The Idea and the Features of Type Space

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Abstract

Several game theoretical topics require the analysis of hierarchical beliefs, particularly in incomplete information situations. For the problem of incomplete information, Harsányi suggested the concept of the type space. Later Mertens & Zamir gave a construction of such a type space under topological assumptions imposed on the parameter space. The topological assumptions were weakened by Heifetz, and by Brandenburger & Dekel. In this paper we show that at very natural assumptions upon the structure of the beliefs, the universal type space does exist. We construct a universal type space, which employs purely a measurable parameter space structure.

We divided this work into two parts. In the first part we introduce the main ideas and features of type space. We follow Heifetz & Samet, however we take some steps out of their work. We also introduce an example to present the usefulness of type space.

In the second part we present a complete universal type space, which contains the works of Mertens & Zamir, Heifetz, and Brandenburger & Dekel as a special case.

The two parts of this work can be read separately, therefore there are some minor parallelism in this paper.

Part I

Basic ideas about type space

0.1 Type space

The basic idea of type space comes from Harsányi[7]. In this part, we keep on the way of Bayesian model, so we use the ideas of probability theory. On the other hand, we define the idea of type space in an exact way, and this definition declares a form of type space is more general than Harsányi's type space. The form of type space, that we use here, comes from Heifetz & Samet[10], however we do not follow strictly their paper.

The main difference between our definitions and [10]'s definitions is that, we do not define exactly what kind of measurable structure is used, so we use an undefined, but fixed measurable structure. Naturally, [10] defines the measurable structure with that, they work, so in some sense we are more general, than [10].

By non-defining exactly the used measurable structure, we demonstrate that, the arguments of [10] are true, in a general concept. Therefore the measurable structure does not matter in our work, so we do not characterize that.

Definition 1 *Let M be the set of players, a type space, which is based on S parameter space, is $< T_{i\in M}, m_{i\in M} >$, shortly $< T, m >$, for which*

α, *$T_0 = S$, T_i measurable $\forall i \in I$*

β, *$m_i : T_i \to \Delta(T)$ measurable $\forall i \in I \setminus \{0\}$, where $T = \otimes_i T_i$, and $\Delta()$ the set of probability measures with the coarsest measurable structure respect to m_is are measurable*

γ, *$m_i(t_i) = \delta_{t_i}$, δ_{t_i} the Dirac-measure concentrated on t_i*

It is important, that there are only measurable ideas in the modell, therefore this definition is purely probabilistic one.

α, in the definition above, is about the main idea of Harsányi, that if we initiate the Nature in the model, then we can transform a non-complete information situation into a non-perfect information one. Point β, pont is the main idea of type space, and pont γ, formulates the fact, that every player knows own type.

A point in T is a state of the world, and a point in T_i is a type of player i. We can find Harsnyi's concept of type space in the definition above, because m mapping gives every point in T a probability measure on T for all i.

It is to see that, m_i mappings can be handled as probability transition functions, therefore functions $m_i : T_i \times T_{-i} \to [0, 1]$ for which

α, $\forall t_i \in T_i$ fixed, $A \to m_i(t_i)(A)$ $A \in T_{-i}$ probability measure

β, $\forall A \in T_{-i}$ fixed, m_i measurable

The probability transition function is a general form of conditional probability, therefore our type space is based on the concept of conditional probality[21].

Let $t_i \in T_i$ be arbitrary, then $m_i(t_i)$ is player i first order belief (about the other players's type, or about any event in T), $m_i(t_i)(\{t_{j'}\})m_j(t_{j'})$ is player i second order belief about player j first order belief about any event in T, etc. Therefore, the above defined type space determines the coherent hierarchies of beliefs, for every player (see. section: An example).

This model contains Harsányi's concept, because:

1. $m_i(t_i)(A)$ exists because, m_i-s measurable
2. fact is that the set $\{t_{j'}\}$ is not necessary measurable, but Harsányi's model is based on the Borel sets of $\mathbb{R}^n$, in where every point is measurable

Harsányi's paper is not a mathematically sophisticated work. The main strength of his work lies in the concept, not in the tools. Therefore, when Harsányi presents the type space as $\mathbb{R}^n$, then he misses some theoretical problems, but he can appropriately introduce the concept of type space.

Definition 2 *The type morphism φ is a measurable function between $< T, m >$ and $< T', m' >$ type spaces, which is generated by $\varphi_i : T_i \to T'_i$ measurable functions, and it has the following properties:*

α, $\varphi_0 = id_S$

β, $m'_i \circ \varphi_i = \widehat{\varphi} \circ m_i \ \forall i \in I \setminus \{0\}$, *where* $\widehat{\varphi}(\mu) = \mu \circ \varphi_i^{-1} \ \mu \in \Delta(T)$

γ, φ_i *is injective (one-to-one)* $\forall i \in M$

φ is type isomorphism, if φ is isomorphism.

The expressions morphism and isomorphism belong to the belief. The point β means that, φ morphism generates $\widehat{\varphi}$ which is a function between sets of probability measures, and $\widehat{\varphi}$ changes the measures in a commutative way.

$$\begin{array}{ccc} T_i & \underset{\longrightarrow}{m_i} & \Delta(T) \\ \varphi_i \downarrow & & \widehat{\varphi} \downarrow \\ T'_i & \underset{\longrightarrow}{m'_i} & \Delta(T') \end{array}$$

The table above is commutative $i \in M$. The commutativity of that table means approximately that, $\widehat{\varphi}$ preserves opinions. The parameter spaces are equivalent up to an isomorphism.

Point γ is about that, types cannot disappear. Therefore, if $\phi :< T, m > \rightarrow$

$< T', m' >$ type morphism exits, then $< T', m' >$ is richer than $< T, m >$ in some sense.

Lemma 3 *$\widehat{\varphi}$ is measurable in Definition 2.*

Proof. Because point β in Definition 2

$$\begin{array}{ccc} T_i & \overrightarrow{m_i} & \Delta(T) \\ \varphi_i \downarrow & & \widehat{\varphi} \downarrow \\ T'_i & \overrightarrow{m'_i} & \Delta(T') \end{array}$$

is commutative $\forall i \in M$. Let $A \in \Delta(T')$ be arbitrary measurable set. $m'_i \circ \varphi_i$ is measurable, so $(m'_i \circ \varphi_i)^{-1}(A) \in T_i$. Because $m'_i \circ \varphi_i = \widehat{\varphi} \circ m_i$, $(\widehat{\varphi} \circ m_i)^{-1}(A) = (m'_i \circ \varphi_i)^{-1}(A) \in T_i$. However, if $\widehat{\varphi}^{-1}(A) \notin \Delta(T)$, then m_i is measurable for $\sigma(\Delta(T) \cup \widehat{\varphi}^{-1}(A))$, therefore $\widehat{\varphi}^{-1}(A)$ must be in $\Delta(T)$. But $A \in \Delta(T')$ is arbitrary measurable, so $\widehat{\varphi}$ is measurable. ■

The main property of $(\varphi, \widehat{\varphi})$ is the keeping of beliefs, but the measurability is also important, we can not leave it out.

