot h_s(1-\beta)}}{\alpha\sqrt{S_R\cdot h_s(1-\beta)}+\sqrt{S_P\cdot(h_s+h_n)(1-\gamma)\alpha^2-S_R\cdot h_n\alpha(1-\alpha)}}=\delta_2(\alpha).$$

The inventory cost function $C_I(\alpha,\delta)$ can be written as

$$C_I(\alpha,\delta)=\begin{cases}\sqrt{2D}\left\{(1-\alpha\delta)\cdot\sqrt{S_P h_s(1-\beta)}+\delta\cdot\sqrt{S_R\left[(h_s+h_n)(1-\gamma)\alpha^2+h_n\alpha(1-\alpha)\right]}\right\} & (\alpha,\delta)\in I\\ \sqrt{2D(S_R+S_P)\left[h_s(1-\beta)(1-\alpha\delta)^2+(h_s+h_n)(1-\gamma)\alpha^2\delta^2+h_n\alpha(1-\alpha)\delta^2\right]} & (\alpha,\delta)\in J\\ \sqrt{2D}\left\{\alpha\delta\cdot\sqrt{S_R(h_s+h_n)(1-\gamma)}+\sqrt{S_P\left[h_s(1-\beta)(1-\alpha\delta)^2+h_n\alpha(1-\alpha)\delta^2\right]}\right\} & (\alpha,\delta)\in K\end{cases}$$

*Example 3*: Let as in examples 2 $D = 1,000$, $h_s = 850$, $h_n = 80$, $\beta = 2/3$, $\gamma = 2/3$, $S_P = 1,960$, $S_R = 440$, $\alpha = 1/2$ and $\delta = 2/3$. Then $A(1/2,2/3) = 55,407.4$, $B(1/2,2/3) = 67,511.1$, $C(1/2,2/3) = 3,911.1$, $D(1/2,2/3) = 17,422.2$, $E(1/2,2/3) = 261,970$. The optimal batch numbers are $m(1/2,2/3) = 1.067$ and $n(1/2,2/3) = 1$. The minimal costs are $C_I(1/2,2/3) = 28,494.1$.

**Figure 4. The representation of sets** *I, J* **and** *K*



### 3.3.6. Minimizing the inventory holding costs for the buyback and use rates

Before minimizing the inventory holding costs $C_I(\alpha, \delta)$ we will prove a simple lemma.

**Lemma 3**: Let values *a, b, c* and *d* be positive. Then the following equality holds

$$\sqrt{(a+b)(c+d)} \geq \sqrt{ac} + \sqrt{bd} \ .$$

*Proof.* Let both sides of the inequality raise to the second power. Then

$$(a+b)(c+d) \geq ac + bd + 2\sqrt{abcd}$$

and after simplifying

$$ad + bc \geq 2\sqrt{abcd}$$

and this inequality holds for all positive *a, b, c* and *d*, because $\left(\sqrt{ad} - \sqrt{bc}\right)^2 \geq 0$ .

Let us apply this result to the strategy with one-one lots:

$$C_I(\alpha,\delta) = \sqrt{2D(S_P + S_R)\cdot\left[h_s(1-\beta)(1-\alpha\delta)^2 + h_s(1-\gamma)\alpha^2\delta^2 + h_n\alpha(1-\alpha)\delta^2\right]}, \quad (\alpha,\delta) \in J \ .$$

Let now

$$\begin{aligned} a &= 2DS_P \\ b &= 2DS_R \\ c &= h_s(1-\beta)(1-\alpha\delta)^2 \\ d &= h_s(1-\gamma)\alpha^2\delta^2 + h_n\alpha(1-\alpha)\delta^2 \end{aligned}$$

Using the result of lemma 3 we have the following inequalities

$$C_I(\alpha,\delta) \geq \sqrt{2DS_P \cdot h_s(1-\beta)(1-\alpha\delta)^2} + \sqrt{2DS_R \cdot \left[h_s(1-\gamma)\alpha^2\delta^2 + h_n\alpha(1-\alpha)\delta^2\right]} \geq$$
$$\geq (1-\alpha\delta)\sqrt{2DS_P \cdot h_s(1-\beta)} + \alpha\delta\sqrt{2DS_R \cdot (h_s + h_n)(1-\gamma)}$$

The last inequality holds because we have reduced the costs with the expression $h_n\alpha(1-\alpha)\delta^2$. With this method it can be shown that over sets *I* and *K*

$$C_I(\alpha,\delta) \geq (1-\alpha\delta)\sqrt{2DS_P \cdot h_s(1-\beta)} + \alpha\delta\sqrt{2DS_R \cdot (h_s + h_n)(1-\gamma)}$$

The last expression is a convex linear combination of the pure strategies, i.e. the recycling and production. The weights are the possible products of marginal use and buyback rates $\alpha\delta$ which is non-negative and not greater than one. This cost expression is always greater than the smaller of the costs of pure strategies:

$$(1-\alpha\delta)\sqrt{2DS_P \cdot h_s(1-\beta)} + \alpha\delta\sqrt{2DS_R \cdot (h_s + h_n)(1-\gamma)} \geq$$
$$\geq \min\left\{\sqrt{2DS_P \cdot h_s(1-\beta)};\ \sqrt{2DS_R \cdot (h_s + h_n)(1-\gamma)}\right\}$$

By this last inequality a proof is given for the

**Theorem 1:** The optimal inventory holding strategy in this production-recycling model is a pure strategy: either to produce to meet the demand ($\alpha^o = \delta^o = 0$) or to buy back and to recycle all used product without production ($\alpha^o = \delta^o = 1$).

*Example 5.* Let $D=1{,}000$, $\beta = \gamma = 2/3$, $S_P = 1960$, $S_R = 440$, $h_s = 850$ and $h_n = 80$. Then the inventory holding costs of recycling is *16,516.7* and that of production *33,326.7*. It is economical to recycle with buyback of all used items.

*Example 6.* Let $D=1{,}000$, $\beta = 2/5$ $\gamma = 2/3$, $S_P = 360$, $S_R = 440$, $h_s = 85$ and $h_n = 80$. Then the inventory holding costs of production is *6,059.7* and that of recycling *6,957.01*. It is more effective to produce and not to recycle.

## 7. Minimizing the total lot-size related and lot-size independent costs

In this section we minimize the sum of the EOQ-related and EOQ independent costs. The cost function is in this case

$$C_T(\alpha,\delta) = C_I(\alpha,\delta) + C_N(\alpha,\delta)$$

where function $C_N(\alpha,\delta) = C_W \cdot (1-\delta)\alpha D + C_R \cdot \delta\alpha D + C_P \cdot (1-\delta\alpha)D + C_B \cdot \alpha D$ is the sum of the linear waste disposal, recycling, production and buyback costs.

The problem to be solved has the form

$$C_T(\delta,\alpha) \to \min$$

subject to

$$\delta \in [0,1], \quad \alpha \in [0,1].$$

In the last section we have seen that

$$C_I(\alpha,\delta) \geq (1-\alpha\delta)\sqrt{2DS_P \cdot h_s(1-\beta)} + \alpha\delta\sqrt{2DS_R \cdot (h_s + h_n)(1-\gamma)}$$

i.e. the inventory holding costs are not greater than the convex linear combination of the pure production and recycling strategies. The non-EOQ related costs can be approximated in the following way

$$C_N(\alpha,\delta) \geq (1-\delta\alpha)D \cdot C_P + \delta\alpha D \cdot (C_B + C_R).$$

To get this inequality, we have reduced the lot-size independent costs with the waste disposal costs $C_W \cdot (1-\delta)\alpha D$ and with costs of bought back but not recycled items $C_B \cdot (1-\delta)\alpha D$.

Using these two approximations we can give a lower bound of the total cost function

$$C_T(\alpha,\delta) \geq (1-\alpha\delta)\left\{\sqrt{2DS_P \cdot h_s(1-\beta)} + D \cdot C_P\right\} + \alpha\delta\left\{\sqrt{2DS_R \cdot (h_s + h_n)(1-\gamma)} + D \cdot (C_B + C_R)\right\}.$$

The right-hand expression is again a convex linear combination of the pure strategies, so

$$\begin{aligned}&(1-\alpha\delta)\left\{\sqrt{2DS_P \cdot h_s(1-\beta)} + D \cdot C_P\right\} + \alpha\delta\left\{\sqrt{2DS_R \cdot (h_s + h_n)(1-\gamma)} + D \cdot (C_B + C_R)\right\} \geq \\ &\min\left\{\sqrt{2DS_P \cdot h_s(1-\beta)} + D \cdot C_P,\ \sqrt{2DS_R \cdot (h_s + h_n)(1-\gamma)} + D \cdot (C_B + C_R)\right\}\end{aligned}.$$

This result proves the next

**Theorem 2**: The optimal production-recycling strategy for the total cost model is either to buyback all sold and used items ($\alpha^o = \delta^o = 1$) or to produce new items without buybacking and recycling ($\alpha^o = \delta^o = 0$).

