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RATIONING RULES AND BERTRAND-EDGEWORTH OLIGOPOLIES

Ph.D. Thesis

Main Results

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Contents

I. Introduction

Oligopoly theory originated with Cournot, who introduced a model in that the oligopolists were able to set their outputs as strategies. While Cournot let a neutral auctioneer determine the market clearing price, Bertrand claimed that it is more natural to let the oligopolists set their prices. Although Bertrand's pricing mechanism is more plausible, the equilibrium behaviour of his model is counterintuitive. The Bertrand paradox states that under certain circumstances the unique equilibrium is given by the two firms charging the competitive price in the duopolistic case. Edgeworth resolved the Bertrand paradox by dropping the assumption that the duopolist offering lower price has to cover the whole market. Edgeworth argued that the company offering lower price is not capable or not interested in covering the entire market. The former behaviour may be due to capacity constraints and the latter to a U-shaped marginal cost function. Edgeworth showed that in case of capacity constraints the Bertrand solution is no longer an equilibrium. His work led to the birth of the so-called Bertrand-Edgeworth model in that both price and quantity are decision variables.

Two problems emerge naturally, in the Bertrand-Edgeworth framework. First, the profit functions of the firms cannot be specified only in the knowledge of the aggregate demand curve. This is because the aggregate demand curve does not itself provide sufficient information to enable the demand of a company offering a higher price to be determined. We have to investigate the rationing problem in order to determine the profit function. This is the topic of chapter 4 of my Ph.D. thesis. Second, we have to investigate the equilibrium behaviour of the Bertrand-Edgeworth model. I address this question in chapter 5 of my Ph.D. thesis.

I have written my Ph.D. thesis in Hungarian with the title "Adagolási szabályok és Bertrand-Edgeworth oligopóliumok" ("Rationing Rules and Bertrand-Edgeworth Oligopolies"). The main results of my research are summarized in the following four papers.

The first one entitled “Implementation of Rationing Rules” has been submitted to an international journal. It develops the mathematical structure required for discussing rationing rules in a uniform framework. To simplify our analysis we focus on duopolistic situations. The extension of the structure to the oligopolistic case is straightforward. We have to analyze the rationing problem at the individual level and at the market level. Suppose that the low-price firm does not satisfy the entire demand of a consumer. Then the high-price firm faces a so-called residual demand. This leads to the notion of the individual rationing rule, which determines the quantities of both firms product that they can sell to a consumer given the consumer’s individual demand curve, the firm’s own price, and its rival’s price as well as the quantity offered by the rival to the consumer. The market rationing rule can be obtained by aggregation of individual rationing rules. Since the residual demand of the market may depend on the order in which the consumers were served by the low-price firm, a probabilistic model seems to be appropriate for determining the residual demand on a market. The two most frequently applied rationing rules in the literature are the so-called efficient and the so-called random rationing rules. Within the developed framework we analyze market situations in which a given rationing rule is applicable.

The second paper in this collection was presented on a Ph.D. conference at the Budapest University of Economic Sciences. “Which rationing rule does a single consumer follow?” considers the individual residual demand of a consumer. Given the utility function of a consumer we can determine its residual demand by solving its utility maximization problem. For general utility functions we cannot obtain the residual demand explicitly. Therefore, we have to consider special types of utility functions. For the Cobb-Douglas utility function we obtain that the consumer behaves according to the combined rationing rule, while for the quasilinear utility function we obtain that the consumer behaves according to the efficient rationing rule. This paper appeared in Blahó,

A. (ed.), *The Future in the Present: - Changing Society, New Scientific Issues.* Budapest: Budapest University of Economic Sciences, 187-200.

The third paper analyzes the capacity constrained Bertrand-Edgeworth game. In general, the Bertrand-Edgeworth game does not have a Nash equilibrium in pure strategies. We obtain that for price elastic demand curves the capacity constrained Bertrand-Edgeworth game possesses a Nash equilibrium in pure strategies for any capacity constraints. While if the demand curve has a price inelastic part, then there are capacity levels for which the Bertrand-Edgeworth game does not have an equilibrium in pure strategies. We can relax the elasticity assumption imposed on the demand curve in case of efficient rationing, if we do not allow a firm to be arbitrarily small in capacity with respect to its rival. Then the demand curve can be even price inelastic at any price level, to guarantee existence of equilibrium in pure strategies. The paper "Existence of Pure Strategy Nash Equilibrium in Bertrand-Edgeworth Oligopolies" appeared in *Economics Letters* 1999 Vol. 63(2), 201-206.

The fourth article investigates a two-stage Bertrand-Edgeworth game in which the firms can select the applied rationing rule. We introduce the combined rationing rule. The two limit cases of the combined rationing rule are the efficient and the random rationing rules. The paper "A two-stage Bertrand-Edgeworth game" appeared in *Economics Letters* 1999 Vol. 65(3), 353-358.

II. Implementation of rationing rules

Abstract:

The Bertrand-Edgeworth duopolistic models may be specified by applying a rationing rule. The present purpose is to discuss a rationing problem within a uniform framework. A system of concepts are introduced for the detailed examination of the rationing rule. We will show that our framework is capable of integrating those well-known market situations in which a certain rationing rule can be implemented. Additionally, we will present a new more general way of implementing the random rationing rule in large but finite markets.

Keywords: Duopoly; Bertrand-Edgeworth; Rationing.
JEL classification: D45; L13.

1 Introduction

The rationing problem examined here arises in Bertrand-Edgeworth-type duopolies. In Bertrand-Edgeworth models the price and the quantity are simultaneously decision variables.

The complete specification of the model is normally given in two ways. One of these assumes that the demand side is given in terms of a representative consumer utility function (see, for example, Benassy, 1986). In this case we have to deal with the consumer's utility maximization problem in conditions of limited supply. Such analyses have been carried out by Howard (1977), Neary and Roberts (1980) and Dixon (1987).

It is another commonly applied means for completely specifying the model, which gives rise to the rationing problem intended for analysis here. In the partial approach the consumer side is given by the aggregate demand curve. In the absence of further information this results in an under-specified model. This is because the aggregate demand curve does not itself provide sufficient information to enable the demand of a company offering a higher price to be determined. We substitute for the lack of information by a rationing rule. It should be noted that knowledge of the aggregate demand curve is sufficient, if the duopolist offering at a lower price covers the whole market. This situation holds in the Bertrand duopoly. In the Bertrand-Edgeworth duopoly, however, the company offering at a lower price is not capable or not interested in covering the entire market. The former behaviour may be due to capacity constraints and the latter to a U-shaped marginal cost function.

There are many conceivable rationing rules. The two most frequently applied rationing rules in the literature are the efficient and the random rationing rules. The random rationing rule has been applied for example by Beckmann (1965) in determining the mixed strategy Nash equilibrium for the Bertrand-Edgeworth duopoly game with capacity constraints in case of a linear demand curve and by Allen and Hellwig (1986) in the asymptotic investigation of the Bertrand-Edgeworth oligopoly model with capacity constraints. Dasgupta and Maskin (1986) demonstrated the existence of mixed strategy equilibrium in the case of random rationing for demand curves which intersect both axes. Using the efficient rationing rule, Levitan and Shubik (1972) have determined the Nash equilibrium for the Bertrand-Edgeworth game in mixed strategies, and Vives (1986) has investigated the asymptotic behaviour of the Bertrand-Edgeworth oligopolistic model. Market situations that implement the efficient or the random rationing rule have been summarized by Davidson and Deneckere (1986).