The measurability of $(\varphi, \widehat{\varphi})$ is a "hidden " information. We mean, the measurability of $(\varphi, \widehat{\varphi})$ assures that, the events are taken into account in the model.

We mentioned earlier, that Heifetz & Samet[10] declares the measurable structure exactly. In that case, the measurability of $(\varphi, \widehat{\varphi})$ is a very strong, and important assumption.

Definition 4 *$< T*, m* >$ is universal type space, if for any $< T, m >$ type space $\exists \varphi$ type morphism from $< T, m >$ to $< T*, m* >$.*

The universal type space is a type space into which, every type space can be embedded by some type morphism (the measurable structure is fixed).

Lemma 5 *$< T*, m* >$ universal type space is unique up to an isomorphism.*

Proof. Indirect: Let $< T^1, m^1 >$, $< T^2, m^2 >$ are two universal type spaces, then $\exists \varphi_1$ type morphism from $< T^1, m^1 >$ to $< T^2, m^2 >$, and $\exists \varphi_2$ type morphism from $< T^2, m^2 >$ to $< T^1, m^1 >$. Therefore $\exists \varphi$ type morphism (because the definition of type morphism) between $< T^1, m^1 >$ and $< T^2, m^2 >$, so $< T^1, m^1 >$ and $< T^2, m^2 >$ are the same up to an isomorphism. ■

The universal type space is a very strongly universal type space. If we look the the type morphisms as binary relations on the space of type spaces based on S parameter space, then this relation is a partial order, and the universal type space is the greatest element of the space of type spaces based on S parameter space.

The universal type space could be defined as the maximal element of the space of type spaces based on S parameter space. However, in this case the uniqueness of universal type space is not a simple question.

0.2 Hierarchy of beliefs

The main problem in modelling an incomplete information situation, is the problem of modelling the hierarchies of beliefs. When Harsányi introduced the concept of type space, he took an object that summarizes the hierarchies of beliefs, instead of building a space from the hierarchies of beliefs. Therefore, we can consider the concept of type space as a set of hierarchies of beliefs. In this case, we must look into the question, that what is the connection between the concept of type space and hierarchies of beliefs. For this reason let us see the following definition.

Definition 6 (Hierarchy of beliefs) *Let $(S, \mathcal{M})$ be measurable parameter space, then the set of first order beliefs is the set of the probability measures on $(S, \mathcal{M})$, denote this set by $\Delta(S, \mathcal{M})$. The set of second order beliefs is $\Delta(S \times \Delta(S, \mathcal{M}))$[1], the set of third order beliefs is $\Delta(S \times \Delta(S, \mathcal{M}) \times \Delta(S \times \Delta(S, \mathcal{M})))$ etc.*

The set of hierarchies of beliefs is a set of probability measures on the set of probability measures on the set of probability measures on ... the set of probability measures on $(S, \mathcal{M})$. Every player has beliefs about other players's beliefs.

Denote M the set of players, then:

$T_0 = S$ *parameter space,*

$T_1 = T_0 \times \Delta[(T_0)]^M,$

.

.

.

$T_n = T_{n-1} \times [\Delta(T_{n-1})]^M = S \times_{i=0}^{n-1} [\Delta(T_i)]^M$

.

.

.

$T_\infty = \times_{n=0}^{\infty} T_n$

[1] The measurable structure of these spaces is not defined, but fixed.

The components belong to a fixed player i of an arbitrary point in T_∞ is called as an hierarchy of beliefs of player i.

A point in T_0 is a possible value of the parameters. A point in T_1 is a parameter value, and the first order beliefs of the players. A point in T_2 is a parameter value, and the first order beliefs of the players, and the second order beliefs of the players (so the beliefs about the other players's first order beliefs and the parameter value), etc.

We are interested in T_∞, because a point in that describes fully the hierarchy of beliefs for every player. A point in T_∞ is called the state of world. This name comes from the fact, that it describes the state of nature, and the state of beliefs. The latest is called as the state of mind.

Remark 7 *Let $t \in T_\infty$ be arbitrary, the components of this point are $(s, \delta_1^1, \delta_1^2, \ldots\ldots, \delta_2^1, \delta_2^2 \ldots.)$, where δ_j^i denotes player i j-th order belief. Therefore, every point in T_∞ describes fully a state, which contains a hierarchy of beliefs of the players.*

T_∞ is a product space, whose points define the beliefs of the players and a possible parameter value. Therefore, every point describes a possible state of wold, the hole space T_∞ describes all possible state of world. Our aim is to construct this product space.

We defined the space of hierarchical beliefs by a recursion. We got the first order beliefs from the set of probability measures on the parameter space, and we got the second order beliefs from the set of probability measures on the space of first order beliefs etc.

This kind of recursive definition of beliefs is not the only way defining the hierarchies of beliefs. It is possible to define hierarchy of beliefs without measurability, or without measures. Examples for the other ways of defining of hierarchies of beliefs: Brandenburger[4], Heifetz[9], and Epstein&Wang[6].

We recommend some logic for the players' beliefs. We do accept only hierarchies of beliefs, which are generated by logical thinking. We define this property by the following definition:

Definition 8 *The coherent hierarchy of beliefs is an hierarchy of beliefs, which answers the following two conditions:*

- $marg_{T_{n-2}} \delta_n^i = \delta_{n-1}^i$
- $marg_{[\Delta(T_{n-1})]^i} \delta_n^i = \delta_{\delta_{n-1}^i}^i$,

$\forall n \geq 2$, $\forall i \in I$, where $\delta_n^i \in \Delta(T_{n-1})^i$, and $marg_{T_{n-2}} \delta_n^i$ denotes, the restriction of δ_n on T_{n-1}, and $\delta_{\delta_{n-1}^i}^i$ is Dirac measure concentrated on the point δ_{n-1}^i.