This result was shown by Richter [10] for another waste disposal model and by Dobos and Richter [3] for a production/recycling model. In the case of linear waste disposal, production, recycling and buyback costs and free choice of buyback and recycling rates between 0 and 1 one of the pure strategies to buy back and recycle or to produce is optimal. The optimal pure strategy can be found by comparing the values $\sqrt{2DS_P \cdot h_s(1-\beta)} + D \cdot C_P$ and $\sqrt{2DS_R \cdot (h_s + h_n)(1-\gamma)} + D \cdot (C_B + C_R)$.

## 8. Conclusions and further research

In this chapter we have investigated a production-recycling model. By minimizing the inventory holding costs it was shown that one of the pure strategies (to produce or to recycle all products) is optimal. A similar proposition can be obtained minimizing the total EOQ and non-EOQ related costs. A similar result was obtained by Richter (1997) in a waste disposal model with remanufacturing and by Dobos and Richter (2003) in a production and recycling model.

Probably these pure strategies are technologically not feasible and some used products will not return or even more as the sold ones will come back, and some of them will be not recyclable. This kind of generalization of this basic model could be the introduction of an upper bound on the buyback rate which is strongly smaller than one. In such a case a mixed strategy would be economical compared to the pure strategy “production”.

An other way to generalize this model is to ask for the quality of the bought back products. In the proposed model we have assumed that all returned items are serviceable. One can be put a question: Who must control the quality of the returned items? If the suppliers examine the quality of the reusable products, then the buyback rate is strongly smaller than one. If the user makes it, then not all returned items are recyclable, i.e. the use rate is smaller than one. Which one of the control systems are more cost advantageous in this case?

# 4. Production Planning in Reverse Logistics

## 4.1. Introduction

Several management problems arise along the reverse material flow. Some important questions are the collection of used products and materials and organization of this process; transportation, storage, and stocking of products, as well as introduction of parts and modules in the production planning process after organization and control of disassembly.

One of the important research and application field is the integration of reuse in the production planning. There are only a few international publications on this field. Most of them are German speaking literature. (Inderfurth (1998), Spengler et al. (1997), Rautenstrauch (1997)) There are some Anglo-Saxon papers on this field. (Ferrer-Whybark (2000), Guide (2000)) As I know, there are no Hungarian publications that investigate this problem. In this chapter I do not examine the organization of return processes, i.e. return management.

This chapter consists of following sections. The second section tries to extend production planning with reuse. It means that I give some insights in the connection of production planning and recycling planning. I create a model to analyze disassembly planning, which can be viewed as a "negative" bill of material. The next section presents the integration of reuse in the MRP production planning and control system. I show a material requirement planning (MRP) item record, and the planning steps till the use of materials. The fourth section summarizes the role of recycling in production planning.

## 4.2. An extension of production planning with reuse

The collection and reuse of used products and materials cause new problems in the production planning, which necessitates a connection between MRP and recycling planning. Involvement of recycling in material flow means a new problem in material management.

Production planning and control systems are developed for traditional production processes, which is not characterized by a cyclical material flow. The role of recycling activities has increased because of decreasing amount of raw materials and of rise in storage prices, which have economic and ecological causes. Strong social pressure and increasing governmental regulation make a current problem from reuse.

In this chapter I define recycling as a return of used products from production process and from outside, in order to reuse of these used products. Internal recycling products are those products that are not necessary in a following production process, i.e. by-products, or wastes. External recycling products are those products that are at the end of life and they originate from the consumer processes. Both internal and external recycling products are reused then in the production process. The aim of a recycling process is to produce new products from recycled products or to manufacture reusable parts and modules for further use. The further not usable parts and materials can be sold in a second hand market or disposed outside.

Material flow extended with recycling processes involves storage of raw materials, semi-finished products, end-products, and recycling products. Uncertainty of wastes and returned products in time, quality and quantity, and uncertainty of duration of reuse process make the recycling planning process uncertain. So the planning becomes a more complex problem, and there are a number of decision variables in the decision making. The first situation is decision about disassembly, reuse and use processes. A second relevant decision is on the field of manufacturing and purchasing, i.e. the substitution between recycled and newly procured products and materials, as alternative possibility of material supply. From this context it is clear that an integration of production and recycling planning is necessary.

Recycling planning, as production planning as well, means a strategic and tactical point of view, and then it has an operative content. This operative content can be divided as an original production planning and control, quantity planning, time and capacity planning, and manufacturing planning. Of course, these activities are extended with recycling activities.

Program planning in recycling means a demand forecasting of type, quantity and duration of recycling products. On the basis of this forecasting the recycling activity can be formed, and on the basis of forecasted and returned products an active planning can be developed. If this forecasting is not taken to be account, then this process is defined as passive recycling planning, because the enterprise reacts only on the known amount of returned recycling products.

### 4.2.1. Connection between production and recycling planning

Integration is necessary between production and recycling planning, because program and quantity plans of production is a basis for forecasting of program plan of recycling products, and quantity plan of recycling influences the raw material requirements in time and quantity. There are three concepts of extension of MRP systems:

1. Integration of recycling and MRP.
2. Disassembly and requirements planning.
3. Integrated material disposition planning.

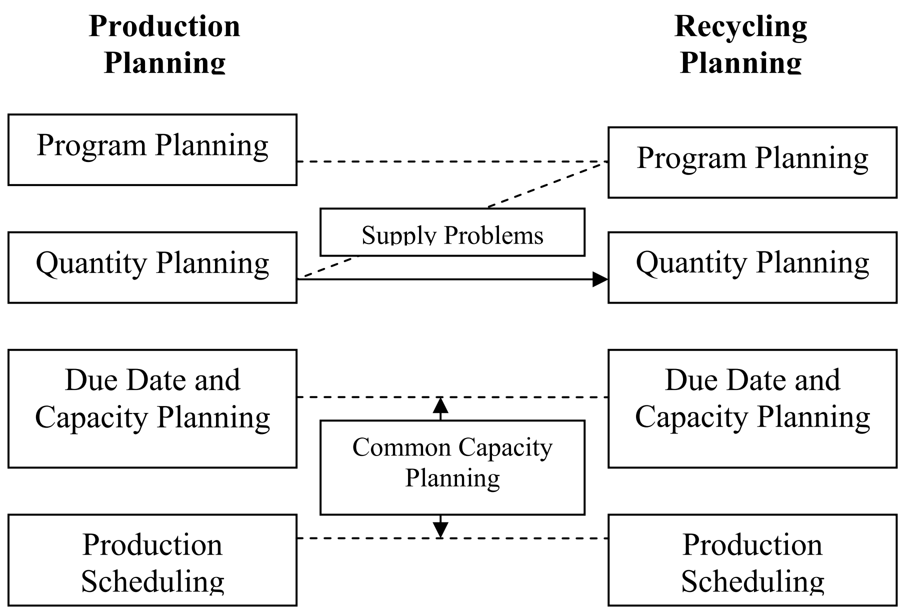



**Figure 1. Connection between production and recycling planning (Corsten-Reiss (1991))**

The first concept does not integrate decision support systems in the MRP extended with recycling in opposite to second and third concepts. It is a deterministic and direct extension of MRP, because it uses only the passive recycling planning and it includes only known dismantling, recycling and material supply strategies. The essence of second and third concepts is summarized in the following section, and I discuss the first concept, i.e. the integration of MRP system and recycling, in an other section. The connection of these two systems is presented in figure 1.

#### 4.2.2. Disassembly and use planning

Disassembly and use planning means often a decision making about disassembly and use measures, as a determination of recycling products in tactical planning and production planning. Disassembly planning means a decision about the deepness of dismantling process, steps and frequency of execution of disassembly process. It must be decided in disassembly planning whether the original product will be recovered, or modules, parts and raw materials will be regained. It is determined in case of recycling of materials and parts that available materials, alternative internal or external possibility of use are applied. There is always an alternative use under traditional methods of recycling. This decision is determined by technological and political conditions, which are fixed by product takeback, disassembly, manufacturing, and use. The following data are necessary to planning: quality of the reusable products or their parts, disassembly, inspection, manufacturing and storage costs, and revenue from sales.