In Tasnádi (1999) we introduced the notion of a combined rationing rule in order to investigate a two-stage Bertrand-Edgeworth game in that each duopolist can select the way it will serve the consumers, if it becomes the low-price firm. The efficient and random rationing rules are also combined rationing rules.

Dixon (1987) already introduced a general framework for the investigation of rationing rules. We extend Dixon's (1987) model by incorporating explicitly into our model the order in which the consumers are served. Furthermore, we want to investigate markets with finite but many consumers. Therefore, we introduce the mathematical structure required for discussing rationing rules in a uniform framework. The precise mathematical foundation of the random rationing rule by Allen and Hellwig (1986) inspired the formulation of the framework presented in this article.

Additionally, we show that in the case of many but finite consumers the random rationing rule will be implemented approximately, if the consumers are served randomly and if none of the consumers' share of the entire demand is significant. Hence, the commonly applied restrictions (see for instance Davidson and Deneckere, 1986; Dixon, 1987) that consumers have identical demand curves or unit demands can be dropped, if we content ourselves with asymptotic results.

The second section introduces the rationing concept and what is understood by the applicability of a rationing rule in a duopolistic market. In the third section we take the combined rationing rule as an example to show that our system of concepts is appropriate for the discussion of the implementation of rationing rules. The fourth section contains an asymptotic implementation of the random rationing rule.

2 Rationing rules

We will consider only duopolies for the sake of simplicity. The producers' decisions can be described by a price-quantity pair. Denote as $A_i = \mathbb{R}^2_+$ $(i \in \{1, 2\})$ the ith duopolists' decision set. The price and quantity pair $(p_i, q_i) \in A_i$ denotes a decision of duopolist $i \in \{1, 2\}$. Furthermore, let $A = A_1 \times A_2$ be the set of decisions. In the following it will be assumed that the producers are indexed so that $p_1 < p_2$. We will disregard cases where $p_1 = p_2$. There is only need for rationing if $q_1 < D(p_1)$, so that we will be restricted to the examination of decisions on the set $A' := \{a \in A \mid p_1 < p_2,\ q_1 < D(p_1)\}$. The demand of the product of a company offering at a higher price is called *residual demand*.

Rationing occurs at individual level and at market level. First, on the level of consumers the residual demand of a consumer has to be determined, if the consumer's entire demand has not been satisfied by the low-price firm. Second, on the level of the market the residual demand that the high-price firm faces has to be determined, which depends on the way that consumers are served and on the consumers' individual residual demands.

Let us denote the consumers's set by Ω. Above the consumers's set there is a measurable space $(\Omega, \mathcal{A}, \mu)$, where μ is assumed to be a finite measure. Each consumer's demand is described by a bounded and measurable function $d : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$, where $d(p, \omega)$ is the demand of consumer $\omega \in \Omega$ at price p. The aggregate demand is thus $D(p) = \int_\Omega d(p, \omega) d\mu(\omega)$. Let us introduce $\mathcal{D} \subset \mathbb{R}_+^{\mathbb{R}_+}$ to denote the set of permitted demand functions.

We have to define the notion of the individual rationing rule:

Definition 2.1. The *individual rationing rule* of consumer $\omega \in \Omega$ is the function giving the quantity of product that a firm can sell to consumer ω given the individual demand curve $d(\cdot, \omega)$, the firm's own price, and its rival's price as well as the quantity offered by the rival to consumer ω. The consumers' individual rationing rules on a duopolistic market can be described by a mapping of the form $\rho : \mathcal{D} \times \mathbb{R}_+ \times \mathbb{R}_+^2 \times \Omega \to \mathbb{R}_+$.

If we denote by $d^{a_1}(p, \omega)$ the residual demand of consumer $\omega \in \Omega$, where $a_1 \in A_1$ and $p_1 < p$, then

$$\rho(d(\cdot, \omega), p_1, p_2, q_2, \omega) = d(p_1, \omega)$$
$$\rho(d(\cdot, \omega), p_2, p_1, q_1, \omega) = d^{a_1}(p_2, \omega).$$

To derive the rationing rule of a consumer we have to solve the consumer's utility maximization problem under constrained supply. Such investigations have been carried out by Neary and Roberts (1980) in general and by Dixon (1987) in context of the Bertrand-Edgeworth game. Shubik (1955) already noted that unless income effects a consumer will behave according to the efficient rationing rule. From Tasnádi (1998) it follows that a consumer with a

Cobb-Douglas utility function of form $u(x, m) = Ax^{\alpha}m^{\beta}$ behaves according to a combined rationing rule (Definition 2.9) with parameter $\frac{\beta}{\alpha+\beta}$, where x is the amount purchased from the duopolists' product, m is the consumption from a composite commodity, $0 < \alpha$, $0 < \beta$ and $\alpha + \beta < 1$. We will focus on analyzing the rationing problem at the market level and refer mainly to Dixon (1987) in considering the level of the individual household.

We will use the following definition for the rationing rule:

Definition 2.2. Let a *rationing rule* be a function giving the quantity of product that a firm can sell when the aggregate demand curve, the firm's own price, and its rival's price and quantity are known. Formally, the rationing rule on a duopolistic market is a mapping of the form $R : \mathcal{D} \times \mathbb{R}_+ \times \mathbb{R}_+^2 \to \mathbb{R}_+$.

The residual demand associated with a decision $a \in A'$ is denoted by D^{a_1}. To determine the residual demand the quantities offered by the low-price firm to each consumer have to be given separately. A bounded and $\mathcal{A}$-measurable function $X : \Omega \to \mathbb{R}_+$ is called an *assignment*, where $X(\omega)$ is the quantity that the low-price firm offers to consumer ω and $X(\omega) \leq d(p_1, \omega)$.

It shall be assumed that for a fixed decision $a \in A'$ there is a probability space $(\Omega', \mathcal{A}', P)$, where Ω' is the set of assignments and $\mathcal{A}'$ is a σ-algebra on Ω'. X_ω is then a random variable of form $\Omega' \to [0, d(p_1, \omega)]$.

Of course only those assignments are interesting that allocate q_1 products to the consumers. Therefore we introduce the notion of an allocation.

Definition 2.3. An assignment $Y \in \Omega'$ is called an *allocation*, if $\int_\Omega Y(\omega) d\mu(\omega) = q_1$.

We collect the above structures in the object defined below.

Definition 2.4. A producer-determined *market situation* can be described by the structure

$$s = \langle (\Omega, \mathcal{A}, \mu), d, \rho, \{1, 2\}, (p_1, q_1, p_2, q_2), (\Omega', \mathcal{A}', P) \rangle$$

where it is assumed that $(p_1, q_1, p_2, q_2) \in A'$.

Assuming that the saleable quantities on the market can be given by a rationing rule R, the residual demand can be given as follows:

$$D^{a_1}(p) = R(D, p, p_1, q_1), \qquad p > p_1.$$

It is important to determine the market situations that a particular rationing rule may apply to. The following two conditions express that we regard only allocations in our model.