The first condition is about that the opinion about something must not change in the hierarchy. This is a very natural assumption.

The second condition recommend that every player knows own type, this is Harsányi's condition.

If we recommend that, the players must be logical, then we are interested in only special points in T_∞. Therefore, we are interested in a subspaces of T_∞. In the following the coherent hierarchies of beliefs in T_∞ is denoted by T_∞^c.

Axiom 9 *The logic of players is common knowledge, so we can concentrate on T_∞^c, on the subspace of T_∞.*

To sum up, we want to build a product space, which contains all players' all coherent hierarchies of beliefs, and their combinations.

0.3 The features of type space

The connection between the idea of type space and the idea of hierarchies of beliefs is the following:

Let $t \in T$ be an arbitrary point in $< T, m >$ type space, then t determines a coherent hierarchy of beliefs for every player, so we can imagine a state of world in place of t. Therefore every type space $< T, m >$ can be considered as set of coherent hierarchies of beliefs.

The question is that: can the set of every coherent hierarchies of beliefs be considered as a type space? In other words, can we build a type space from T_∞^c, from the set of all coherent hierarchies of beliefs?

By now we can compose the problem exactly if we want to build a type space from the coherent hierarchies of beliefs:

Under what conditions is T_∞^c a type space? In other words, if we have a product space, and we have probability measures on every sub-product space, which consists finite many member of the original spaces, and these measures are coherent, then does such a measure on the product space exist, which restricted on a arbitrary sub-product space which consist finite many member of original spaces, is the original measures on that sub-product space? This problem comes in other fields of mathematics as well.

Let us see a sequence of random variables. We know every finite joint-distribution of these random variables. Does such a joint-distribution of these random variables exist, marginals of which are the finite joint-distribution of the original random variables? In other words, is there a stochastic process, which's members are the random variables above, and the whose finite dimension marginal distributions are the prefixed joint-distributions? Kolmogorov [13] answered this question positively.

To sum up we need a generalized Kolmogorov's [13] Existence(Extension) Theorem for the building of a type space from the coherent hierarchies of beliefs.

The general forms of Kolmogorov's Existence(Extension) Theorem are the theorem of existence of measure inverze limits. Therefore, we have to look into the mathematics of inverz systems, and the problem of existence of inverz limits, to research the question of building type space form hierarchies of beliefs.

Definition 10 *A $< T, m >$ type space is sound, if every type in $< T, m >$ responds to a coherent hierarchy of beliefs for every player.*

The expression "sound" comes from mathematical logic. In this topic, we mean a type space be sound, if the language(type space) responds to the reality(hierarchies of beliefs). In other words, if something is type, then it determines the hierarchies of beliefs(see section: An example).

Definition 11 *A $< T, m >$ types space is complete, if every coherent hierarchy of beliefs is type.*

Using the definitions above, we would like to build a model which is complete and sound. From the definition of type space (see 1) our type is sound. Therefore, the problem is the completeness of the type space.

In non-Bayesian model, also the completeness is the main problem. See Brandenburger[4], Meier[14].

0.4 An example

So far we discussed the need of examining of the hierarchies of beliefs in the case of incomplete information situations. We mentioned as direct corollary of the definition of type space that, every type determines a hierarchy of beliefs for every payer. In this section we present an example how can a type contain the hierarchy of beliefs for every player.

This example comes from Aumann & Heifetz[2], but we modified that a little bit.

Let Anna and Robert are two players. Anna has three possible type:

$Q = \{AA, AB, AC\}$.

Robert has also three type:

$Q = \{RA, RB, RC\}$.

Let $PAnna$ denote Anna's opinion about what the state of world is, and let $PRobert$ denote the Robert's opinion about the same as at Anna.

The following tables summarize the above mentioned beliefs.

Q=	N=	RA	RB	RC
	AA	1/2	1/2	0
	AB	1/4	1/4	1/2
	AC	1/4	1/4	1/2

PAnna

Q=	N=	RA	RB	RC
	AA	1	0	0
	AB	0	1/2	1/2
	AC	0	1/2	1/2

PRobert

These tables contain conditional probabilities(see Rényi[21]).

For example the third number in the first row of table $PAnna$ means that, if Anna's type is AA, then she believes that the probability of that, Robert's type is RC, is 0.

An other example, the second number in the first row of table $PRobert$ means that, Robert beliefs that, if Anna's type is AA, then she believes that, the probability of Robert's type is RB, is 0.

Let the state of the world be $\{AA, RB\}$.

Every player knows own type, so Anna knows that, her type is AA. Anna's first order belief about Robert's type (which is included in the first row of table $PAnna$):

$P_1^A(RA) = P_1^A(RB) = 1/2$, $P_1^A(RC) = 0$.

Anna's second order belief about what Robert believes about her type, therefore the belief about what is Robert's belief about Anna's type:

$P_2^A(AA) = P_1^A(RA)P_1^R(AA) + P_1^A(RB)P_1^R(AA) + P_1^A(RC)P_1^R(AA) = 1/2$,

$P_2^A(AB) = P_1^A(RA)P_1^R(AB) + P_1^A(RB)P_1^R(AB) + P_1^A(RC)P_1^R(AB) = 1/4$,

$P_2^A(AC) = P_1^A(RA)P_1^R(AC) + P_1^A(RB)P_1^R(AC) + P_1^A(RC)P_1^R(AC) = 1/4$.

Anna's third order belief about what Robert believes about what Anna believes about Robert's type:

$P_3^A(RA) = P_2^A(AA)1/2 + P_2^A(AB)1/4 + P_2^A(AC)1/4 = 3/8$,

$P_3^A(RB) = P_2^A(AA)1/2 + P_2^A(AB)P_1^A 1/4 + P_2^A(AC)P_1^A 1/4 = 3/8$,

$P_3^A(RC) = P_2^A(AA)0 + P_2^A(AC)1/2 + P_2^A(AC)1/2 = 1/4.$
e.t.c.
Therefore the types determine the hierarchies of beliefs.
Remark:

1. The prior opinions of the two players do not coincide, so there is no only one probability distribution on the set of the states of the world, which's marginal distributions are the probability distributions of Anna and Robert[2].

Let us connect the idea of probability transition function and the example above to each other.