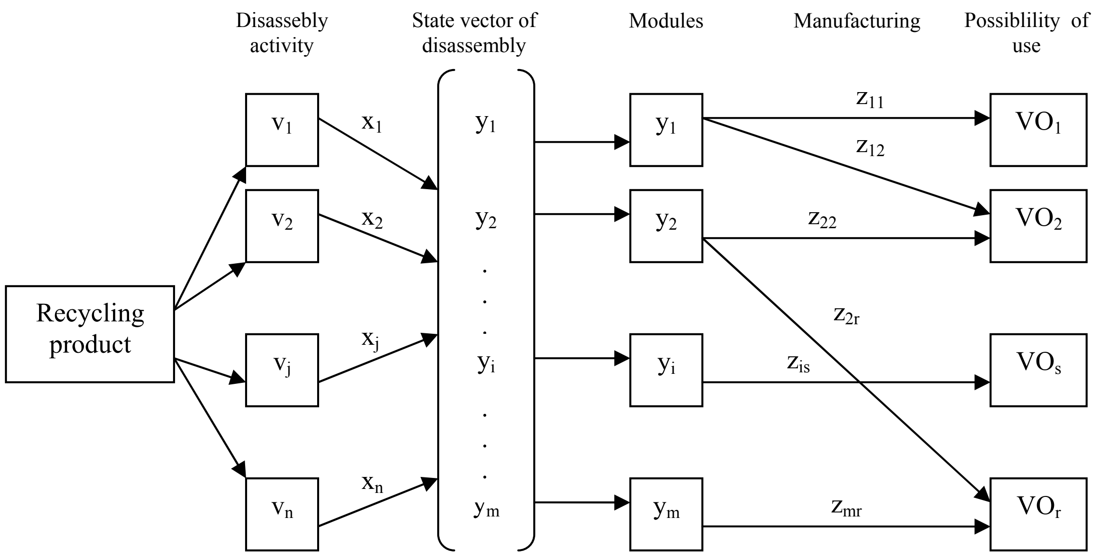



**Figure 2. Simultaneous handling of disassembly and reuse activities (Inderfurth (1998))**

Spengler et al. (1997) have determined the exact use capacities with a simultaneous disassembly and use plan. The planning problem is described with an activity analytical model, which is a mixed integer linear programming model in this case. They have created a disassembly graph, which contains alternative disassembly steps of a product ($v_j$ , where j=1,…..n). The product can be dismantled in m different components, which can be further decomposed, or have different methods of use that can be landfilling, as well. The execution frequency of different disassembly activities ($x_{j,}$, where j=1,….n) that is given by number of treatable products determines the number of components ($y_{ji}$, where j=1,… m), which are usable for further dismantling. The usable quantities determine the number of manufacturing and preparation steps ($z_{is}$, where s=1,….r) that are revenues and costs in use.

The objective is to maximize the profit through the disassembly and use activity variables $x_j$ and $z_{is}$. An overview of this planning system is shown in figure 2.

#### 4.2.3. Integrated material disposition

The main point of integrated material disposition is that it connects the return of reusable products, or components of these products with appropriate levels of manufacturing process. It is a difficult problem that is caused by satisfaction of product requirements of manufacturing and use with returned products, while the time requirements of these processes are different. The problem of disposition is to co-ordinate traditional production, recycling, and waste disposal activities, further to minimize the expected (manufacturing, recycling, storage, and transportation) costs in a planning period. There are two possible solutions of the disposition problem of storage:

1. continuous control of decision process, and
2. periodical control of decision process.

The uncertainty problems can be cleared by calculation of product requirements and of returned recycling products. In general, the storage of all products can be solved, and there is a choice between manufacturing and waste disposal of recycling products.

Ordering restricted strategies are characterized by three storage disposition:

1. storage restriction in traditional production,
2. restriction on recycling, and
3. restriction on waste disposal in a landfilling site.

If the storage of recycling products is not possible, then restriction of recycling and waste disposal is in keeping with this fact. (Inderfurth (1998))

### 4.3. Integration of reuse planning in MRP

#### 4.3.1. Rise and groups treatable products

New products are manufactured in a production process using input products, but by-products are originated from the production process that is not excluded in industrial production. It means that such goods originate in a manufacturing process, which does not occur in a production plan. By-products are fully excluded, if production of planned goods is stopped. Quantity of by-products can be reduced with steps in product planning process, and with appropriate steps in fields of purchasing, production, and quality management.

Wastes can be categorized in two groups: subjective and objective wastes. Subjective wastes are those materials, which are not further used by the owners, and there is no information about the reusability of these materials. Objective wastes are those materials, which are not reusable and they must disposed in a landfilling site. Wastes that are reusable are named by Corsten and Reiss (1991) as recycling goods. They have grouped these goods as follows:

1. *By-products* are those materials and energy, which occur in the end-products. By-products can be grouped in an other way: rests and wastes. Rests are those materials, which are reusable, and they can be results of a reuse process. Wastes are not reusable, or the reuse process is not realized in an economical way.
2. *Substandard goods* originate by products and by-products in a production process. The reuse form of these three categories is named as recycling. If these materials are not directly used, then they become inventory and it leads to an inventory decision problem.
3. *Used products* are the end-of-life or end-of-use products.

A fault of this grouping is that all by-products are recycling goods, although these wastes are objective wastes, and they can not be object of recycling. The idea of recycling goods does not contain objective wastes. A grouping of wastes is shown in figure 3.

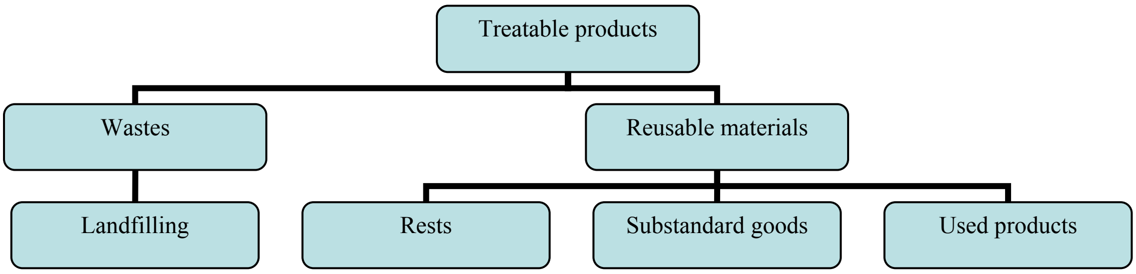



**Figure 3. Groups of reusable products (Becher-Roseman (1993))**

### 4.3.2. Process of collection and return

The reuse is come before collection of wastes in a firm. The collection of used products realizes through physical and information connection of sources and destinations.

#### 4.3.2.1. Collection

The first element of a return process is collection. Collection means the transportation of used products to a collection place. Collection process is based on the planning information. Data collection is a part of collection of used products, which is an information process. The collection requirements are determined in this process including address of consumers and due dates of transportation of collected goods and appliance. Further information is necessary about type, age, and quality of used appliances. This information are the basis of a vehicle routing planning, and disassembly and reuse planning process, i.e. these activities are the planning basis of the collection.

There are three types of collection:

1. The collectors transport the used products to a collection place, that can be a disassembly factory or a transfer place.
2. The owners of used products transport the materials to the collection place.
3. The combination of the above mentioned two strategies.

In general, the collection process is realized by waste transportation companies, although the retailer firms take back the used products nowadays, if the consumer purchases a new one.

Collection of data and used products has different problems:

1. There are parallel channels for consumers to collect data, and the supply is not appropriate concrete.
2. Taking an order per telephone or data collection is not always possible. If it is possible, then the waiting time is very long and after a number of trials.
3. The time between registration and collection can be more than a week because of the attainability of these places.
4. The given due date can not be kept.
5. The collection means only the availability of products collected in containers, and not the products stored in a house or in a cellar.
6. It is used such a vehicle, which capacity is not fully used.
7. The increase in collection systems leads to a competition for used products. The goal is to maximize the utilization of collection systems and of reuse capacity. In this case the collection routes become longer, and this means greater transportation costs and environmental burden.

#### 4.3.2.2. Loading

Loading is all transportation and storage process, which is loading and unloading vehicle with products, and it emerges in case of change of transportation facilities. Loading is necessary in a number of cases, in order to reduce the concentration of material flow. The loading is not optional activity since there are direct and gradual return processes. Loading are realized mainly with hand, which leads to high loading costs and to cause damages in used products.