Definition 2.5. A market situation s satisfies the *supply condition*, if

$$P(\{Y \in \Omega' \mid \int_\Omega Y(\omega) d\mu(\omega) = q_1\}) = 1.$$

To state this in words: an assignment is an allocation with probability one.

We weaken the supply condition to prepare our framework to handle asymptotic results.

Definition 2.6. A sequence $(s^{(n)})_{n=1}^{\infty}$ of market situations satisfy the *asymptotic supply condition*, if $p_1^{(n)} = p_1$, $p_2^{(n)} = p_2$, $q_1^{(n)} = q_1$, $q_2^{(n)} = q_2$, and

$$\forall \varepsilon > 0 : \lim_{n\to\infty} P^{(n)} \left(\left\{ Y \in \Omega'^{(n)} \ : \ \left| \int_{\Omega^{(n)}} Y(\omega) d\mu(\omega) - q_1 \right| < \varepsilon \right\} \right) = 1.$$

Now, we are ready to define the implementability of rationing rules. We introduce two types of implementations.

Definition 2.7. We say that in a market situation s a rationing rule R is *implemented* if s satisfies the supply condition and

$$P \left(Y \in \Omega' \mid R(D, p_2, p_1, q_1) = \int_\Omega \rho(d(\cdot, \omega), p_2, p_1, Y(\omega), \omega) d\mu(\omega) \right) = 1$$

provided the integral exists.

To investigate the applicability of a rationing rule in large but finite markets we define the notion of asymptotic implementation.

Definition 2.8. A sequence $(s^{(n)})_{n=1}^{\infty}$ of market situations *asymptotically implements* a rationing rule R, if the sequence $(s^{(n)})_{n=1}^{\infty}$ satisfies the asymptotic supply condition and for all $\varepsilon > 0$:

$$\lim_{n\to\infty} P^{(n)}\left(\left\{Y \in \Omega'^{(n)} \ : \ \left|\frac{\int_{\Omega^{(n)}} \rho^{(n)}(d^{(n)}(\cdot,\omega), p_2, p_1, Y(\omega), \omega) d\mu(\omega)}{R(D, p_2, p_1, q_1)} - 1\right| < \varepsilon\right\}\right) = 1$$

provided the integral exists.

An implementable rationing rule is also asymptotically implementable, because if the market situation s implements rationing rule R, then the sequence $(s, s, \dots, s, \dots)$ of market situations asymptotically implements rationing rule R.

We will investigate combined rationing rules.

Definition 2.9. A function $R : \mathcal{D} \times \mathbb{R}_+ \times \mathbb{R}_+^2 \to \mathbb{R}_+$ is called a *combined* rationing rule with parameter $\lambda \in [0,1]$, if the demand faced by the firm $j \in \{1,2\}$ is given by

$$R_j(D, p_j, p_i, q_i) := \begin{cases} D(p_j), & \text{if} \quad p_j < p_i; \\ \frac{q_j}{q_1+q_2} D(p_j), & \text{if} \quad p_j = p_i; \\ \max\left(D(p_j) - \alpha(p_i, p_j) q_i, 0\right), & \text{if} \quad p_j > p_i; \end{cases}$$

where $\alpha(p_i, p_j) = (1-\lambda)\frac{D(p_j)}{D(p_i)} + \lambda$ and $i \neq j$.

The efficient and the random rationing rules are also combined rationing rules. We can see this by selecting for λ in Definition 2.9 the values 1 and 0 respectively.

3 Implementations of the combined rationing rule

We will demonstrate that our concept of implementability is capable of discussing rationing rules in detail. We will consider combined rationing rules. We already mentioned in Tasnádi (1999) that the implementations of a combined rationing rule can be obtained by combining implementations of the efficient and the random rationing rules on a certain market. An overview of the implementations of the efficient and the random rationing rules can be found for instance in Davidson and Deneckere (1986).

3.1 Identical individual demand curves

Let the set of consumers be $\Omega = \{1, \ldots, I\}$. We assume that every consumer's demand curve $d(\cdot)$ is identical. Furthermore, we assume that every consumer's individual rationing rule is the efficient rationing rule. Let the first producer's price be the lower, i.e. $p_1 < p_2$. In the case of interest to us, we assume that $q_1 < I \cdot d(p_1)$.

A combined rationing rule with parameter $\lambda \in [0, 1]$ can be implemented briefly in the following way: let $1 - \lambda$ portion of the low-price firm's supply be sold to randomly selected consumers by satisfying their entire demand, and let the remaining λ portion of the low-price firm's supply uniformly distributed amongst the unsatisfied consumers.

Clearly $\lfloor q_1/d(p_1) \rfloor$ persons can be supplied completely at the low price. Suppose, that the low-price firm supplies only $m := \lfloor (1 - \lambda)q_1/d(p_1) \rfloor$ consumers entirely. The remaining consumers obtain $q := \frac{q_1 - md(p_1)}{I - m}$ amount of the product. This market situation is described by the structure

$$s^I = \langle (\Omega, \mathcal{P}(\Omega), \zeta), d^*, \rho, \{1, 2\}, (p_1, q_1, p_2, q_2), (\Omega', \mathcal{A}', P) \rangle$$

where ζ is now the counting measure, $d^*(p, \omega) = d(p)$ for all $\omega \in \Omega$, ρ is the efficient rationing rule for every consumer, and P is the measure of the uniform distribution on set

$$\{f : \Omega \to [0, d(p_1)] \mid |f^{-1}(d(p_1))| = m \text{ and } |f^{-1}(q)| = I - m\}.$$

The supply condition is satisfied because

$$\int_\Omega X(\omega) d\zeta(\omega) = md(p_1) + (I - m)q = q_1.$$

Let us denote by B the set of those consumers, who have been only partly satisfied at the low price by allocation X. A λ combined rationing rule is implemented asymptotically in market situation s, because

$$\begin{aligned}
\int_\Omega \rho(d^*(\cdot,\omega),p_2,p_1,X(\omega),\omega)d\mu(\omega) &= \int_\Omega (d^*(p_2,\omega)-X(\omega))^+\zeta(\omega) = \\
&= \int_B (d^*(p_2,\omega)-X(\omega))^+\zeta(\omega) = \\
&= (I-m)(d(p_2)-q)^+ = \\
&= (D(p_2)-md(p_2)-(q_1-md(p_1)))^+ \approx \\
&\approx (D(p_2)-q_1(1-\lambda)\frac{D(p_2)}{D(p_1)}-q_1\lambda)^+,
\end{aligned}$$

holds with probability one, if I is sufficiently large.

3.2 Consumer's with unit demand

We regard the other frequently applied market in that every consumer's demand is either a unit of the good or nothing. Let the consumers side be given by the $([0,1],\mathcal{B}([0,1]),\mu)$ measure space, where μ denotes in this subsection the Lebesgue-Borel measure on the unit interval. In fact we could have chosen any nonatomic finite measure. Let the consumer's reservation prices be given by the measurable and strictly decreasing function $r:[0,1]\to\mathbb{R}_+$. Then the consumers demand curves are

$$d(p,\omega)=\begin{cases}1, & \text{if} \quad p\le r(\omega)\ ;\\ 0, & \text{if} \quad p>r(\omega).\end{cases}$$

At price level p the consumers in interval $[0,D(p)]$ are those who are demanding a unit of the good, and their demand equals to $D(p)$, because r is decreasing. Let $p_1<p_2$ and $q_1<D(p_1)$.