Let the state of the world be $\{AC, RB\}$.
Let $\mathbf{A} = \{\{AA, RA\}, \{AB, RB\}, \{AC, RC\}\}$ be an event.
Let us see Robert's hierarchy of beliefs about A.

$$P^1(\mathbf{A}) = \frac{P(\mathbf{A} \cap ((\prod_i \Theta_{i\in(N\setminus\{j\})}) \times \{RB\}))}{P((\prod_i \Theta_{i\in(N\setminus\{j\})}) \times \{RB\})} =$$

$$= \frac{P(\{AB, RB\})}{P(\{AA, RB\}, \{AB, RB\}, \{AC, RB\})} = \frac{1}{2}$$

This is Robert's first order belief about event $\mathbf{A}$. Therefore, Robert beliefs that the probability of event $\mathbf{A}$ is $\frac{1}{2}$.

$$P^2(\mathbf{A}) = \int_{\{AA,AB,AC\}} P^1(\mathbf{A}) dP^1 = \frac{1}{2} * 0 + \frac{1}{4} * \frac{1}{2} + \frac{1}{4} * \frac{1}{2} = \frac{1}{4}$$

This is Robert's second order belief about $\mathbf{A}$. So, Robert beliefs that, Anna beliefs that, the probability of $\mathbf{A}$ is $\frac{1}{4}$.

$$P^3(\mathbf{A}) = \int_{\{RA,RB,RC\}} P^2(\mathbf{A}) dP^1 = \frac{1}{2} * 0 + \frac{1}{4} * \frac{1}{2} + \frac{1}{4} * \frac{1}{2} = \frac{1}{4}$$

This is Robert's third order belief, therefore Robert beliefs that, Anna beliefs that, Robert beliefs that, the probability of $\mathbf{A}$ is $\frac{1}{4}$.

e.t.c.

[2] The Harsányi's Doctrine is not valid in this case.

Part II

Complete universal type space

(This part is based on [18])

0.5 Introduction

Modelling rationally behaving actors in a multi-person decision problem involves the analysis of players' information about all aspects, which have influence on the decision making. During the decision making process the rational players use all available information, so its analysis is necessary for modelling the actors' behavior. Aumann[1] introduced a formal definition for the idea of common knowledge. The distinction between common knowledge and knowledge leads to, among others, the research of hierarchies of beliefs.

The problem of incomplete information is related to the problem of hierarchical beliefs. In an incomplete information situation, some parameters of the model are not common knowledge. If something is not common knowledge, we must deal with hierarchies of beliefs, that is, we have to consider arguments like what every agent believes about what every agent believes about what every agent believes and so on, which makes the model very complicated.

Harsányi[7] assumed a ready-made type space, which includes all possible types of players, and hence, their knowledges, beliefs as well. Simultaneously he assumed a probability measure, defined on the product of the parameter space and the type spaces. This probability measure induces hierarchies of beliefs, so we can consider this probability measure as a "summary of hierarchies of beliefs". However, the opposite question remains: how can we build a type space from hierarchies of beliefs?

A very important step in this direction was made by Mertens & Zamir[17] who built a universal type space based on a compact parameter space. Later, Heifetz[8] relaxed the compactness, but other topological assumptions were retained. Almost parallel Brandenburger & Dekel[5] proved the existence of a universal type space in presence of a complete, separable metric (Polish) parameter space. More recently, Mertens & Sorin & Zamir[16] gave an elegant proof for the existence of a universal type space in cases of parameter spaces with various structures. Ultimately, all of the above proofs are based on the Kolmogorov's Existence Theorem and its generalizations.

In 1998 Heifetz & Samet[10] proved the existence of a universal type space, which possesses a purely measurable structure. In contrast to our paper, the authors make a distinction between universal type space, and space of coherent hierarchies of beliefs. They also gave an illuminating discussion on the problem of type spaces, beliefs spaces. The same authors gave a counterexample showing that in general circumstances, coherent beliefs are not always types (see Heifetz & Samet[11]).

Quite recently, Meier[15] investigated the problem of the existence of a universal type spaces, his model is based on finitely additive measures. By regarding the opinions as finitely additive measures, the problem of existence of σ-additive measures on type spaces can be eliminated. On the other hand, the author discusses how "rich" the structure of a universal type space can be. This work brings to the surface that, the problem of existence of σ-additive measures on type spaces is not only the problem of σ-additivity.

There are some works, which do not connect to our paper directly, but which include some important issue in the research of beliefs, and type spaces. The first is the work of Epstein & Wang[6], in this work, the authors build type spaces, where the opinions are not beliefs, but preferences. The second one is the work of Battigalli & Siniscalchi[3]. Although, this work contributes to the understanding of the hierarchies of beliefs, but as we see, the main contribution of this paper is in the better understanding of rationality, and common certainty of rationality.

Mertens & Zamir[17], Heifetz[8], Brandenburger & Dekel[5], and Mertens & Sorin & Zamir[16] use the concept of projective limit for proving the existence of a universal type space. In all four papers the structure of beliefs is inherited from the topology of lower ranked beliefs spaces or the parameter space, moreover beliefs are modelled by compact regular probability measures.

Earlier works on the subject are of two types. The first group contains the papers [7] and [10], where a type space is not constructed but is taken to be ready-made. The papers [17], [8], and [5] belong to the other group, where a universal type space is constructed by coherent hierarchies of beliefs, so these works are stronger than existence theorems in some sense.

Our main goal is to build a universal type space, that is apparently "purely measurable", and in which every coherent hierarchy of beliefs is a type. The structure on the beliefs is naturally generated by the Baire sets of the pointwise convergence topology. For metric spaces Baire sets and Borel sets coincide. However, in non-metrizable cases (for instance when the cardinality of the players is greater than countable), our approach results in a weaker then Borel structure, but this structure allows the players to distinguish between any pair of beliefs (i.e. regular probability measures) yet.

An other new idea in this paper is that we cut the parameter space off the beliefs' space. This truncated space has a sufficiently good topological structure (i.e. a projective system of completely regular topological spaces), so the measure projective limit exists. After this, we re-fit the parameter space to the measure projective limit, and we construct the universal type space. It is clear that the existence of a measure projective limit crucially depends on topological assumptions. However, if we remove finitely many elements of the projective system of measure spaces, it does not influence the existence of the measure projective limit.

In the next section we build up our model. In section 3, we prove the main result of our paper, finally, in section 4 an illustrative example is provided.

0.6 The Model

If something is common knowledge, then everybody knows that, everybody knows that everybody knows that, and so on. So, common knowledge is more than knowledge, it is some kind of knowledge that is the strongest knowledge in the situation. If something is common knowledge, then somebody's knowledge of this fact does not influence the situation. If something is not common knowledge, then the rational players must concern with the beliefs of other players, beliefs about beliefs of other players and so on.