#### 4.3.2.3. Transportation

Transportation is determined as to get used products to a collection place or to a central collection place. In a one-step return process transport means transportation of products to assembly factory, and it is transportation of goods to next collection center in a several-step return process. Other types of vehicles are used to transport than collect goods, in order to reduce transportation costs. Transportation is not a forced activity in the return process, since source and destination are the same places, if the distances are not so long and the collection tour is ending at the destination. Transport is fulfilled by trucks.

The arising questions are the following:

1. The number of automotive and more effective loading processes is limited.
2. The used products can be damaged at loading and unloading processes, as in transport process.
3. Moisture leads to corrosion, which reduces the disassembly of the products.
4. Auxiliary materials used in transportation are not storable, so there is no possibility to store them in a place economical way.

#### 4.3.2.4. Storage

Storage is a planned placing of usable products. The aim of storage is

1. to prevent the fluctuation of purchasing, transportation, and production,
2. to balance the difference between supply and demand,
3. to reduce the uncertainty of unknown supply and demand divergences,
4. to choice assortment.

There is output and input oriented storage. Output oriented storage concentrates on the sources of used products, i.e. on owners, who will sell the used products. The input oriented storage concentrates on the destination of used products, which is a disassembly factory. This factory produces new inputs for the manufacturing process with dismantling.

#### 4.3.2.5. Selection and assort

Selection of collected used products is the assort of products according to special disassembly or reuse operation. The documentation of transportable used products is stored in addition to concrete dismantling at this place This disassembly information plays an important role to estimate the factory capacity and sales from the dismantled parts and modules. A preliminary disassembly can be made in the selection process, so the efficiency of the transport process can be increased and the disassembled quantity can be reduced. These two facts lead to a better utilization of capacity. These supplement activities increase the demand for assort factory providing special services. Assort is not a selection of products after type of machines and modules, but it is an activity to support disassembly.

#### 4.3.2.6. Packaging

Packaging has a function to prevent, to store, to transport, to identify, and to inform others about the goods. This function necessitates the sale and use of products. One of the aims of packaging activities is to achieve such a state, which do not pollute the environment in process of packaging. It can be attained by use of such transportation facilities that requires less or no packaging materials, i.e. containers.

The return process is influenced by the improvement programs and service level and quality. It is important to know the consumer needs in return processes, for example service quality. This determines the use level of built system and network, i.e. the demand for reverse logistics services, which can reduce the costs of these services. (Waltemath, 2001)

### 4.3.3. Definition and types of recycling

Reuse is return of solid, liquid and gas state rests, substandard goods, and used products in the manufacturing process. All firms are such systems that emit goods and wastes in the environment, as output and absorb raw and other materials, and energy, as input. Jahnke (1986) distinguishes internal, among firm, and external recycling.

1. *Internal recycling* means that products intended for recycling return to the manufacturing firm. There is direct and indirect recycling. In case of direct recycling the products are returned to the manufacturing process that they produced. Recycling is indirect, if the return is preceded by a handling before production process.
2. *Recycling among firm* is defined as reuse of used products of other firms.
3. *External recycling* is the case when the product recycling is made by other firms.
4. *Cooperative recycling* can be defined, if not only used recycling products flow among firms, but also planning and organizational information. This is a special case of external and among firm recycling.
5. *Manufacturing recycling* is defined, as reuse of recycling products by their emittents.

This is a special case of internal and among firm recycling.

6. Reuse of products created in the production process is called as *primary recycling*, in other cases the reuse is a secondary recycling. The connection between primary and *secondary recycling* is shown in figure 4. (Rautenstrauch (1997))

| | **Direct** | **Indirect** |
|---|---|---|
| **Primary** | Reuse | Further use |
| **Secondary** | Resale | Further sale |

**Figure 4. Groups of recycling (Rautenstrauch (1997))**

*4.3.3.1. An other groups of recycling*

1. Rests or substandard goods are returned in the same manufacturing process without any handling, as an input.
2. Rests or substandard goods are returned in an other manufacturing process without any handling, as an input.
3. Rests and substandard goods are handled that is a disassembly or a transformation.
4. After transformation the materials are returned in the same manufacturing process, as input.
5. The recycling products are returned in an other production process after product recovery (regain and reuse). After use they are stored and transported in a landfilling site.
6. The recycling products are sent to an other firm for handling, and used after that as input.
7. There is a possibility of handling goods in a landfilling site.
8. Points 1 to 6 are internal recycling, and points 7 and 8 are among firm recycling.

Heterogeneity of rest materials and low concentration of rest materials are problems in reuse process, which means the measure of use, corrosion, and level of hazardousness. Both factors make more difficult the collection, storage, transportation, assort, and handling of materials. (Corsten H., Reiss M., 1991)

*4.3.3.2. Groups of recycling after processes*

1. *Production waste recycling* is an internal recycling that touches the rests and substandard products of manufacturing process.
2. *Recycling during product use* is a use of products in order to make the product partly reusable.
3. *Used material recycling* is a different form recycling during product use. The difference is that the product can not be as new recycled. These products are dismantled and recovered, and as raw or other materials are returned in the production process.

**4.3.4. Objectives, conditions, tools and restrictions of recycling**

**4.3.4.1. Objectives**

The objectives of recycling are the reduction of raw material and energy requirements, load of environment, and saving of storage capacity through decrease and liquidation of wastes and rest materials. The aim of a private enterprise is to reduce the requirements of raw materials, and to lengthen the lifetime of recycling goods in a long run, in order to slow down the creation rate of these goods, and to decrease the uncertainty.

Minimization of expensive capital investment causes the decrease of lifetime in development of production process. This minimization includes the recycling, logistics, manufacturing, planning, and transaction costs.

**4.3.4.2. Conditions**

The firm must consider a number of exogenous facts in reuse process, which restrict the activity of the enterprise, for example:

1. the full recovery of raw materials from recycling goods is often not possible,
2. recycling goods are not reused with an intended frequency ,
3. not all recycling products are reusable economically,
4. environment injuring products originate in a recycling process,
5. reuse of certain goods is regulated by a law, so the use of these products is not a decision problem.

Recycling can be seen, as a temporary relieve of use or consume of primary raw materials in order to lengthen the lifetime of a product.

#### 4.3.4.3. Tools

To investigate the tools used in recycling, there is a difference between application and handling of recycling products:

1. Some of the recycling goods are reusable without any remanufacturing by the help of assorts, transportation, and storage.
2. A selection or transformation procedure must be made in a handling process, which can be a biological-technical or chemical-technical process, in order to make the recycling products reusable.

A recycling decision model can be applied as a decision and planning model, if relevant information about the recycling goods are available, e.g. price, quantity, and quantity of recycling products. One of the factors influencing the decision complexity is the fundamental goals, which can be a multidimensional goal function. In case of multidimensional goals there is a conflict among the goals, and not only conflicts between ecological and economic objectives.

The different groups of factors determine the next levels of complexity.

1.) Production process:

a) the lower and upper bounds are given for a usable quantity of recycling goods,
b) the fitting requirements of production and recycling process are known,

c) the return of recycling products in the same and other manufacturing processes is known.

2.) Recycling products:

a) only rests, substandard goods, or used products or both of them are examined,
b) the beginning of recycling products can be continuous or discontinuous in time,
c) the storability of goods after heterogeneity, i.e. cleanness, form, color, or heat-resistance etc.,
d) separability of parts and modules, material substitution.

3.) Reuse process:

a) depth of reuse process, i.e. level of dismantling and processing,
b) by-products of a reuse process, i.e. usable and unusable, damages and losses in reuse,
c) losses in quality.

The aim of a production planning and control system is to determine the production and purchasing quantities in time under consideration of capacity restrictions, and handling steps for deviation of planning, realization and control, in order to catch the fundamental goals.

**4.3.4.4. Restrictions**

1. *Technical restrictions:* recycling products are not usable without any frequency, because the quality of goods deteriorates with the frequency of reuse. Further, recycling goods are not fully reusable, because their decomposability is restricted and dismantling is technically not possible.
2. *Economic restrictions:* costs caused by recycling can exceed its result, and the savings in primary materials.
3. *Ecological restrictions:* recycling necessitates energy to transport recycling goods, and often primary materials to refurbish their quality. Recycling is not useful from ecological point of view, since recycling can cause environmental burden, which exceeds the environmental utility.
4. *Psychological restrictions:* products manufactured from used products seem to have a lower quality than new one, so the market reacts abstained with these products. (Rautenstrauch (1998))

### 4.3.5. An MRP system

Fundamental objective of an MRP system is to minimize the influenceable costs, i.e. production, transportation, storage, and capacity costs. The time and quantity aims of these systems are the following:

a) minimal throughput time
b) great accuracy,
c) low inventory level,
d) maximal capacity utilization.