In order to implement a combined rationing rule with parameter λ we will assign to the consumers in interval $[0,D(p_2)]$ a higher probability of purchasing from the low-price firm, than to those consumers in interval $(D(p_2),D(p_1)]$. We denote by π_H the probability that a consumer in $[0,D(p_2)]$ and by π_L

the probability that a consumer in $(D(p_2), D(p_1)]$ obtains the good at the low price. Let

$$\pi_H := \min\left\{(1-\lambda)\frac{q_1}{D(p_1)} + \lambda\frac{q_1}{D(p_2)}, 1\right\}$$

and

$$\pi_L := \begin{cases} (1-\lambda)\frac{q_1}{D(p_1)}, & \text{if } (1-\lambda)\frac{q_1}{D(p_1)} + \lambda\frac{q_1}{D(p_2)} \leq 1; \\ \frac{q_1 - D(p_2)}{D(p_1) - D(p_2)}, & \text{if } (1-\lambda)\frac{q_1}{D(p_1)} + \lambda\frac{q_1}{D(p_2)} > 1. \end{cases}$$

Denote by M the set of $[0,1] \to \{0,1\}$ functions which take the value zero on the interval $(D(p_1), 1]$. The value $f(x)$ $(x \in [0,1])$ of a function $f \in M$ expresses whether the consumer x has secured a supply of the product at the lower price.

Consider base set $\{0,1\}$ and the σ-algebra $\mathcal{P}(\{0,1\})$. Take the probability measures

$$P_t(\{1\}) = \begin{cases} 0, & \text{if } t \in (D(p_1), 1]; \\ \pi_L, & \text{if } t \in (D(p_2), D(p_1)]; \\ \pi_H, & \text{if } t \in [0, D(p_2)]; \end{cases}$$

and $P_t(\{0\}) = 1 - P_t(\{1\})$, where $t \in [0,1]$. Let the probability measures P_t be independent. It follows that (see, for example Theorem 9.2 and its consequences in Bauer, 1991) on the measurable space $(\Omega', \mathcal{A}') := (\{0,1\}^{[0,1]}, \mathcal{P}(\{0,1\})^{[0,1]})$ there exists a probability measure P for which:

$$\forall T \subset [0,1] : |T| < \infty, Pr_T(P) = \prod_{t \in T} P_t,$$

where Pr is the projection operator. The measure P cannot be given explicitly, but this is not necessary to solve the problem. Denote by $X_t : \Omega' \to \{0,1\}$ independent random variables for which $X_t(f) = f(t)$, where $t \in [0,1]$ and $f \in \Omega'$.

Proposition 3.1. *The market situation described above implements a λ combined rationing rule.*

Proof. First, we suppose that $(1-\lambda)\frac{q_1}{D(p_1)} + \lambda\frac{q_1}{D(p_2)} \leq 1$. By Kolmogorov's strong law of large numbers,

$$Y_n := \sum_{i=1}^{n} X_{t_i} \frac{D(p_1)}{n} = \sum_{i=1}^{m_n} X_{t_i} \frac{D(p_1)}{n} + \sum_{i=m_n+1}^{n} X_{t_i} \frac{D(p_1)}{n} \to \\ \to \pi_H D(p_2) + \pi_L (D(p_1) - D(p_2)) = q_1 \tag{1}$$

with probability one, if $n \to \infty$, where $m_n := \left\lfloor \frac{D(p_2)}{D(p_1)} n \right\rfloor$, $t_i \in [\frac{i-1}{n} D(p_1), \frac{i}{n} D(p_1))$. The limit of Y_n is a quantity which approaches a Riemann integral. Thus if $f \in \{0,1\}^{[0,1]}$, the Riemann integral of f exists with probability one and equals to q_1 with probability one. The value of $\int_0^{D(p_1)} f d\mu$ gives the demand of those consumers, who obtain product at price p_1. Hence, the supply condition is satisfied.

We determine the demand of those consumers, who are served by the low-price firm:

$$Z_n := \sum_{i=1}^{n} X_{t_i} \frac{D(p_2)}{n} \to \pi_H D(p_2) = (1-\lambda) q_1 \frac{D(p_2)}{D(p_1)} + \lambda q_1, \tag{2}$$

where $t_i \in [\frac{i-1}{n} D(p_2), \frac{i}{n} D(p_2))$. The limit of Z_n is a quantity which approaches a Riemann integral. Thus if $f \in \{0,1\}^{[0,1]}$, the Riemann integral of f on interval $[0, D(p_2)]$ exists with probability one and equals to $(1-\lambda) q_1 \frac{D(p_2)}{D(p_1)} + \lambda q_1$ with probability one. The value of $\int_0^{D(p_2)} f d\mu$ gives the demand of those consumers with higher reservation prices, who obtain product at price p_1. Therefore, we conclude that the current market situation implements a combined rationing rule with parameter λ.

Second, we investigate the case of

$$(1-\lambda) \frac{q_1}{D(p_1)} + \lambda \frac{q_1}{D(p_2)} > 1. \tag{3}$$

Regarding (1) by Kolmogorov's strong law of large numbers,

$$Y_n \to D(p_2) + q_1 - D(p_2) = q_1$$

with probability one, if $n \to \infty$. Hence, the supply condition is satisfied.

Similarly to (2) and its consequences we obtain that the demand of those consumers who are served by the low-price firm Z_n approaches to $D(p_2)$ with probability one. Therefore, the residual demand equals zero. A combined rationing rule with parameter λ gives the same value because of (3). $\square$

4 An implementation of the random rationing rule

In this section we present an asymptotic implementation of the random rationing rule. In comparison to previous results (configure Davidson and Deneckere, 1986; Dixon, 1987) on markets with finitely many consumers we drop the assumption of equal demand curves.

We are looking for an asymptotic implementation, therefore we will regard a sequence of market situations $(s^{(n)})_{n=1}^{\infty}$. Let the set of consumers in the nth market situation be given by the measure space $(\{1,\dots,n\},\mathcal{P}(\{1,\dots,n\}),\zeta^n)$ where ζ^n denotes the counting measure on set $\{1,\dots,n\}$. We denote by $d^{(n)}$ the demand curves of the nth market situation. The prices and quantities set by the duopolists are the same in each market situation, i.e. $p_1 := p_1^{(n)}$, $p_2 := p_2^{(n)}$, $q_1 := q_1^{(n)}$ and $q_2 := q_2^{(n)}$ for all $n \in \mathbb{N}$. Let $p_1 < p_2$ and $q_1 < D(p_1)$. If a consumer is served by the low-price firm, then its entire demand will be served. Thus, the set of assignments in the nth market situation is

$$\Omega^{(n)} = \{f : \{1,\dots,n\} \to \mathbb{R}_+ \mid \forall i \in \{1,\dots,n\} : f(i) \in \{0, d^{(n)}(p_1,i)\}\}.$$

Let the demand curves $d^{(n)}$ be decreasing. Furthermore, suppose that the market demand remains the same in every market situation, i.e. $D(p) = \sum_{i=1}^{n} d^{(n)}(p,i)$ for all prices $p \geq 0$ and market situations $n \in \mathbb{N}$. We assume that in any market situation the individual demands are insignificant with respect to the entire market demand.