Therefore, if we have a parameter space S, and this includes all parameters of the game, then we are about to construct a space generated by S, that includes all reasonable beliefs, beliefs about beliefs and so on. This space is called the beliefs' space.

Definition 12 *The parameter space is a measurable space $(S, \mathcal{A}_S)$, where $\mathcal{A}_S$ is a σ-algebra defined on S.*

This space S contains all parameters, which have impact on the game. We assume only measurability on this space. The players think in ideas like probability, events, thus a purely measure theoretic model seems to be adequate. However, as is well known from Heifetz & Samet[11], a purely measure theoretic universal type space does not exist in our context.

Definition 13 *Let $\Delta(S, \mathcal{A}_S)$ denote the space of the probability measures on $(S, \mathcal{A}_S)$, and put $d(\mu_1, \mu_2) = \sup_{A \in \mathcal{A}_S} |\mu_1(A) - \mu_2(A)|$. Then $(\Delta(S, \mathcal{A}_S), d)$ or briefly (Δ, d) is a metric space. The collection of all Baire sets of (Δ, d) is denoted by $B(\Delta, d)$*

We use metric spaces to model the structure of possible opinions of the players (beliefs). This metrics is very natural, we mean, this does not recommend any special assumption. Important to see that, in metric spaces the Borel and Baire sets coincide. The difference between these two ideas will be important later. An other important thing is that, all measures on Baire sets are regular (not compact regular). We use this feature very extensively.

If it will not lead to misunderstanding, instead of $\Delta(S, \mathcal{A}_S)$ we use the shorter notation $\Delta(S)$ or simply Δ. Analogously, $B(\Delta(S), d)$ is replaced by $B(\Delta(S))$.

Definition 14 *Let us define a sequence of spaces recursively, where M stands for set of the players:*
$T_0 = (S, \mathcal{A}_S)$
$T_1 = T_0 \otimes (\Delta(T_0)^M, B(\Delta(T_0)^M))$
$T_2 = T_1 \otimes (\Delta(T_1)^M, B(\Delta(T_1)^M)) = T_0 \otimes (\Delta(T_0)^M, B(\Delta(T_0)^M)) \otimes$
$\otimes (\Delta(T_1)^M, B(\Delta(T_1)^M))$
$\vdots$
$T_n = T_{n-1} \otimes (\Delta(T_{n-1})^M, B(\Delta(T_{n-1})^M)) = T_0 \otimes \otimes_{j=0}^{n-1} (\Delta(T_j)^M, B(\Delta(T_j)^M))$
$\vdots$
where $\otimes$ denotes the product measurable structure.

A point in T_0 is called parameter value, simply a parameter of the game. A point in T_1 is a combination of a parameter value and a 1-st order beliefs (the players' beliefs on the parameter values), and so on.

Consider the infinite product $T_\infty = S \times \times_{j=0}^{\infty} \Delta(T_j)^M$. If $t \in T_\infty$ then it has the form $t = (s, \mu_1^1, \mu_1^2, \ldots, \mu_2^1, \mu_2^2, \ldots)$, where μ_j^i means the "i" player's j-th order belief. So, every element of T_∞ describes an *hierarchy of beliefs* i.e. $(\mu_1^i, \mu_i^2, \ldots)$ for all players and a possible parameter, therefore it is a possible state of the world. We call beliefs' space the spaces of type of T_∞.

Remark 15 *The elements of $(s, \mu_1^1, \mu_1^2, \ldots, \mu_2^1, \mu_2^2, \ldots)$ can be regarded as members of a generalized sequence, where the ordering is: the least element is s, and $\mu_j^i < \mu_k^l$ iff $j < k$.*

Definition 16 *Fix an $i \in M$. A hierarchy of beliefs $(\mu_1^i, \mu_i^2, \ldots)$ is coherent if $n \geq 2$*

- $marg_{T_{n-2}} \mu_n^i = \mu_{n-1}^i$
- $marg_{[\Delta(T_{n-2})]^i} \mu_n^i = \mu^i_{\mu_{n-1}^i}$,

where μ_n^i is taken from $[\Delta(T_{n-1})]^i$ (which is the ith copy of $\Delta(T_{n-1})$), furthermore, $marg_{T_n}$ denotes the marginal distribution on T_n, and $\mu^i_{\mu_{n-1}^i}$ stands the Dirac measure concentrated on the "point" μ_{n-1}^i.

The first condition declares the fact that the beliefs over some aspects of the game do not change in the hierarchy. The second condition states that the players know exactly their own beliefs (cf. Harsányi[7]). These two conditions describe the "logic" of the players, we assume this logic to be *common knowledge*.

Remark 17 *The measurable structure on $[\Delta(T_{n-1})]^i$ $\forall i, n$ is defined by the Baire sets, which coincide with Borel sets in the case of metric spaces, hence any singleton is measurable.*

Consider an element $(s, \mu_1^1, \mu_1^2, \ldots, \mu_2^1, \mu_2^2, \ldots)$ from T_∞ such that the hierarchies of beliefs $(\mu_1^i, \mu_i^2, \ldots)$ are coherent for every $i \in M$. The set all those elements is denoted by T_∞^c and called the *coherent subspace* of T_∞. (The superscript ^{c} will be used in the same context throughout the paper.)

Definition 18 *Fix an $i \in M$ and set*

$$T^i = (\times_{k=0}^{\infty}[\Delta(T_k^c)]^i)^c.$$

T^i is called the type space for player i. A point in T^i is a possible type of player i.

The type space of player i consists of all coherent hierarchies of beliefs. In particular, if $t \in T^i$, then $t = (\mu_1^i, \mu_2^i, \mu_3^i, \ldots)$, and t is coherent.

Corollary 19 *T^i is metrizable since it is a subspace of a countable product of metric spaces. This metric is given by $d_p(\mu, \mu') = \sum_n \frac{1}{2^n} d(\mu_n, \mu_n')$ where $\mu, \mu' \in T^i$, and $\mu_n, \mu_n' \in [\Delta(T_{n-1}^c)]^i$ (d is given in Definition 13).*

Remark 20 *If the cardinality of M is more than countable, then the Baire structure of $\Delta(T_n)^M$ is weaker than the Borel structure. On the other hand, this structure (Baire sets) coincides with the product measurable structure $\otimes_{m \in M} B(\Delta(T_n))^m$. It is worth noting that our construction very similar to a purely measurable type space, because no topology is used to make a stronger measurable structure for product spaces.*

Corollary 21 *For a given $i \in M$,*

$$(((T_n^c, B(T_n^c), \mu_{n+1}^i), pr_{mn})_{m<n}) \tag{1}$$

is a projective system of measure spaces, where pr_{mn} is the coordinate projection from T_n^c to T_m^c, and $(\mu_1^i, \ldots, \mu_{n+1}^i, \ldots) \in T^i$.