Figure 5 shows the endogenous and influenceable elements of an MRP system.

| Object<br><br>Goal | Capacity | Order |
|---|---|---|
| Time | Utilization of capacity | Throughput time |
| Quantity | Work force and assets | Transportability |
| Costs | Costs of capacity | Costs of shortage and storage |

**Figure 5. Objectives of MRP (Corsten-Reiss (1991))**

In order to integrate reuse processes in an MRP system, it is necessary to collect the relevant information to planning about recycling products and recycling processes. An enterprise environmental information system must be built to attain this information in an appropriate form. Laws and instructions must be followed with attention in this information system; steps to reduce emission must be introduced, which contain environmental statistics and information about waste disposal, purchasing methods, quality and material and energy balance considering different inputs and outputs. Environmental pollution and quantity of this pollution are examined in a production process, since the pollution tax is paid on this basis.

Horizontal extension of an MRP system with a reuse process means three possible generalizations:

1) recycling program planning,
2) recycling capacity planning,
3) recycling throughput planning.

I.) *Breadth and depth of a recycling program is specified.*

The following extension possibilities are in an MRP system:

1) Transportation and storage capacities must be considered, and priority rules have to determine, in order to use the restrictedly storable products first.
2) Revision of transportation is necessary to consider the quantity and due date of transportable recycling products. Immediate inclusion of not or only restrictedly storable recycling products in the production process is very important. It is necessary further to consider the manufacturing steps, quantity of recycling products, and due dates in a manufacturing process. Emission limits have to be paid attention in case of handling of not reusable by-products. Quantitative and qualitative criteria must be kept in a further manufacturing.

II.) *Quantity planning*

Extension and transformation of an MRP system is necessary with quantity planning. Net requirements of parts and raw materials are reduced by rests and wastes, and these reused input goods are used as inputs in the production process, but rise of recycling goods means a great uncertainty. Gross requirements are determined from material requirement planning. There is an extension requirement in data handling and processing. It contains collection, storage, refreshing, and processing of data connected with:

a) reliability (production time, quantity, and quality),
b) machines (waiting time),
c) work force (absence and presence time) and,
d) materials (shortage, and availability of materials at production places).

Of course, these materials must be available not only in the production, but also in the reuse process. It is necessary a plan to group of recycling goods after quantity and type.

III.) *Recycling throughput planning*

The planning levels depend on one-sided each other in an MRP system, they follow each other consecutively. Recycling process has a circular character, i.e. their processes are independent from each other. Different steps of production planning are built up linearly, so independence on planning levels is not considered. Performability of each steps depends on precedents, i.e. each decision levels are conditions for the following decision levels. Production program and capacity are connected with each other. If activities are realized with independent throughput time from capacity, then this leads to an inconsistency in planning, since quantities are fixed in quantity plan, and due date and capacity plans are not held for a given due date, because due dates given in contracts do not correspond with necessary due date. Since rest and waste materials return in not constant, but unsystematical quantities, recycling makes more complex the estimation of due dates in a traditional MRP system. Linearity of planning and cyclicality of planning object make more difficult the timeliness of planning.

Reused products are used for net requirement. Factory and ordered products, spare parts and safety stocks are used to cover gross requirement under recycling goods.

The measure of centralized decision must be mentioned in planning. If rests, wastes, substandard goods, and used products appear in a recycling process, then the recycling process is a multiple process, and the MRP system becomes more centralized. There is a positive correlation between complexity and centralizedness. If a recycling process is more uncertain, then an extended MRP system is less centralized. (Corsten-Reiss (1991))

### 4.3.6. MRP item record extended with recycling

The first part of an MRP item record extended with recycling is similar to that of traditional MRP item record, although there is a recycled product stock line, which means that traditional inventory line is extended with an alternative inventory. These stocks contain the spare parts and materials recovered from returned items, and they become inventory. Then there is no difference between reused and newly manufactured materials. Let us assume that the length of planning horizon is 6 weeks, safety stock is 15 units, and lead time is 2 weeks. The data of this example contain figure 6. The material flow is shown in figure 7.

The stock line is equal to the sum of produced, recycled, and initial stock level reduced with gross requirements. The safety stock is 15 units, which is considered at the stock. The expected level of returned items is 4 units. Recycling stock level is given. Recycling requirements are equal to 4 units, which are the number of returned items. Recycling order comes from recycling requirements with a leg of 2 weeks lead time. Handling requirement is the stock level of recycling process reduced with recycling order. Planned order receipt is the net requirement reduced with recycling requirement. Planned order release is planned order receipt with a leg of lead time that is 2 weeks in our example.

| | **0** | **1** | **2** | **3** | **4** | **5** | **6** |
|---|---|---|---|---|---|---|---|
| Gross requirement | | 10 | 10 | 10 | 10 | 10 | 10 |
| Produced stock | | 8 | 14 | | | | |
| Recycled stock | | 5 | 4 | | | | |
| Stock | 9 | 12 | 20 | 15 | 15 | 15 | 15 |
| Net requirement | | 3 | 0 | 5 | 10 | 10 | 10 |
| Planned return | | 4 | 4 | 4 | 4 | 4 | 4 |
| Recycling stock | 7 | 4 | 4 | 4 | 4 | 4 | 4 |
| Recycling requirement | | - | - | 5 | 4 | 4 | 4 |
| Recycling order | | 5 | 4 | 4 | 4 | - | - |
| Handling requirement | | 2 | 0 | 0 | 0 | - | - |
| Planned order receipt | | - | - | 0 | 6 | 6 | 6 |
| Planned order release | | 0 | 6 | 6 | 6 | - | - |

**Figure 6. MRP item record extended with recycling (Inderfurth-Jensen (1998))**

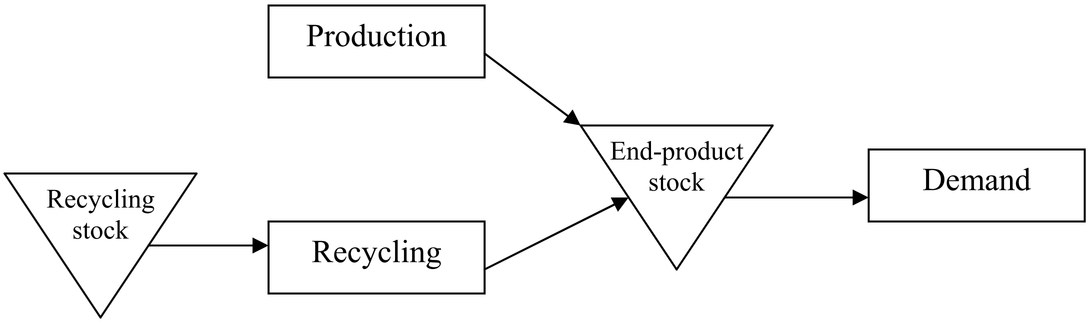



**Figure 7. Material flow in MRP item record**

### 4.4. Summary

Environmental protection is become more important in enterprise praxis nowadays. Till environmental protection is not a great business for firm, i.e. reverse logistics is not a factor of competitive advantage; it can not be treated at a strategic level. It becomes competitive, if society acknowledges environmental conscious activity supported with environmental audits and the members of society purchase environment friendly products. Some of methods are presented in this chapter, which help firm reduce the use of primary raw materials and energy, as well environmental pollution. Firm can choose the best appropriate method from this checklist. View of society must change in interest of environment. The government plays an important role to influence the corporate strategy of firms. The saving of natural resources and lengthen of use of natural resources support to achieve a sustainable development path in interest of future generations.

# 5. Summary and Further Research

Reverse logistics and its application in production planning are investigated in this work. Reverse logistics can completely built in material requirements planning (MRP) systems. The extension is that classical MRP item records contain the files of returned and reusable products in addition to traditional information. Last rows of MRP item records show the requirements of previous production phases and/or purchasing. The inventory problem occurs at this level, whether the decision maker must unite production and/or purchasing lots, or not. It is used heuristics in traditional MRP systems to determine the lot sizes, as Groff-algorithm, or Silver-Meal heuristics, and so on. These heuristics apply the optimality conditions of EOQ model. This property is that the ordering and/or setup costs are equal to the inventory holding costs in cot minimum. The question is now, whether EOQ-type reverse logistic inventory models are applied to lot sizing in extended MRP systems.

To answer the last question, I have presented six reverse logistic EOQ-type inventory models. The models have common conditions, that are the exclusion of shortage. The cost structure is similar to that of classical EOQ model, i.e. there are cycle fix purchasing/production costs, inventory holding costs for new products, and fix and holding costs for reusable items.

