CORVINUS UNIVERSITY OF BUDAPEST
Doctoral School of Economics, Business and Informatics

THESIS COLLECTION

to the Ph.D. Thesis titled

Some problems in discounted stochastic games

written by

Imre Balog

Supervisors:

Ágoston Kolos Csaba PhD

Pintér Miklós PhD

Budapest, 2025

1 Research background and relevance of the chosen field

Stochastic games (also known as *Markov games*) is a powerful mathematical framework for analysing decision-making in dynamic environments. Since the theory of stochastic games is a broad branch of mathematics, we focus on *discrete-time* stochastic games with a *finite number of players*.

Stochastic games generalise *Markov decision processes*, which can be viewed as stochastic games with a single player. They also generalise *repeated games*, because at each stage players do not necessarily encounter the same underlying game; state transitions can generate different strategic environments.

Different problems motivate different game models, and one of the key distinctions concerns whether a stochastic game unfolds in *discrete time* or in *continuous time*. This thesis concentrates primarily on stochastic games with *a finite number of players* played in *discrete time*. This framework gives particular attention to *discounted stochastic games*.The cases of *total reward* and *long-run average reward* also receive consideration, though with somewhat less emphasis.

The applications of stochastic games are diverse, including the analysis of arms races (Winston, 1978), the study of tax evasion (Raghavan, 2006), and the modelling of conflicts over fishery resources (Levhari and Mirman, 1980). In addition to these examples, many other applications have been examined in the literature, with detailed discussions found in a range of studies and monographs; see, for instance, White (1993); Puterman (1994); Amir (2003); Solan and Vieille (2015); Chen (2019).

In recent years, many studies have focused on discounted stochastic games and considered different discounting methods Wu, Wang and Kong (2021); Wu, Tang and Medina (2022); Yu, Guo and Xia (2022).

This thesis investigates three problems in the theory of discounted stochastic games. For each problem, we formulate a structured hierarchy of research questions. We examine each problem in a separate chapter.

<table>
<tr><th>Chapter</th><th>Game model</th><th>Discounting</th><th>Approach</th></tr>
<tr><td>2</td><td>Finite stochastic games</td><td>Generalised</td><td>Continuous generalized games</td></tr>
<tr><td>3</td><td>Zero-sum stochastic games</td><td>Separable</td><td>Supergames</td></tr>
<tr><td rowspan="2">4</td><td rowspan="2">Finitely additive Markov decision processes</td><td>Ripple</td><td rowspan="2">Superprocesses</td></tr>
<tr><td>Separable</td></tr>
</table>

Table 1: Summary of the problems addressed and the approaches employed in each chapter.

The main challenge in each chapter, as indicated in the first column of Table 1, arises from the combination of two elements: the *Game model* and the *Discounting*, which are listed in the second and third columns of the same table.

The final column of Table 1 presents our approaches to examine each problem. While these methods are explained in detail in their respective chapters, it is important to highlight a key distinction here.

Chapters 3 and 4 utilise the concepts of *supergames* and *superprocesses*. These approaches extend the original stochastic game into a larger game by replacing the state space with a *position space*, which is often constructed by indexing states over time or representing them using the set of histories. This transformation is significant because it makes a strong connection between these games: the new game remains equivalent to the original one regarding our research goals.

In contrast, Chapter 2 takes a different approach by introducing a broader class of games that includes the original game as a special case.

With respect to Table 1, it must be emphasised that, although supergames (or superprocesses) are a well-established tool in stochastic game theory (see, for instance, Filar and Vrieze (1992)), they remain relatively underutilised in the literature.

1.1 Finite stochastic games with generalised discounting

Chapter 2 studies finite stochastic games with generalised discounting. The research builds on the results of Fink (1964), Takahashi (1964), and Sobel (1971), who established that every finite stochastic game admits a stationary Nash equilibrium with exponential discounting.