Proof. For the definition of projective systems we refer to M. M. Rao[19] p. 117.

- $pr_{mn} = pr_{mk} \circ pr_{kn}$ $\forall m < k < n$, by the definition of coordinate projections.

- $pr_{nn} = id_{T_n^c}$ $\forall n$ follows from the definition of coordinate projections.
- pr_{mn} is measurable $\forall m < n$, because of the definition of product measurable spaces.
- $\mu^i_{n+1}(pr^{-1}_{mn}(A)) = \mu^i_{m+1}(A)$ $\forall m < n$ and $\forall A \in B(T^c_m)$ is a consequence of the coherency of beliefs.

■

The above Corollary establishes the connection between the idea of projective system and beliefs' space. The main question is that, whether or not a proper projective limit of the above defined system exists.

0.7 The main result

Before we take the next step, we clarify the role of Baire sets in our model. In Mertens & Zamir[17], the opinions were modelled by regular probability measures on Borel sets of a compact space. However, if there is a compact regular probability measure on the Baire sets of a topological space, then it can uniquely be extended to the Borel sets as a compact regular measure. So, there is one-to-one correspondence between compact regular probability measures on Baire sets and on Borel sets. In conclusion, regular probability measures are compact regular measures on a compact topological space hence, there is a bijection between opinions in Mertens & Zamir[17] and opinions in our model.

At Brandenburger & Dekel[5], the opinions are compact regular probability measures on the Borel sets of a Polish (separable, complete, metric) space. As is well known, Borel sets and Baire sets coincide in the case of metric spaces, and all regular probability measures on Borel sets of a Polish space are compact regular. Therefore, the opinions in Brandenburger & Dekel[5] and the opinions in our model are related the same way as Mertens & Zamir[17] and our model, respectively.

In Heifetz[8], and Mertens & Sorin & Zamir[16] the opinions are compact regular probability measures on different kinds of spaces. According to our previous discussion, all compact regular probability measures on Borel sets are regular probability measures on Baire sets, but there may be regular probability measures on Baire sets, which are not necessarily compact regular. In an informal way we may say that the set of opinions in our model is, in a certain context broader than that in Heifetz[8], or Mertens & Sorin & Zamir[16].

As we have seen, the collection of Baire sets is essentially smaller than the collection of Borel sets if the cardinality of M is more than countable. In this case, a point is not measurable in T_n^c $n > 0$ space. We can interpret this phenomenon as the players' inability of knowing what the others' beliefs exactly are. The players can concentrate on countably many players' beliefs only. We often meet the following argument: "I don't know who, but I'm sure somebody believes that ....!". In the language of probability theory: "Mr. X believes that ...." is the outcome, "somebody believes that ..." is the event. In this example, we mean that the players cannot make an argument like "Mr. i believes that ..., Mr. j believes that ..., " for all players, but our players can argue that "Mr. 1 believes that ..., Mr. 2 believes that ..., ..., somebody believes that ...". This feature of our model is typically a pure measure theoretic feature.

In the next proposition we show that, the central question in our model is the σ-additivity of μ^i in the projective limit (definition is given in the Appendix).

Proposition 22 *Let $i \in M$ be fixed. The projective limit $(T, \mathcal{A}_T, \mu^i) = \varprojlim(((T_n^c, B(T_n^c), \mu_{n+1}^i), pr_{mn})_{m<n})$ of the projective system (1) exists. Further, $T = T_\infty^c$, $\mathcal{A}_T$ is a field and μ^i is an additive set function on $\mathcal{A}_T$.*

Proof. The proof essentially follows the ideas of Rao[19] p. 118.

Since every pr_{mn} is a coordinate projection we deduce that T is not empty and $T = T_\infty^c$. Pick an $A \in \mathcal{A}_T$, then there is an index n, and $B \in B(T_n^c)$, $A = p_n^{-1}(B)$. Moreover, if $B \in B(T_n^c)$, then also $\complement B \in B(T_n^c)$, so $\complement A = p_n^{-1}(\complement B) \in \mathcal{A}_T$. If $A_1, ..., A_m \in \mathcal{A}_T$, then for every $1 \leq j \leq m$ there exists an index n_j such that $A_j = p_{n_j}^{-1}(B_j)$. Let k be the maximal element of $\{n_1, ..., n_m\}$, and let $K_j = p_{n_j k}^{-1}(B_j)$, we know $K_j \in B(T_k^c)$ $\forall j$, so $\cup_j K_j \in B(T_k^c)$. Making use of $A_j = p_k^{-1}(K_j)$ we obtain $\cup_j A_j \in \mathcal{A}_T$. Thus, $\mathcal{A}_T$ is an algebra.

Since every p_{nm} is a coordinate projection, we conclude that p_n is onto. This implies that p_n^{-1} is one-to-one. Therefore, the set function μ^i defined by the equality $\mu^i \circ p_n^{-1} = \mu_n^i$ is uniquely defined.

Take $A_1, ..., A_m \in \mathcal{A}_T$ disjoint sets, then $\cup_j A_j \in \mathcal{A}_T$. For each $1 \leq j \leq m$ select B_j and K_j as above. We know K_js are disjoint, and therefore, $\sum_j \mu_{k+1}^i(K_j) = \mu_{k+1}^i(\cup_j K_j)$, and $\sum_j \mu^i(A_j) = \sum_j \mu^i(p_k^{-1}(K_j)) = \sum_j \mu_{k+1}^i(K_j) = \mu_{k+1}^i(\cup_j K_j) = \mu^i(\cup_j p_k^{-1}(K_j))$, hence μ^i is finitely additive $\forall i$. ■

Proposition 1 concentrates on the additivity of μ^i. Generally, the problem of existence of a proper measure projective limit is twofold: the first problem is the "richness" of the projective limit set (Heifetz & Samet[11] addresses this problem), the second is the problem of σ-additivity of μ^i. We use the idea of coordinate projections in the projective system, which ensures that the projective limit set is "rich" enough. The second problem demands regularity (but not compact regularity).