I have investigated the examined models under these conditions, I have shown that the models lead to a meta-model presented in appendix. I can simplify the cost function after its construction with two equalities. Either the number of lots or the lot sizes can be substituted in the cost function. If the number of batches is substituted in the cost function, then the variables are the lot sizes and the problem that the number of lots is integer can not be studied in the model further. This is the reason why it is more simply to substitute the lot sizes in cost function. With the help of this method I have shown such an example, where purchasing and reuse lot numbers are strictly greater than one.

I have examined those cases, when EOQ-type and non EOQ-type linear purchasing/manufacturing, reuse, and waste disposal costs are included in the cost function. For this type of models I have shown that waste disposal is neglected in the optimal solution, i.e. all returned and reusable items are economical to reuse. The necessary and sufficient condition for reusability of all used items is that one of the pure strategies, i.e. only purchasing/manufacturing and complete reuse, is cost minimal.

The presented inventory models can be the basis for those heuristics, which are applicable in extended MRP systems. As I know, there is no publication on this field. The generalization of Wagner-Whitin dynamic lot size model with reuse was undertaken by Richter-Sombrutzki (2000), Richter-Weber (2001) and Richter-Gobsch (2005).

I present the generalized model of Richter-Sombrutzki (2000), which is the extension of model of Schrady (1967) for the case of time varying demand and return. In this model there is no waste disposal activity. The parameters and variables are different that of published article, and they are the same, as they were introduced by Schrady (1967).

The stock-flow identity relations are the following:

$$\begin{aligned} I_t &= I_{t-1} + Q_t^P + Q_t^R - D_t \\ i_t &= i_{t-1} - Q_t^R + R_t \end{aligned}, \qquad (t = 1,2,\ldots,T)$$

$$\begin{aligned} &I_t \geq 0, \quad i_t \geq 0, \\ &Q_t^P \geq 0, \quad Q_t^R \geq 0. \end{aligned}, \qquad (t = 1,2,\ldots,T)$$

where $I_0 = i_0 = 0$. The first equality is the balance connection in *t*th period for new products, i.e. the inventory level at the end of period *t* is equal to the sum of initial inventory level, purchasing and repair reduced with demand. The second equation includes the returned items and the items taken in reuse process. The last inequalities are the nonnegativity conditions of variables.

The cost function is

$$\sum_{t=1}^{T} \left( A_P \cdot sign\, Q_t^P + h_1 \cdot I_t + A_R \cdot sign\, Q_t^R + h_2 \cdot i_t \right) \rightarrow \min .$$

The cost function includes the ordering, setup, and inventory holding costs. Function *sign* is zero, if its argument is nonpositive, and one in other cases.

Let us now summarize the parameters and variables of the model.

*Parameters of the model:*

- $D_t$ demand for new products in period $t$, nonnegative,
- $R_t$ returned used items in period $t$, nonnegative,
- $I_0$ initial inventory level of new products,
- $i_0$ initial inventory level of returned items,
- $A_P$ fixed procurement cost, per order,
- $A_R$ fixed repair batch induction cost, per batch,
- $h_1$ holding cost of new products,
- $h_2$ holding cost of used items,
- $T$ length of planning horizon.

*Variables of the model:*

- $I_t$ inventory level of new product in period $t$, nonnegative,
- $i_t$ inventory level of used items in period $t$, nonnegative,
- $Q_P$ procurement quantity, nonnegative,
- $Q_R$ repair batch size nonnegative.

Richter and Sombrutzki (2000) have proven some properties of the model.

**Lemma (Richter-Sombrutzki (2000)):**

It holds in optimal solution:

i) $Q_t^P \cdot Q_t^R = 0, \qquad (t = 1,2,\ldots,T)$

ii) $I_{t-1} \cdot (Q_t^P + Q_t^R) = 0, \qquad (t = 1,2,\ldots,T).$

I do not prove these properties, it can be found in above mentioned paper. Condition (i) says that both repair and purchasing can not occur in a period in optimal solution. The second equation expresses that purchasing or repair occurs only those periods, when initial inventory level is zero. If the initial inventory level is zero, then purchasing or repair must occur. This property is similar that of Wagner-Whitin dynamic lot size model, i.e. procurement/production occurs in periods with zero initial inventory level. As we see, the inventory holding policy offered by Schrady (1967) applies these two properties. This dynamic model can be solved with method of dynamic programming, but the solution is very time consuming for relatively small problem, so construction of heuristics to create suboptimal solution is necessary.

A need for further research is obvious, but the question is whether EOQ-type reverse logistic models are appropriate to construct a suboptimal solution of extended Wagner-Whitin models, i.e. to create with it satisfactory lot sizes for repair and purchasing. And now a question is how to construct such an algorithm.

The next group of question is; if there are effective heuristics, then how they function. For which cost and system parameters offer the algorithms a satisfactory solution? To test the effectiveness of algorithms, it is necessary to make simulations. Without any numerical analysis we can not answer this question. These offered heuristics can be used in production planning and MRP systems.

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## Appendix

### A. The meta-model

The meta-model is a fractional program of minimizing the following function for arbitrary real inputs,

$$S(m,n) \rightarrow min$$
$$(m,n) \in R_G = \{(m,n): m,n \in \{1,2,...\}\},$$

i. e. the problem of finding an optimal *(m,n)* is discussed below. The problem studied by Dobos and Richter (1999) will be shortly called ”integer problem”.

The relaxed fractional program

$$S(m,n) \rightarrow min$$
$$(m,n) \in R = \{(m,n): (m,n) \geq 1\}$$

has been studied by Richter (1994, 1996a, 1996b, 1996c, 1997) and Richter and Dobos (1999). It will be called the ”continuous problem”. First some properties found by Richter (1994, 1996a, 1996b, 1996c, 1997) will be presented here.

#### A.1. The existence of optimal solutions for the continuous and the integer problem

Both the problems have optimal solutions at the same time.

**Lemma 1** (Richter (1997)): The function *S(m,n)* is bounded on *R* and on $R_G$ if and only if

$$C \geq 0 \wedge D \geq 0 \wedge A+C \geq 0 \wedge B+D \geq 0. \quad (1)$$

Let below the relations (1) be fulfilled:

**Lemma 2** (Richter (1997)): Provided the relations (6) hold the integer and the relaxed problems have an optimal solution if and only if

$\{A \leq 0 \wedge B \leq 0\} \vee \{A+C > 0 \wedge B+D > 0\}$. (2)

**A.2. The structure of the optimal solution for the continuous problem**

Let us assume that parameters *A* and *B* are positive.

**Lemma 3** (Richter (1997)): There are two curves $M(n) = n\sqrt{\frac{B}{A+Cn}}$ and $N(m) = m\sqrt{\frac{A}{B+Dm}}$ of local minima in *m* or *n* for *n* and *m*, respectively, with the values

$$S(M(n),n) = 2\sqrt{(A+Cn)B} + Dn + E$$

and

$$S(m,N(m)) = 2\sqrt{(B+Dm)A} + Cm + E$$

for the function *S(m,n)*. The function *S(m,n)* is monotonously increasing along these two curves.

The level set of a function is defined as $lev_F S = \{(m,n) > 0: S(m,n) \leq F\}$ for an arbitrary *F*. The function *S(m,n)* is called quasi-convex if the level sets $lev_F S$ are convex for all feasible *F*. An equivalent definition of quasi-convexity for a function *f* is

$$f(\lambda\ x + (1-\lambda)\ x') \leq \max\{f(x), f(x')\}$$
$$\forall \lambda\ \in (0,1),\ \ \forall x, x' \in X$$

(see Arrow and Enthoven (1961), Takayama (1985)). A function *f* is strictly quasi-convex if

$$f(\lambda\ x + (1-\lambda)\ x') < \max\{f(x), f(x')\}$$
$$\forall \lambda\ \in (0,1),\ \ \forall x, x' \in X$$

Below some example of the level set for a quasi-convex function is given.

**Theorem 1:** If the relation $A > 0, B > 0, C+D \geq 0$ holds then function $S(m,n)$ is strictly quasi-convex.

*Proof:* To prove the theorem, let us check the conditions of strictly quasi-convexity. It follows from the definition of quasi-convexity, that the function $F(\lambda) = f(x' + \lambda(x - x'))$ has no maximum between *0* and *1*. Let us investigate our problem in the following form:

$$G(\lambda) = A\frac{m + \lambda \Delta m}{n + \lambda \Delta n} + B\frac{n + \lambda \Delta n}{m + \lambda \Delta m} + C\left(m + \lambda \Delta m\right) + D\left(n + \lambda \Delta n\right) + E,$$

where $(m,n)>(0,0)$ is an arbitrary point, and $(\Delta m,\Delta n)$ a feasible direction. We must now prove, that function $G(\lambda)$ has no maximum for every $(m,n)$ and $(\Delta m,\Delta n)$.