With this in mind, we consider the following research questions:

(RQ1) *Does a finite stochastic game with generalised discounting admit a Nash equilibrium?*

(RQ2) *Assuming the answer to Research Question (RQ1) is affirmative, is there a particular kind of equilibrium strategy profile?*

1.2 Zero-sum stochastic games with separable discounting

Chapter 3 examines two-person zero-sum stochastic games with *separable discounting*. It is known that every finite zero-sum stochastic game with exponential discounting admits a value and that both players have 0-optimal (i.e. optimal) stationary strategies (Shapley, 1953).

With this in mind, we introduce the following research questions:

(RQ3) *Does a zero-sum stochastic game with separable discounting admit a value?*

(RQ4) *Assuming the answer to Research Question (RQ3) is affirmative, does there exist a 0-optimal strategy for each player?*

(RQ5) *Assuming the answer to Research Question (RQ4) is affirmative, are there* 0*-optimal strategies that are either Markov or stationary for each player?*

For the sake of precision, it should be noted that Research Question (RQ5) addresses the existence of optimal stationary and Markov strategies.

1.3 Discounted Markov decision processes in the finitely additive framework

Chapter 4 investigates *finitely additive Markov decision processes* with *ripple discounting* and *separable discounting*. The starting point of our research is the following result by Sudderth (2016): in any finitely additive Markov decision process with *exponential discounting*, Player 1 always has an optimal stationary strategy.

In the case of separable discounting, we address the following research questions:

(RQ6) *Does a 0-optimal strategy exist for the player?*

(RQ7) *Assuming the answer to Research Question (RQ6) is affirmative, does a Markov or stationary 0-optimal strategy exist for the player?*

In the case of separable discounting, the analysis addresses the following research questions:

(RQ8) *Does a 0-optimal strategy exist for the player?*

(RQ9) *Assuming the answer to Research Question (RQ8) is affirmative, does a Markov or stationary 0-optimal strategy exist for the player?*

2 Methodology

This thesis discusses discounted stochastic games using various mathematical branches, including, among others, *general topology*, *functional analysis*, *descriptive set theory*, and the *theory of charges*.

Furthermore, each chapter of the thesis makes use of specific game-theoretic concepts and tools. Chapter 2 discusses finite stochastic games, drawing on Parthasarathy and Babu (2020, Chapter 2) and Solan (2022, Chapter 4). For continuous generalised games, the analysis relies on the *Glicksberg Fixed Point Theorem* (Glicksberg, 1952). Chapter 3 focuses on zero-sum stochastic games, following the works of Parthasarathy and Babu (2020, Chapter 3), Flesch, Predtetchinski and Sudderth (2020), and Nowak (1984A,B). Chapter 4 presents the model of Sudderth (2016) on finitely additive Markov decision processes.

3 The main results of the thesis

3.1 Finite stochastic games with generalised discounting

The scientific results presented in this thesis were published in Balog and Pintér (2025). These findings are the outcome of the joint and inseparable work of the co-authors. The thesis does not introduce any new results beyond those already presented in that work.

1. We introduce the class of *continuous generalised games* and prove that every such game admits a Nash equilibrium.

2. We show that every finite stochastic game with generalised discounting is a continuous generalised game.

3. In response to Research Question (RQ1), we show that every finite stochastic game with generalised discounting has a Nash equilibrium.

4. In response to Research Question (RQ2), we present an example of a finite stochastic game with generalised discounting that does not admit a stationary Nash equilibrium.

3.2 Zero-sum stochastic games with separable discounting

The thesis introduces the technique of *separable discounting* for zero-sum stochastic games and presents the concept of *supergames* as a tool for their analysis. Building on the work of Nowak (1984A,B), it further identifies three special classes of zero-sum infinite stochastic games with separable discounting: *Borel*, *Suslin*, and *Nowak* stochastic games.

In response to Research Question (RQ3), the thesis establishes the following results:

1. Every zero-sum *finite* stochastic game with separable discounting admits a value.

2. A zero-sum *countable* stochastic game with separable discounting admits a value, *provided* that at least one of the players has only finitely many available actions in every state.