The previous proposition was about, that the sequence of opinions of every player can be extended to an additive set function on the beliefs space. Therefore, we use the projective limits for every player separately.

In the next proposition, we take preliminary steps for proving our main result.

Proposition 23 *Let us define the following sequence of truncated spaces (c.f. Definition 14):*

$C_0 = (\Delta(T_0)^M, B(\Delta(T_0)^M)$

$C_1 = C_0 \otimes (\Delta(T_1)^M, B(\Delta(T_1)^M)) = (\Delta(T_0)^M, B(\Delta(T_0)^M)) \otimes$
$\otimes (\Delta(T_1)^M, B(\Delta(T_1)^M))$

$\vdots$

$C_n = C_{n-1} \otimes (\Delta(T_{n-1})^M, B(\Delta(T_{n-1})^M)) = \otimes_{j=0}^{n-1} (\Delta(T_j)^M, B(\Delta(T_i)^M))$

$\vdots$

Consider the projective limit

$$(C, \mathcal{A}_C, \nu^i) = \varprojlim(((C_n^c, B(C_n^c), \nu_n^i), pr_{mn})_{m<n}),$$

where $\nu_n^i = marg_{C_n^c} \mu_{n+2}^i$. *Then* ν^i *is* σ*-additive for every* $i \in M$.

Proof. The proof based on M.M. Rao[20] p. 357-358.

Let $i \in M$ be fixed and arbitrary.

The preceding proposition tells us that $\mathcal{A}_C$ is a field, and ν^i is an additive set function on it for each i. Furthermore, $\mathcal{A}_C \subset B(C)$ because all p_n are continuous with respect to the product topology on C (which is the weakest topology for which all p_n are continuous).

Since the topological product of completely regular spaces is completely regular, it follows that C enjoys complete regularity. It is not hard to verify that ν^i is inner regular set function.

The completely regular topological spaces are characterized by the fact, that they can be embedded into a compact space as a dense set (Čech-Stone compactification). Let I be the one-to-one function, which embeds C into a K compact space, and let $\nu_K^i = \nu^i \circ I^{-1}$ be a set function on $\mathcal{A}_K$, the subsets of K, which are defined by $\mathcal{A}_K = \{X \subseteq K | I^{-1}(X) \in \mathcal{A}_C\}$. The direct corollary of this definition that, ν_K^i is inner regular, so (inner) compact regular as well.

As is well known, if an additive set function is compact regular, then it is σ-additive as well. Hence, ν_K^i is σ-additive. On the other hand, C contains the support of ν_K^i, and ν^i is the restriction of ν_K^i on C, hence ν^i is σ-additive as well.

Let $\cup_j A_j$, $A_j \in \mathcal{A}_C$ $\forall j$ be a sequence of disjoint sets, then there are $B_j \in \mathcal{A}_K$, and $A_j = I^{-1}(B_j)$ $\forall j$. Furthermore, B_js are quasi-disjoint sets, we mean $\nu^i_K(B_j \cap B_k) = 0$ $\forall j,k$. If μ^i is not σ-additive, then there are a sequence of disjunct sets (A_j), on which $\sum_j \nu^i(A_j) \neq \nu^i(\cup_j A_j)$. In this case $\sum_j \nu^i_K(B_j) \neq \nu^i_K(\cup_j B_j)$, but this is a contradiction, because we saw earlier that, ν^i_K is σ-additive.

Consequently, ν^i is σ-additive on $\mathcal{A}_C$ $\forall i$. ■

Remark 24 *The role of compact regularity in the proofs of existence theorems of measure projective limit is twofold. First, compact regularity ensures σ-additivity. On the other hand, every compact regular measure can uniquely be extended from Baire sets to Borel sets. This later proves to be very important in the case of stochastic processes (the measurability of the sample function), but it is not relevant in our problem. We do not want to introduce events into our model that cannot be deduced directly by probabilistic logic.*

The next theorem is our main result.

Theorem 25 *T^i is universal type space, so there exists a homeomorphism $f : T^i \to (\Delta(\mathcal{A}_T), \tau_p)$, where $(\Delta(), \tau_p)$ means the pointwise (setwise) topology on $\Delta()$.*

The proof of the theorem is basically divided into two parts.

Definition 26 *Let $g : \Delta(\mathcal{A}_T) \to T^i$ that associates with every measure μ a point $t = (\mu^i_1, \mu^i_2, \ldots, \mu^i_n, \ldots)$ in T^i, where*

$$\mu^i_n = marg_{T_{n-1}} \mu$$

for every integer n.

Lemma 27 *Let $(M, \mathcal{A}_M, \mu_M), (N, \mathcal{A}_N, \mu_N)$ be probability measure spaces, and let μ be an additive set function on $\mathcal{A}_M \otimes \mathcal{A}_N$, and let p_M and p_N denote the coordinate projections. If $\mu \circ p_M^{-1} = \mu_M$ and $\mu \circ p_N^{-1} = \mu_N$, then μ is σ-additive on the field $\mathcal{A}$ generated by the cylinder sets.*

Proof. It is easy verify that every element of $\mathcal{A}$ has the form $\cup_j M_j \times N_j$, where $j < \infty$, $M_j \in \mathcal{A}_M$, $N_j \in \mathcal{A}_N$. It is well known ([12]) that, μ is σ-additive on $\mathcal{A}$ iff for a sequence $A_{n+1} \subseteq A_n$, $\cap_n A_n = \emptyset \Longrightarrow \lim_{n\to\infty} \mu(A_n) = 0$. For every finite intersection $\cap_n A_n = \cup_j M_j \times N_j$, for a finite set of indices j. Therefore, if the countable intersection $\cap_n A_n = \emptyset$, then the corresponding $M_j \times N_j = \emptyset$. Let us divide the sets $M_j \times N_j$ into two groups. Let the first group contain those products $M_j \times N_j$ where $M_j = \emptyset$, and let the second contain the others. Let us take the union of the members of the first group, it has the form $\emptyset \times \cup_j N_j$. Similarly, the union of the elements of the second group can be expressed as $\cup_j M_j \times \emptyset$. We have $\mu(\emptyset \times \cup_j N_j) = \mu(\cup_j M_j \times \emptyset) = 0$, from the additivity of μ, $\mu(\emptyset \times \cup_j N_j) + \mu(\cup_j M_j \times \emptyset) = \mu(\emptyset)$, which implies $\lim_{n\to\infty} \mu(A_n) = 0$, hence μ is σ-additive on $\mathcal{A}$. ■

Lemma 28 *g is a bijection.*

Proof. First we show that g is injective. If $\mu \in \Delta(\mathcal{A}_T)$ is given, then μ determines its marginals, in other words, it determines a unique point in T^i.