Assumption 4.1. There exists a positive real value c such that for any $n \in \mathbb{N}$ we have that

$$d^{(n)}(p,i) < \frac{c}{n} D(p)$$

for all consumers $i \in \{1,\dots,n\}$ and prices $p \geq 0$.

Suppose that every consumers' chance of obtaining the low-price product equals $q = q_1/D(p_1)$. Hence, we assume that $P^{(n)}$ is the uniform distribution with parameter $1/2^n$ on the set $\Omega^{(n)}$, i.e. $P^{(n)}(f) = 1/2^n$ for all $f \in \Omega^{(n)}$.

Proposition 4.2. *The sequence $s^{(n)}$ of market situations defined above implements the random rationing rule asymptotically.*

Proof. First, we show that the sequence $s^{(n)}$ of market situations satisfies the asymptotic supply condition. Let us denote by $X_i^{(n)}$ the random variable, which tells us the amount of product obtained by consumer i in the nth market situation. For any $n \in \mathbb{N}$ the random variables have independent characteristic distributions of parameter q. The expected value of allocated products is

$$\mathrm{E}\left(\sum_{i=1}^{n} X_i^{(n)}\right) = q\sum_{i=1}^{n} d^{(n)}(p_1, i) = qD(p_1) = q_1. \tag{4}$$

Furthermore, the variance of allocated products is

$$\begin{aligned}\mathrm{Var}\left(\sum_{i=1}^{n} X_i^{(n)}\right) &= q(1-q)\sum_{i=1}^{n}\left(d^{(n)}(p_1, i)\right)^2 < \\ &< q(1-q)\sum_{i=1}^{n}\left(\frac{c}{n}D(p_1)\right)^2 = q(1-q)\frac{c^2}{n}D^2(p_1)\end{aligned} \tag{5}$$

because of Assumption 4.1. Therefore, the asymptotic variance is

$$\lim_{n\to\infty} \mathrm{Var}\left(\sum_{i=1}^{n} X_i^{(n)}\right) = 0. \tag{6}$$

The asymptotic supply condition is satisfied because of the Chebyshev inequality, equation (4), and equation (6).

Second, we have to show that the residual demand equals the value suggested by the random rationing rule. We can calculate the expected residual demand in the following way

$$\begin{aligned}E(D^{a_1}(p_2)) = E\left(\sum_{i=1}^{n} \rho(p_2, p_1, X_i^{(n)}, i)\right) &= \\ = (1-q)\sum_{i=1}^{n} d^{(n)}(p_2, i) &= D(p_2) - \frac{q_1}{D(p_1)}D(p_2),\end{aligned} \tag{7}$$

because $\rho^{(n)}(p_2, p_1, X_i^{(n)}, i)$ equals $d^{(n)}(p_2, i)$ with probability $1-q$, if consumer i has not been served, and 0 with probability q, if consumer i has been served at the low price. Similarly to (5), the variance of the residual demand can also be determined:

$$\begin{aligned}\operatorname{Var}\left(\sum_{i=1}^{n} \rho(p_2, p_1, X_i^{(n)}, i)\right) &= q(1-q)\sum_{i=1}^{n}\left(d^{(n)}(p_2, i)\right)^2 < \\ &< q(1-q)\sum_{i=1}^{n}\left(\frac{c}{n}D(p_2)\right)^2 = \\ &= q(1-q)\frac{c^2}{n}D^2(p_2).\end{aligned} \tag{8}$$

Hence, by the Chebyshev inequality, equation (7), and equation (8) we conclude that the random rationing rule is implemented asymptotically by the sequence $(s^{(n)})_{n=1}^{\infty}$. □

5 Summary

If the consumers' aggregate demand curve is known, a rationing rule is capable of giving the quantities that can be sold by duopolists. Since the duopolists serve individuals, the residual demand of a duopolist offering at the higher price actually depends on the consumers' individual rationing rules and the way that the consumers are served. The problem is to establish what kinds of individual rationing rules and modes of service cause the residual demand of the duopolist offering at the higher price to coincide with the saleable product quantities given by particular rationing rules. It was in order to examine this question that the concept of the implementability of rationing rules was introduced.

The implementability of the combined rationing rule was examined in detail. Additionally, we presented an implementation of the random rationing rule in the case of many but finitely many consumers without employing the commonly imposed assumption of equal individual demand curves.

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III. Which rationing rule does a single consumer follow?

Abstract:

We will investigate the amount of the residual demand in a market consisting of only one consumer and two producers. Because there is only one consumer, we can not really speak about a rationing rule, but we can pose the question that, which rationing rule is adequate to the utility maximizing behaviour. We will show that, if the consumer has a Cobb-Douglas utility function, then the amount purchased by the consumer from the high-price firm lies between the values determined according to the efficient rationing rule and the random rationing rule. We will show further, that if the consumer has a quasi-linear utility function, then in the economically interesting case his residual demand function will be equal to the residual demand function under efficient rationing.

Keywords: Rationing; Bertrand-Edgeworth.
JEL classification: D45; L13.

1 Introduction

In Bertrand-Edgeworth duopolies quantities and prices are both decision variables. At first sight the simultaneous admittance of these two control variables leads to an underspecified model. Particularly, in the context of partial equilibrium analysis, where the consumers' side of the duopoly market is given by the aggregate demand curve, we can not determine the quantity demanded from the high-price firm. The missing item in the model is called a rationing rule. The aggregate demand function and the rationing rule together contain enough information on the determination of the sales of both duopolists. Let me mention that the only case in which the knowledge of the aggregate demand curve suffice is when the low-price firm covers the entire market. In

Bertrand-Edgeworth type duopolies the low-price firm typically is not able or not interested in covering the entire market at the low-price. The cause for this behaviour can be either capacity constraints or a U-shaped marginal cost functions.

There are many applicable rationing rules, but the two most frequently used rationing rules are the so called random rationing rule and the efficient rationing rule.

2 Rationing rules

First we give a formal definition of a rationing rule in a duopolistic environment. Let us denote the set of the admissible demand curves with $\mathcal{D} \subset \mathbb{R}_+^{\mathbb{R}_+}$.

Definition 2.1. A function is called a *rationing rule* if it assigns to every admissible demand function and to the duopolists' every quantity and price choices the saleable amount of products. Formally a rationing rule is a $h : \mathcal{D} \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \to \mathbb{R}_+^2$ function.

It is of main interest to find reasonable rationing rules. We will only discuss the two main rationing rules, namely the random and the efficient one.

In case of the random rationing rule the ratio of the satisfied demand at the low-price to the entire demand remains constant for all price levels above the low-price. In fact form the definition below the ratio is $1 - q_i/D(p_i)$.