(i) $\Delta m < 0, \Delta n > 0$

In this case the function

$$\frac{m + \lambda \Delta m}{n + \lambda \Delta n} = \frac{m - \frac{\Delta m}{\Delta n} n}{n + \lambda \Delta n} + \frac{\Delta m}{\Delta n}$$

is convex because the numerator is positive, and the function

$$\frac{n + \lambda \Delta n}{m + \lambda \Delta m} = \frac{n - \frac{\Delta n}{\Delta m} m}{m + \lambda \Delta m} + \frac{\Delta n}{\Delta m}$$

is also convex, because value $\Delta m$ is negative. The other functions are linear and convex, and the function $G(\lambda)$ is convex and has no maximum for every $\lambda>0$. The case $\Delta m>0, \Delta n<0$ can be handled similarly.

(ii) $\Delta m > 0, \Delta n > 0, n\Delta m\text{-}n\Delta m > 0$

Let us now investigate the derivative of function $G(\lambda)$. We will show that the function has a minimum, if any.

$$G'(\lambda) = A\frac{n\Delta m - m\Delta n}{\left(n+\lambda\Delta n\right)^2} + B\frac{m\Delta n - n\Delta m}{\left(m+\lambda\Delta m\right)^2} + C\Delta m + D\Delta n$$

The case is considered when the function $G(\lambda)$ is monotonously non-decreasing. Then

$$A\left(n\Delta m - m\Delta n\right)\frac{\left(m+\lambda\Delta m\right)^2}{\left(n+\lambda\Delta n\right)^2} \geq -\left(C\Delta m + D\Delta n\right)\left(m+\lambda\Delta m\right)^2 + B\left(n\Delta m - m\Delta n\right)$$

The left-hand side function is monotonously increasing, and the quadratic function is monotonously decreasing because of the non-negativity of parameters $C$ and $D$. (It is easy to check with derivation.) There exists one and only one $\lambda^0$ satisfying the equality, if

$$A\left(n\Delta m - m\Delta n\right)\frac{m^2}{n^2} \geq -\left(C\Delta m + D\Delta n\right)n^2 + B\left(n\Delta m - m\Delta n\right).$$

It means, that on the interval $\lambda>\lambda^0$ the function is monotonously increasing, and monotonously decreasing in other case. If value $\lambda^0$ does not exist, then the function $G(\lambda)$ is monotonously increasing for every nonnegative $\lambda$. And we have proved of the theorem. □

In case of quasi-convex programming the following theorem provides a necessary and sufficient condition of optimality. A variable is called relevant, if it can take on a positive value without necessarily violating the constraints.

**Theorem 2.** (Arrow and Enthoven (1961), Takayama (1985)): Let $f(x)$ be a differentiable quasi-convex function of the n-dimensional vector $x$, and let $g(x)$ be an m-dimensional differentiable quasi-convex vector function, both defined for $x \geq 0$. Let $x^0$ and $\lambda^0$ satisfy the Kuhn-Tucker-Lagrange conditions, and let one of the following conditions be satisfied:

(a) $f_{x_{i_0}} > 0$ for at least one variable $x_{i_0}$;

(b) $f_{x_{i_1}} < 0$ for some relevant variable $x_{i_1}$;

(c) $f_x \neq 0$ and *f(x)* is twice differentiable in the neighborhood of $x^0$;

(d) *f(x)* is convex.

Then $x^0$ minimizes *f(x)* subject to the constraints $g(x) \leq 0$, $x \geq 0$.

Proof is omitted; see Arrow and Enthoven (1961).

Let us now check condition (c) of Theorem 2 to our problem.

**Lemma 4:** Let point $(m_0, n_0) \geq (1, 1)$. Then

$$\text{(a)} \quad \begin{pmatrix} \dfrac{\partial S}{\partial m} \\ \dfrac{\partial S}{\partial n} \end{pmatrix}_{(m_0 \quad n_0)} = \begin{pmatrix} A\dfrac{1}{n_0} - B\dfrac{n_0}{m_0^2} + C \\ -A\dfrac{m_0}{n_0^2} + B\dfrac{1}{m_0} + D \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

$$\text{(b)} \quad \begin{pmatrix} \dfrac{\partial^2 S}{\partial m^2} & \dfrac{\partial^2 S}{\partial m \partial n} \\ \dfrac{\partial^2 S}{\partial n \partial m} & \dfrac{\partial^2 S}{\partial n^2} \end{pmatrix}_{(m_0 \quad n_0)} = \begin{pmatrix} 2B\dfrac{n_0}{m_0^3} & -A\dfrac{1}{n_0^2} - B\dfrac{1}{m_0^2} \\ -A\dfrac{1}{n_0^2} - B\dfrac{1}{m_0^2} & 2A\dfrac{m_0}{n_0^3} \end{pmatrix}.$$

The proof of the lemma is easy and is left to the reader.

As it is shown, our continuous auxiliary problem is a quasi-convex, and the function *S(m,n)* satisfies the condition (c) of Theorem 2. This condition guarantees the optimal solution. An example of the function *S(m,n)* in shown in Fig. 1.

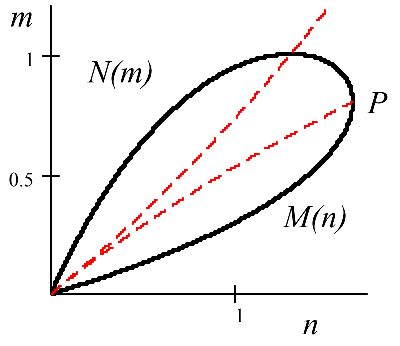


**Fig. 1.:** **The curves of local minima and the level set of the function *S(m,n)* with $A = 25$, $B = 10$, $C = 10$, $D = 5$, $E = 0$ and $F = 48.73$**

Some of the properties of the function *S(m,n)* provided by Richter (1997) are collected in Tab. 1.

| case | *A* | *B* | *C+D* | *A+C* | *B+D* | properties of *S(m,n)* |
|---|---|---|---|---|---|---|
| *a)* | $> 0$ | $> 0$ | $\geq 0$ | | | convex in *m* and in *n*, strictly quasi-convex in *(m,n)* |
| *b)* | $\leq 0$ | $> 0$ | | $> 0$ | $> 0$ | increasing in *m* , convex in *n* |
| | $> 0$ | $\leq 0$ | | $> 0$ | $> 0$ | increasing in *n*, convex in *m* |
| *c)* | $\leq 0$ | $\leq 0$ | | $\geq 0$ | $\geq 0$ | increasing in *m* and in *n* |

**Tab. 1. Properties of the function *S(m,n)***

The explicit solution of the continuous problem is given by

**Theorem 3** (Richter (1996a)) If the relations (6) - (7) hold there are three cases of optimal solutions *(m,n)* and minimum cost expressions $S$ for the function (5) on *R*:

(i) $B \geq A+C$ $\quad (m^*,n^*) = \left(\sqrt{\frac{B}{A+C}}, 1\right)$, $\quad S^* = 2\sqrt{B(A+C)} + D + E$,

(ii) $A\text{-}D \le B \le A+C$ $\quad (m^*n^*) = (1,1),$ $\quad S^* = A+B+C+D+E$

(iii) $A \ge B+D$ $\quad (m^*,n^*) = \left(1, \sqrt{\dfrac{A}{B+D}}\right),$ $\quad S^* = 2\sqrt{A(B+D)} + C + E$.

### A.3. The optimal solution for the integer problem

#### A.3.1. The cases a) in Tab. 1

**Lemma 5:** Let in Theorem 1 to the condition (i) additionally $49A \le 527C$ or to (iii) additionally $49B \le 527D$ be fulfilled. Then the optimal integer solution is on the line $n = 1$ or on the line $m=1$, respectively.