Now we verify that g is onto. Let a point $t \in T^i$ be given. From Proposition 22 and 23 we have that $\mathcal{A}_S \times \mathcal{A}_C \subset \mathcal{A}_T$. Let us define $q_1 : (T, \mathcal{A}_T) \to (S, \mathcal{A}_S)$, and $q_2 : (T, \mathcal{A}_T) \to (C, \mathcal{A}_C)$ as coordinate projections. Define μ on the cylinder sets by the equalities:

$$\mu = \mu_1^i \circ q_1, \quad \text{and} \quad \mu = \nu^i \circ q_2$$

(see Definition 14 and Proposition 23). On the cylinder sets, μ and μ^i coincide (μ^i is taken from the projective limit, see Proposition 22) and μ^i is additive set function, hence we can extend μ to the field generated by the cylinder sets, in the way that, μ and μ^i coincide on this field. From Lemma 27 μ is σ-additive set function on this field, so it can be extended uniquely onto $\mathcal{A}_T$. We prove that $\mu = \mu^i$ on $\mathcal{A}_T$. Indeed, if there were an $A \in \mathcal{A}_T$ with $\mu(A) \neq \mu^i(A)$, then there would exist a k, and $B \in B(T_k^c)$ such that $A = p_k^{-1}(B)$. We know μ_{k+1}^i is σ-additive, hence $\mu = \mu^i$ on T_k^c, which is a contradiction. Thus, g is a bijection. ■

Definition 29 *Set $f = g^{-1}$.*

Lemma 30 *f is a homeomorphism.*

Proof. f is continuous ($t_k \xrightarrow{d_p} t \Longrightarrow f(t_k) \xrightarrow{p} f(t)$): $t_k \xrightarrow{d_p} t$ means $\forall l$, $\forall A_l \in B(T_l^c)$ $t_k^l(A_l) \to t^l(A_l)$, moreover $p_l^{-1}(A_l) \in \mathcal{A}_T$, and $f(t_k) \circ p_l^{-1}(A_l) = t_k^l(A_l)$, hence $f(t_k) \xrightarrow{p} f(t)$ on $\mathcal{A}_T$.

f^{-1} is continuous ($\mu_k \xrightarrow{p} \mu \Longrightarrow f^{-1}(\mu_k) \xrightarrow{d_p} f^{-1}(\mu)$): $\mu_k \xrightarrow{p} \mu$ on $\mathcal{A}_T$, which means the marginals of μ_k converge to μ pointwise, so $f^{-1}(\mu_k) \xrightarrow{d_p} f^{-1}(\mu)$. ■

Proof. of the Theorem

Let f be defined by Definition 29.

From Lemma 28, f is a bijection.

From Lemma 30 f is a homeomorphism. ■

Remark 31 *We proved the homeomorphism for $\mathcal{A}_T$, but not for $\sigma(\mathcal{A}_T)$, because the homeomorphism is not valid in the latter case. Our theorem can be extended to the $\sigma(\mathcal{A}_T)$, if the structure of $\sigma(\mathcal{A}_T)$ is induced by the pointwise convergence topology on $\mathcal{A}_T$.*

Remark 32 *This Theorem shows the importance of pointwise convergence topology. If T is a topological space, then the weak or weak* topology is less then our structure on $\Delta(\sigma(\mathcal{A}_T))$.*

0.8 Conclusion

The main advantage of this model comes from the pointwise convergence topology on beliefs, that is independent of the topology of the original space. This space is a completely regular topological space, so we can use Kolmogorov's Existence Theorem in a general form (Proposition 2, Theorem 1).

Let us see an example for the usage of this model.

Example 33 *Let there be two players, every player has two strategies. This game in normal form is a point in $\mathbb{R}^8$. There are two random variables, which determine the payoffs of the players. Therefore, the parameter space: $S = \mathbb{R}^{8^{\mathbb{R}^2}}$ (the parameters are functions from $\mathbb{R}^2$ to $\mathbb{R}^8$). S is not compact, nor Polish, so Mertens & Zamir's and Brandenburger & Dekel's construction do not work in this case. Let the measurable structure of S be the Borel sets of S. In our model, the opinions are the probability measures on S, but these are not necessarily compact regular, so Heifetz's, Mertens & Sorin & Zamir's models are less general, than ours.*

It seems that, our model performs better, than the previous ones. On the other hand, recently, Simon[22] showed that, there may be problem with the existence of measurable equilibrium of the games with incomplete information. Hence, a model, in which , the beliefs of the players are modelled by probability measures, is not necessarily appropriate for some problems.

We think the existence of measurable equilibrium is out of the scope of our paper, hence we refer to this problem as an open problem in general, so in the case of our model as well.

0.9 Appendix: Definition of measure projective limit

We define the idea of projective limit of measure spaces for completeness. Rao's[19] definition is little bite different from us.

Definition 34 *Let* $(((M_n, \mathcal{M}_n, \mu_n), p_{mn})_{m<n}, I)$ *be a projective system, where* $(M_n, \mathcal{M}_n, \mu_n)$*s are measure spaces,* p_{mn}*s are the measurable projections, and* I *is a directed set. The projective limit of* $(((M_n, \mathcal{M}_n, \mu_n), p_{mn})_{m<n}, I)$ *is* $(M, \mathcal{M}, \mu) = \varprojlim(((M_n, \mathcal{M}_n, \mu_n), p_{mn})_{m<n}, I)$*, where*

- $pr_n : \times_n M_n \to M_n$ *coordinate projection,*
- $M = \{\omega \in \times_n M_n | pr_m(\omega) = p_{mn} \circ pr_n(\omega),\ \forall m < n \in I\}$*,*
- $p_n = pr_n|_M$*,*
- $\mathcal{M} = \cup_n \Sigma_n$*, where* $\Sigma_n = \{p_n^{-1}(A) | A \in \mathcal{M}_n\}$*,*
- μ *is on* $\mathcal{M}$*, defined by the equality* $\mu \circ p_n^{-1} = \mu_n\ \forall n$*, and it is unique.*

The main difference between our and Rao's definition is in the properties of μ. Rao recommends μ to be σ- additive, we do not. Our definition makes the discussion more clear.

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