Definition 2.2. An $h : \mathcal{D} \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \to \mathbb{R}_+^2$ rationing rule is called *random*, if $\forall j \in \{1, 2\}$:

$$h_j(D, p_1, p_2, q_1, q_2) := \begin{cases} D(p_j) & \text{if} \quad p_j < p_i, \ i \neq j; \\ \frac{q_j}{q_1+q_2} D(p_j) & \text{if} \quad p_j = p_i, \ i \neq j; \\ \max\left((1 - \frac{q_i}{D(p_i)}) D(p_j), 0 \right) & \text{if} \quad p_j > p_i, \ i \neq j. \end{cases}$$

By the efficient rationing rule the consumer with a higher reservation price is served before a consumer with a lower reservation price. Therefore if we shift the demand curve leftward by the amount of sales at the low-price, then we will obtain the residual demand curve. This rationing rule is called efficient because at given prices and quantities it maximizes consumer surplus (see Tirole (1988)). Let us give also a formal definition for the efficient rationing rule.

Definition 2.3. An $h : \mathcal{D} \times \mathbb{R}^2_+ \times \mathbb{R}^2_+ \to \mathbb{R}^2_+$ rationing rule is called *efficient*, if $\forall j \in \{1, 2\}$:

$$h_j(D, p_1, p_2, q_1, q_2) := \begin{cases} D(p_j) & \text{if} \quad p_j < p_i, \;\; i \neq j; \\ \frac{q_j}{q_1+q_2} D(p_j) & \text{if} \quad p_j = p_i, \;\; i \neq j; \\ \max(D(p_j) - q_i, 0) & \text{if} \quad p_j > p_i, \;\; i \neq j. \end{cases}$$

For market situations in which the application of the efficient or the random rationing rule is reasonable see for example Allen and Hellwig (1986), Gelman and Salop (1983), Tirole (1988) and Wolfstetter (1993).

3 The behaviour of a single consumer market

Now we turn to the case, where the demand side of the market contains only one consumer. We have now two possibilities to determine the residual demand of the single consumer. First, given the microeconomic theory of consumer behaviour, we can formulate and solve the adequate consumer's utility maximizing problem explicitly for a given type of utility function. Second, we can determine the consumer's residual demand from the consumer's individual demand with the help of an explicitly chosen rationing rule. Of course the first method gives the right solution for the residual demand. But it is an interesting task to compare the results of the two methods. Of course we can not really speak about a rationing rule in a single consumer market, but we can pose the question that, which rationing rule is adequate with the utility maximizing behaviour. Let us remark, that Howard (1977) and Neary and Roberts (1980) investigated the utility maximizing decision of a consumer under supply constraints, but they did not compare the optimal decision to the rationing rules.

Now we turn to the formulation of our problem. Consistent with the main oligopolistic literature our analysis will be of partial nature. Our consumer's utility function is $U(x, m)$, where x is the amount consumed from the duopolists' product and m is his consumption from a composite commodity, which we call from now on simply money. Furthermore we assume that U is twice continuously differentiable, $U_x > 0$, $U_m > 0$. We denote with $\overline{m}$ our single consumer's amount of money and assume that this value is strictly positive. Our consumer's utility maximizing problem assuming that the first firm is the low-price firm ($p_1 < p_2$) takes the form as below:

$$\begin{aligned} U(x_1 + x_2, \overline{m} - p_1 x_1 - p_2 x_2) &\rightarrow \max \\ x_1 &\leq q_1 \\ p_1 x_1 + p_2 x_2 &\leq \overline{m} \\ x_1, x_2 &\geq 0 \end{aligned} \tag{1}$$

The purchased amount of products from firm 1 and 2 are denoted by x_1 and x_2.

From our consumer's utility function we can derive his demand function. So we can determine the residual demand function belonging to a given rationing rule. To compare the residual demand function obtained by the second method with the solution of problem (1) is quite demanding, perhaps even impossible, because for general utility functions we can not solve problem (1) explicitly. But we can get positive results for special types of utility functions.

3.1 Cobb-Douglas utility function

Let us for example investigate our consumer's behaviour in case of the Cobb-Douglas utility function. In the Cobb-Douglas case we can relate the solution to the two main rationing rules. The results are summarized in the next proposition.

Proposition 3.1. *There is only one consumer on a duopol market. His utility function is* $u(x,m) = Ax^{\alpha}m^{\beta}$*, where* $0 < \alpha$, $0 < \beta$ *and* $\alpha + \beta \leq 1$*. His money stock is positive and denoted by* $\overline{m}$*. The duopolists' prices are given, and let* $0 < p_1 < p_2$*. The low-price firm is offering* $q_1 > 0$*. Then the there exists a unique solution to the consumer's utility maximizing problem. Furthermore*

1. *if*

$$\overline{m} > p_1 q_1 + \frac{\beta}{\alpha} p_2 q_1, \tag{2}$$

then the optimal solution will be $x_1^* = q_1$,

$$x_2^* = \frac{\alpha\overline{m} - q_1(\alpha p_1 + \beta p_2)}{(\alpha + \beta) p_2} \tag{3}$$

and x_2^* *is lying between the values suggested by the efficient and the random rationing rule;*

2. *if* $\overline{m} \leq p_1 q_1 + \frac{\beta}{\alpha} p_2 q_1$*, then* $x_2^* = 0$*.*

Proof. Our utility maximizing consumer has to solve the following problem

$$\begin{aligned} A(x_1 + x_2)^{\alpha}(\overline{m} - p_1 x_1 - p_2 x_2)^{\beta} &\rightarrow \max \\ x_1 &\leq q_1 \\ p_1 x_1 + p_2 x_2 &\leq \overline{m} \\ x_1, x_2 &\geq 0 \end{aligned} \tag{4}$$

We can check that the object function is strictly concave because of our restrictions imposed on the parameters α and β. So the uniqueness is guaranteed. The Lagrangian belonging to problem (4) is $\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2) =$

$$A(x_1 + x_2)^{\alpha}(\overline{m} - p_1 x_1 - p_2 x_2)^{\beta} - \lambda_1(x_1 - q_1) - \lambda_2(p_1 x_1 + p_2 x_2 - \overline{m})$$

and the appropriate Kuhn-Tucker conditions (5) are the following.

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial x_1} = \; & A\alpha(x_1+x_2)^{\alpha-1}(\overline{m}-p_1x_1-p_2x_2)^{\beta}- \\
& A\beta p_1(x_1+x_2)^{\alpha}(\overline{m}-p_1x_1-p_2x_2)^{\beta-1}-\lambda_1-\lambda_2p_1\le 0 \\
\frac{\partial\mathcal{L}}{\partial x_2} = \; & A\alpha(x_1+x_2)^{\alpha-1}(\overline{m}-p_1x_1-p_2x_2)^{\beta}- \\
& A\beta p_2(x_1+x_2)^{\alpha}(\overline{m}-p_1x_1-p_2x_2)^{\beta-1}-\lambda_2p_2\le 0 \\
\frac{\partial\mathcal{L}}{\partial \lambda_1} = \; & q_1-x_1\ge 0 \\
\frac{\partial\mathcal{L}}{\partial \lambda_2} = \; & \overline{m}-p_1x_1-p_2x_2\ge 0 \quad \text{and} \\
& x_1\ge 0, \quad x_2\ge 0, \quad \lambda_1\ge 0, \quad \lambda_2\ge 0 \quad \text{and} \\
& x_1\frac{\partial\mathcal{L}}{\partial x_1}=0, \quad x_2\frac{\partial\mathcal{L}}{\partial x_2}=0, \quad \lambda_1\frac{\partial\mathcal{L}}{\partial \lambda_1}=0, \quad \lambda_2\frac{\partial\mathcal{L}}{\partial \lambda_2}=0.
\end{aligned} \tag{5}$$

Notice that the Kuhn-Tucker conditions are not defined on the

$$S := \{(x_1,x_2)\in\mathbb{R}^2_+ | p_1q_1+p_2q_2=\overline{m}\}\cup\{(0,0)\} \tag{6}$$

set. The values in S cannot be optimal, because their associated utility level is zero, but positive utility levels are obviously attainable. Therefore $\lambda_2^*=0$.