*Proof*: Let the case (iii) be considered. Let us assume, that $S(1,n) \ge S(1,n+1)$ and the optimal continuous solution is $(1,n^*)$. Let $n+1 = n^*+\delta$. It can be shown by elementary operations that

$$S(1,n+1) = 2\sqrt{A(B+D)} + C + E + \sqrt{A(B+D)}\frac{\delta^2}{n^*\left(n^*+\delta\right)},$$

where $0 < \delta < 0.5$. Let the following problem be investigated:

$$\sup_{n^* \ge 1.5,\ 0<\delta<0.5} \left\{ \frac{\delta^2}{n^*(n^*+\delta)} \right\}.$$

The function is monotonously increasing in $\delta$ and monotonously decreasing in $n^*$. Then

$$\frac{\delta^2}{n^*\left(n^*+\delta\right)} \le \frac{1}{12},$$

and

$$S(1,n+1) \leq \frac{25}{12}\sqrt{A(B+D)}+C+E. \quad (10)$$

Any other integer solution with $m \geq 2$ attains no smaller value than $S(2,n_2)$ where $n_2 = M(2)$ and $S(2, 2\sqrt{\frac{A}{B+2D}}) = 2\sqrt{A(B+2D)}+2C+E$ according to Lemma 3. The inequality $S(1,n+1) \leq \frac{25}{12}\sqrt{A(B+D)}+C+E \leq 2\sqrt{A(B+2D)}+2C+E = S(2,n_2)$ holds if and only if $\frac{25}{12}\sqrt{A(B+D)}-2\sqrt{A(B+2D)} \leq C$ is fulfilled. The indicated condition of the lemma secures this inequality. If $S(1,n) \leq S(1,n+1)$ then $n=n^*-\delta$ and the same estimation will be found.

*Remark:* Let, for instance $A=20.25$, $B=1$, $C=0.04$, $D=0.0001$, $E=5$. Then the optimal continuous solution is given by $(1,n^*) = (1;4.5)$ with $S(1,n^*) = 14.04$, while $S(1,4) = 14.103 > S(1,5) = 14.091 > S(2,9) = 14.081$, i. e. the optimal integer solution is not on the line $m=1$. It is also clear that the case b) from Tab. 1 the same effect might occur (see Fig. 2.).

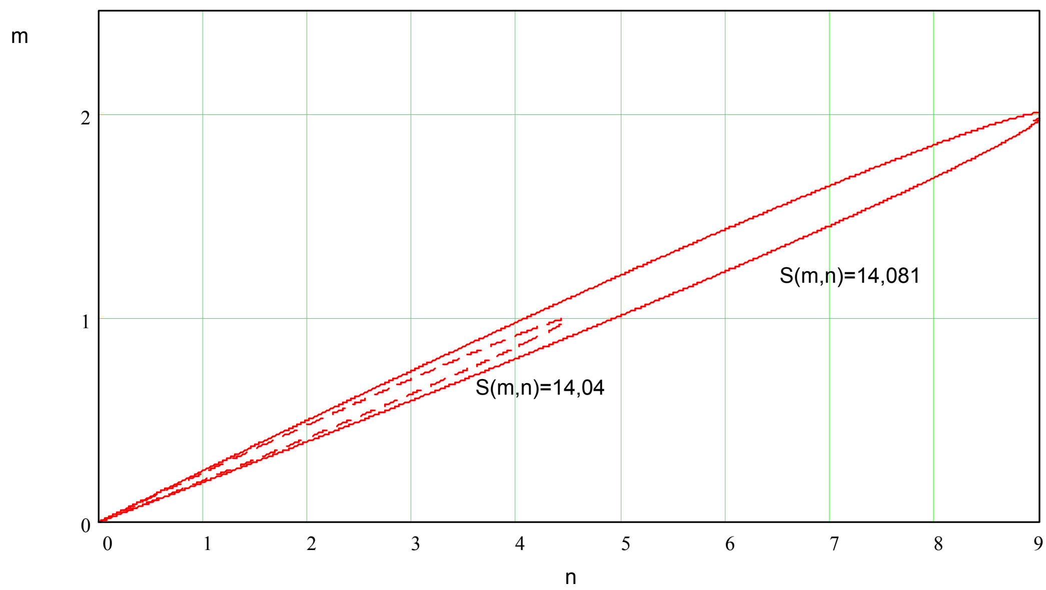


**Fig. 2: The level set of the function *S(m,n)* for continuous and integer solution with *A = 20.25, B = 1, C = 0.04, D = 0.0001, E = 5***

### A.3.2. The cases b) in Tab. 1

**Lemma 6:** If the continuous optimal solution is not integer then the optimal integer solution is

(i) $B \geq A+C \Rightarrow$ $(m^g,1)$ where $m^g$ is one of nearest integers of $\sqrt{\frac{B}{A+C}}$

(iii) $A \geq B+D \Rightarrow$ $(1,n^g)$ where $n^g$ is one of nearest integers of $\sqrt{\frac{A}{B+D}}$.

*Proof:* (i) Then $A \leq 0 < B$. Since the function $S(m,n)$ is at the same time convex in $n$ and concave and increasing in $m$, one of the mentioned solutions is optimal. □

### A.3.3. The case c) in Tab. 1

This case occurs only for $A\text{-}D \leq B \leq A+C$ in Theorem 1 and the continuous optimal solution is automatically integer.

An optimal integer solution on the line $n = 1$ or on the line $m= 1$ found in the previous Lemma will be called *boundary*, and below the question if boundary optimal integer solutions can be found for a significant part of the repair and waste disposal problems will be discussed.

**Theorem 4** (Richter and Dobos (1999)) Let the conditions of the Lemmas 4 or 5 be fulfilled. Then the following boundary optimal solutions for the discrete problem can be found:

$$\begin{aligned} &(i)\ B \geq A+C \Rightarrow m^g = \left\lfloor \sqrt{\frac{B}{A+C}+\frac{1}{4}}+\frac{1}{2} \right\rfloor,\ n^g = 1 \\ &(ii)\ A-D \leq B \leq A+C \Rightarrow m^g = n^g = 1, \\ &(iii)\ A \geq B+D \Rightarrow m^g = 1,\ n^g = \left\lfloor \sqrt{\frac{A}{B+D}+\frac{1}{4}}+\frac{1}{2} \right\rfloor \end{aligned} \qquad (11)$$

Proof: (iii) It is clear from the lemma that one of the two integer solutions $(1,n)$, $(1,n+1)$ with $n < n^* < n+1$ is optimal. Then $S(1,n) \leq S(1,n+1)$ holds if and only if $A\frac{1}{n(n+1)} \leq B+D$, or, if $n^2 + n - \frac{A}{B+D} \geq 0$, or, if $n \geq \left\lceil \sqrt{\frac{A}{B+D} + \frac{1}{4}} - \frac{1}{2} \right\rceil = \left\lfloor \sqrt{\frac{A}{B+D} + \frac{1}{4}} + \frac{1}{2} \right\rfloor$.

This right hand side is not less than $\lfloor n^* \rfloor$ and not greater than $\lceil n^* \rceil$. If $n$ does not fulfill that inequality then $n+1$ does and it is optimal. The first case can be studied similarily. □

**Theorem 5:** Let $S_G$ denote the minimal value for the integer problem. Then for the boundary solution (11) the relative error $dS_G = \frac{S(m^g, n^g) - S_G}{S_G} \leq \frac{1}{24}$ holds.

*Proof:* Let the case (iii) be considered and let the solution (11) not be optimal.

a) $n^g = \lceil n^* \rceil$. Then $S(1,n^g) - S_G \leq S(1,n^g) - S(1,n^*)$ and

$$\frac{S(1,n^g) - S_G}{S_G} \leq \frac{S(1,n^g) - \left(2\sqrt{A(B+D)} + C + E\right)}{2\sqrt{A(B+D)} + C + E}.$$

Then using inequality (10):

$$dS_G \leq \frac{\frac{1}{12}\sqrt{A(B+D)}}{2\sqrt{A(B+D)} + C + E} = \frac{1}{24}\frac{2\sqrt{A(B+D)}}{2\sqrt{A(B+D)} + C + E}.$$

b) If $n^g = \lfloor n^* \rfloor$ then the same estimation will be found. The case (i) can be treated similarly. □

If the boundary property is not guaranteed the following Lemma holds.

**Lemma 7:** The optimal solutions of the integer problem fulfill

(i) $n^g = 1$, $m^g = \lfloor m^* \rfloor$ or $m^g \geq \lceil m^* \rceil$ and (iii) $m^g = 1$, $n^g = \lfloor n^* \rfloor$ or $n^g \geq \lceil n^* \rceil$.

*Proof:* (i) If $m^g < \lceil m^* \rceil$ and $F_g = S(m^g, 1)$ then $(m^g, 1) \in lev_{F_g} S$, but $(m^g + 1, 1) \notin lev_{F_g} S$ and any other solution is not optimal because of quasi-convexity of function $S$. The case (iii) can be discussed similarly. □

*Remark:* As a consequence from that Lemma it can be noted that the optimal integer solution follows the changes of the optimal continuous solution. If $m^*$ ($n^*$) increases, the appropriate lower bounds of the components of the optimal integer solution will not decrease!