3. Every zero-sum *Borel*, *Suslin* and *Nowak* stochastic game admit a value.

For the sake of simplicity, let us refer to the player who maximises the separable discounted reward as the *Attacker*, and the other player as the *Defender*. The answers to Research Questions (RQ4) and (RQ5) are as follows:

1. In every zero-sum *finite* stochastic game with separable discounting, both players have an *optimal Markov strategy*.

2. The *Defender* always has an optimal Markov strategy in zero-sum *Borel*, *Suslin* and *Nowak* stochastic games.

3. For zero-sum *countable* stochastic games, the *Defender* has an optimal Markov strategy provided that at least one player has only finitely many actions available in each state.

4. In zero-sum *Borel*, *Suslin* and *Nowak* stochastic games, the *Attacker* has an ε-optimal Markov strategy for every $\varepsilon > 0$.

5. For zero-sum *countable* stochastic games, the *Attacker* has an ε-optimal Markov strategy provided that at least one player has only finitely many actions available in each state.

6. We present an example of a zero-sum *finite* stochastic game with separable discounting in which no player has an optimal *stationary* strategy.

The thesis uses the idea of *supergames* to obtain these results. In developing this approach, it draws on the work of Flesch et al. (2020) on positive zero-sum countable stochastic games, and also on the works of Nowak (1984A,B) on zero-sum stochastic games with a general state space.

3.3 Discounted Markov decision processes in the finitely additive framework

The thesis introduces the notions of *separable* and *ripple discounting* in finitely additive Markov decision processes and presents the framework of *superprocesses* to study their solutions.

In response to Research Questions (RQ6) and (RQ7), as well as Research Questions (RQ8) and (RQ9), the thesis establishes the following results:

1. In every finitely additive Markov decision process, the player has an *optimal* strategy under *ripple discounting*.

2. In every finitely additive Markov decision process, the player has an *optimal Markov* strategy under *separable discounting*.

3. We present an example of a finitely additive Markov decision process with *separable discounting* in which the player does *not have an optimal stationary* strategy. A similar counterexample can be constructed for ripple discounting as well.

To establish these results, the thesis applies the approach of *superprocesses*, relying extensively on the work of Sudderth (2016) on negative finitely additive Markov decision processes with total reward.

4 Main references

Amir, R. (2003): *Stochastic Games in Economics and Related Fields: An Overview*. In: Neyman, A. and Sorin, S. (eds.), Stochastic Games and Applications, NATO Science Series C, Mathematical and Physical Sciences, Vol. 570, Kluwer Academic Publishers, Dordrecht, Chapter 30, pp. 455–470.
https://doi.org/10.1007/978-94-010-0189-2_30

Chen, B. S. (2019): *Stochastic Game Strategies and their Applications*. CRC Press
https://doi.org/10.1201/9780429432941

Filar, J. A. and Vrieze, O. J. (1992). *Weighted reward criteria in Competitive Markov Decision Processes*. ZOR Zeitschrift für Operations Research Methods and Models of Operations Research, 36:343–358.
https://doi.org/10.1007/BF01416234

Fink, A. M. (1964): *Equilibrium in a stochastic n-person game*. Journal of Science of the Hiroshima University, 28(1):89–93.
https://doi.org/10.32917/hmj/1206139508

Flesch, J., Predtetchinski, A. and Sudderth, W. (2020): *Positive Zero-Sum Stochastic Games with Countable State and Action Spaces*. Applied Mathematics & Optimization, 82:499–516.

https://doi.org/10.1007/s00245-018-9536-3

Glicksberg, I. L. (1952): *A Further Generalization of the Kakutani Fixed Point Theorem, with Application to Nash Equilibrium Points*. Proceedings of the American Mathematical Society, 3(1):170–174. https://doi.org/10.2307/2032478

Levhari, D. and Mirman, L. J. (1980): *The Great Fish War: An Example Using a Dynamic Cournot-Nash Solution*. The Bell Journal of Economics, 11(1):322–334. https://doi.org/10.2307/3003416

Nowak, A. S. (1984A): *On zero-sum stochastic games with general state space I*. Probability and Mathematical Statistics, 4(1):13–32.

Nowak A. S. (1984B): *On zero-sum stochastic games with general state space II*. Probability and Mathematical Statistics, 4(2):143–152.

Parthasarathy, T. and Babu, S. (2020): *Stochastic Games and Related Concepts*. Springer Singapore https://doi.org/10.1007/978-981-15-6577-9

Puterman, M. L. (1994): *Markov Decision Processes: Discrete Stochastic Dynamic Programming*. John Wiley & Sons, Inc. https://doi.org/10.1002/9780470316887

Raghavan, T. E. S. (2006): *A Stochastic Game Model of Tax Evasion*. In: Haurie, A., Muto, S., Petrosjan, L.A., Raghavan, T.E.S. (eds) Advances in Dynamic Games. Annals of the International Society of Dynamic Games, vol 8. Birkhäuser Boston. Chapter 21, pp. 397–420.
https://doi.org/10.1007/0-8176-4501-2_21

Shapley, L. S. (1953): *Stochastic games*. Proceedings of the National Academy of Sciences of the United States of America, 39(10):1095–1100.
https://doi.org/10.1073/pnas.39.10.1095

Sobel, M. J. (1971): *Noncooperative Stochastic Games*. The Annals of Mathematical Statistics, 42(6): 1930–1935.
https://doi.org/10.1214/aoms/1177693059

Solan, E. (2022): *A Course in Stochastic Game Theory*. Cambridge University Press
https://doi.org/10.1017/9781009029704

Solan E. and Vieille N. (2015): *Stochastic games*. Proceedings of the National Academy of Sciences of the United States of America, 112(45):13743–13746.
https://doi.org/10.1073/pnas.1513508112

Sudderth, W. D. (2016): *Finitely Additive Dynamic Programming*. Mathematics of Operations Research, 41(1):92–108.
https://doi.org/10.1287/moor.2015.0717

Takahashi, M. (1964): *Equilibrium points of stochastic non-cooperative n-person games*. Journal of Science of the Hiroshima University, 28(1):95–99.
https://doi.org/10.32917/hmj/1206139509

White, D. J. (1993): *A Survey of Applications of Markov Decision Processes*. Journal of the Operational Research Society, 44(11):1073–1096.
https://doi.org/10.1057/jors.1993.181

Winston, W. (1978): *A Stochastic Game Model of a Weapons Development Competition*. SIAM Journal on Control and Optimization, 16(3):411–419.
https://doi.org/10.1137/0316026

Wu, X., Wang, Q. and Kong, Y. (2021): *Two-person zero-sum stochastic games with varying discount factors*. AIMS Mathematics, 6(10):11516–11529.
https://doi.org/10.3934/math.2021668

Wu, X., Tang, Y. and Medina, R. (2022): *Numerical Calculation of Optimal Policy Pairs in Zero-sum Stochastic Games with Varying Discount Factors*. Discrete Dynamics in Nature and Society, 2022(1):1–10.
https://doi.org/10.1155/2022/7474566

Yu, Z., Guo, X. and Xia, L. (2022): *Zero-sum semi-Markov games with state-action-dependent discount factors*. Discrete Event Dynamic Systems, 32(4):545–571.

https://doi.org/10.1007/s10626-022-00366-4

5 List of publications

Balog, I. (2023): *Biztosítási díj meghatározása összetett kockázati modellben általánosított exponenciális hasznosságfüggvény esetében*. Szigma, 54(1):55–79.

Balog, I. and Pintér, M. (to appear): *Folytonos általánosított játékok*. Alkalmazott Matematikai Lapok