1. First, let us assume that the optimal solution x_2^* is positive. We will show that the positivity of x_2^* implicates the positivity of x_1 and λ_1. Let us assume that $\lambda_1=0$. Hence if the first condition in (5) is satisfied, then the second condition will hold as strict inequality. So $x_2=0$ would follow, which is a contradiction. Therefore, we conclude $\lambda_1>0$. Now $\lambda_1>0$ implies $x_1>0$ because $x_1=q_1$ holds by the third complementary condition. Therefore, we have equalities in the first three condition of (5). We now have to look for nonnegative λ_1 fulfiling the first two equalities. From the second equality in (5) we can get

$$\frac{A\alpha}{p_2}(q_1+x_2^*)^{\alpha-1}(\overline{m}-p_1q_1-p_2x_2^*)^{\beta}-A\beta(q_1+x_2^*)^{\alpha}(\overline{m}-p_1q_1-p_2x_2^*)^{\beta-1}=0$$

From the first equality in (5) the existence of nonnegative λ_1 follows because of $p_1<p_2$. We can express x_2^* from the second equality in (5) and we will get (3). We can check that condition (2) is equivalent to the positivity of x_2^* given in (3). It can be verified that x_2^* satisfies the budget constraint.

Now, we show that the value in (3) lies really between the values which the efficient ($x_2^e := D(p_2) - q_1$) and the random ($x_2^r := D(p_2) - q_1 \frac{D(p_2)}{D(p_1)}$) rationing rules would suggest. We need the demand function of the Cobb-Douglas utility function

$$D(p) = \frac{\alpha \overline{m}}{p(\alpha + \beta)} \tag{7}$$

which is well known (see for example Varian (1992)). So

$$x_2^* = D(p_2) - q_1 \left(\frac{\alpha}{\alpha + \beta} \frac{D(p_2)}{D(p_1)} + \frac{\beta}{\alpha + \beta} \right) \tag{8}$$

Now using the fact that $D(p_2) < D(p_1)$ because of $p_1 < p_2$, we can verify that $x_2^e < x_2^* < x_2^r$ regarding the equalities below.

$$1 > \frac{\alpha}{\alpha + \beta} \frac{D(p_2)}{D(p_1)} + \frac{\beta}{\alpha + \beta} > \frac{D(p_2)}{D(p_1)} \tag{9}$$

To complete the proof of the first part of the proposition we still have to show that if $x_2 = 0$ is a solution of (4), then (2) cannot hold. We have to consider three cases: $x_1 = 0$, $0 < x_1 < q_1$ and $x_1 = q_1$.

(i) $x_1 = x_2 = 0$ cannot be a solution to (4) because $u(0, \overline{m}) = 0$ and positive utility level is attainable.

(ii) If $0 < x_1 < q_1$, then from (5) $\lambda_1 = 0$ will follow immediately. We already know that $\lambda_2 = 0$. Solving now for x_1, we will get $x_1 = \frac{\alpha \overline{m}}{(\alpha+\beta)p_1}$. But substituting this into the third inequality in (5) we will get a contradiction to (2).

(iii) If $x_1 = q_1$, then from the budget constraint we will obtain $p_1 q_1 \leq \overline{m}$. This is in contradiction to (2).

2. Controversially, let us assume that $x_2 > 0$ is a solution and (2) does not hold. We already saw in the first part that if $x_2 > 0$ is a solution, then x_2 must take the value given by (3). But by our assumption this is positive. Hence (2) must hold. So we have got to a contradiction. □

Remark 3.2. Considering equation (8) we can see that if β is close to zero, than our consumer will act approximately according to the random rationing rule, while if α is close to zero, than our consumer will act approximately according to the efficient rationing rule.

3.2 Quasilinear utility function

Now we consider the case of another frequently used utility function. We assume that the single consumer has a quasilinear utility function, particularly his utility function is $U(x,m) = u(x) + m$. Furthermore, we assume that u is twice continuously differentiable, $u' > 0$ and $u'' < 0$. Our consumer's utility maximizing problem assuming that the first firm is the low-price firm ($p_1 < p_2$) takes the form as below:

$$\begin{aligned} u(x_1+x_2) + \overline{m} - p_1x_1 - p_2x_2 &\rightarrow \max \\ x_1 &\leq q_1 \\ p_1x_1 + p_2x_2 &\leq \overline{m} \\ x_1, x_2 &\geq 0 \end{aligned} \tag{10}$$

Let us write down the Lagrangian belonging to problem (10): $\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2) =$

$$u(x_1+x_2) + \overline{m} - p_1x_1 - p_2x_2 - \lambda_1(x_1 - q_1) - \lambda_2(p_1x_1 + p_2x_2 - \overline{m})$$

The object function is twice continuously differentiable, strictly concave and the constraint functions are convex. Furthermore Slater's condition is satisfied because of the assumptions $q_1 > 0$ and $\overline{m} > 0$. Therefore the Kuhn-Tucker conditions (11) are equivalent to our problem (10).

$$\begin{array}{lll} \frac{\partial\mathcal{L}}{\partial x_1} = u'(x_1+x_2) - p_1 - \lambda_1 - \lambda_2 p_1 \leq 0 & \text{and } \frac{\partial\mathcal{L}}{\partial x_1} = 0, & \text{if } x_1 > 0; \\ \frac{\partial\mathcal{L}}{\partial x_2} = u'(x_1+x_2) - p_2 - \lambda_2 p_2 \leq 0 & \text{and } \frac{\partial\mathcal{L}}{\partial x_2} = 0, & \text{if } x_2 > 0; \\ \frac{\partial\mathcal{L}}{\partial \lambda_1} = q_1 - x_1 \geq 0 & \text{and } \frac{\partial\mathcal{L}}{\partial \lambda_1} = 0, & \text{if } \lambda_1 > 0; \\ \frac{\partial\mathcal{L}}{\partial \lambda_2} = \overline{m} - p_1x_1 - p_2x_2 \geq 0 & \text{and } \frac{\partial\mathcal{L}}{\partial \lambda_2} = 0, & \text{if } \lambda_2 > 0. \end{array} \tag{11}$$

We can obtain the demand function for the quasilinear utility function easily. The individual demand function of a consumer with a quasilinear utility function is

$$d(p) = \begin{cases} (u')^{-1}(p), & \text{if } \; u'(\overline{m}/p) < p, \\ \overline{m}/p, & \text{if } \; u'(\overline{m}/p) \geq p. \end{cases} \tag{12}$$

Now let us turn back to our original problem (11). As a first step let us regard the following proposition:

Proposition 3.3. *There is only one consumer on a duopol market. His utility function is $U(x, m) = u(x) + m$, where $u \in \mathcal{C}^2(\mathbb{R}_+)$, $u' > 0$ and $u'' < 0$. His money stock is positive and denoted by $\overline{m}$. The duopolists' prices are given, and let $0 < p_1 < p_2$. The low-price firm is offering $q_1 > 0$. There exists a unique solution to problem (10) and let x_1^*, x_2^* its solution. Then*

1. *if $x_2^* > 0$, $u'(q_1 + x_2^*) > p_2$*
   (a) *and $u'(\frac{\overline{m}}{p_1}) \geq p_1$, then the consumer's behaviour is following the random rationing rule;*
   (b) *and $u'(\frac{\overline{m}}{p_1}) < p_1$, then the consumer will demand even more than the random rule would suggest;*
2. *if $x_2^* > 0$ and $u'(q_1 + x_2^*) = p_2$, then the consumer's behaviour is following the efficient rationing rule.*

Proof. Let x_1^*, x_2^*, λ_1^* and λ_2^* be a solution of problem (11). Our assumptions about u assures the existence and the uniqueness of the solution x_1^*, x_2^* of problem (10), because the constraint set is nonempty, compact and convex and the object function is strictly concave.

First, we will show that the positivity of x_2^* implicates the positivity of x_1 and λ_1. Let us assume that $\lambda_1 = 0$. Hence, if the first condition in (11) is satisfied, then the second condition will hold as strict inequality. So $x_2 = 0$ would follow, which is a contradiction. Therefore we conclude $\lambda_1 > 0$. Now $\lambda_1 > 0$ implies $x_1 > 0$ because $x_1 = q_1$ holds by the third complementary condition. Thus, we have only to consider two cases depending on the relation in the last inequality of (11).

We have to examine under which conditions there are existing $\lambda_1^* > 0$, $\lambda_2^* \geq 0$ such that together with x_1^*, x_2^* they are a solution of (11). Let us write down the first two equalities.

$$\begin{bmatrix} u'(q_1 + x_2) - p_1 \\ u'(q_1 + x_2) - p_2 \end{bmatrix} = \begin{bmatrix} 1 & p_1 \\ 0 & p_2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \tag{13}$$

This equality system has to be solvable for positive λ_1 and nonnegative λ_2. The matrix of (13) is invertible and we can obtain the next equivalent system.

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \frac{1}{p_2} \begin{bmatrix} p_2 & -p_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u'(q_1 + x_2) - p_1 \\ u'(q_1 + x_2) - p_2 \end{bmatrix} \tag{14}$$

The positivity of λ_1 results from the assumptions $p_2 > p_1$ and $u' > 0$. For λ_2 we have to consider two cases.

1. In the first point of the proposition we made the following assumption:

$$u'(q_1 + x_2^*) > p_2 \tag{15}$$

The positivity of λ_2 is equivalent to assumption (15) by (14). Furthermore, $\lambda_2 > 0$ implies that equality holds in the last inequality of (11) by the complementary conditions. This means, that our consumer does spend his entire money stock. Solving the last two equalities we get $x_1^* = q_1$ and $x_2^* = \frac{\overline{m} - p_1 q_1}{p_2}$.

(a) Now we will show, that if (15) and $u'(\frac{\overline{m}}{p_1}) \geq p_1$ holds, then our consumer is acting according to the random rationing rule. Because $\overline{m}/p_2 < q_1 + x_2$ we conclude $u'(\overline{m}/p_2) > p_2$ from condition (15). Now using (12), we obtain $d(p_2) = \overline{m}/p_2$. If we use again (12) we will get $d(p_1) = \overline{m}/p_1$ because of assumption $u'(\overline{m}/p_1) \geq p_1$. Therefore, the

$$x_2^* = \frac{\overline{m} - p_1 q_1}{p_2} = \frac{\overline{m}}{p_2}\left(1 - \frac{q_1}{d(p_1)}\right) = d(p_2)\left(1 - \frac{q_1}{d(p_1)}\right) = x_2^r \tag{16}$$

equalities holds. We can verify that $q_1 \leq D(p_1)$, because otherwise we would get $x_2^* = 0$. We can recognize in (16) the random rationing rule.

(b) Now we assume that (15) and $u'(\overline{m}/p_1) < p_1$ holds. This second assumption is equivalent to $\overline{m}/p_1 > (u')^{-1}(p_1)$. We use again (12). Hence $d(p_1) = (u')^{-1}(p_1)$. Thus, we obtain the inequality stated below.

$$x_2^* = \frac{\overline{m} - p_1 q_1}{p_2} = \frac{\overline{m}}{p_2}\left(1 - \frac{q_1}{\overline{m}/p_1}\right) > \frac{\overline{m}}{p_2}\left(1 - \frac{q_1}{d(p_1)}\right) = d(p_2)\left(1 - \frac{q_1}{d(p_1)}\right) \tag{17}$$

This is what we wanted to prove. As a limiting case we will get, that if the amount of money spent at the low-price firm's product is almost negligible in relation to the money stock, then the residual demand would almost be equal to the demand.

Condition (15) means, that our consumer's marginal utility is greater than the high price. So our consumer is ready to spend his entire money stock on the product.

2. We have to consider the case of $\lambda_2 = 0$. In the second point of the proposition we assumed that the optimal solution satisfies the following equality.

$$u'(q_1 + x_2^*) = p_2 \tag{18}$$

This is exactly equivalent to $\lambda_2 = 0$ by (14). Hence $x_2^* = (u')^{-1}(p_2) - q_1$. Therefore, our consumer is acting according to the efficient rationing rule. Condition (18) means that our consumer's marginal utility is equal to the price set by the high-price firm. We saw that the utility of holding money could hinder our consumer to spend his entire money stock on the product, because the equality in the forth condition in (11) is not assured. □

At the first look the condition for the occurrence of the efficient rationing rule is more plausible. If we accepted that money means in this context a composite commodity, then it would be quite unrealistic to assume that our consumer would consume only the product offered by our duopolists, which he actually would, if he acted according to the random rationing rule by the proposition. In defense we could bring forward the extreme case, that the product sold by the duopolists is the only basic good for survival and that our consumer is too poor to spend money on other goods. Another way to explain, why our consumer will act according to the random rationing rule is to speak really of money instead of a composite good. Then one could say, what is quite realistic, that our consumer first decides how much money he is willing to spend at most on the product offered by the duopolists. Thus, $\overline{m}$ would mean that value. Of course this behaviour contradicts to global utility maximization.

Without calculating too much one would surely suggest the efficient rationing rule to be applied for the following reasoning. The individual demand function tells our consumer how many products he will buy at the high price, particularly $d(p_2)$. At the low price he bought q_1. Now he obviously wants to buy $\max\{d(p_2) - q_1, 0\}$ products from the high-price firm. The only pitfall in this way of arguing is, that we neglect the income effect, which results from the fact that he bought the first q_1 products cheaper and so we must not use directly $d(p_2)$ to calculate his extra demand at the price level p_2. In fact in the case of a quasilinear utility function, there will be only an income effect, if the consumer's budget constraint is binding.

The next proposition summarizes the entire solution of problem (10).

Proposition 3.4. *Under the assumptions of proposition 3.3 the explicit solution of problem (10) is the following:*

1. *if* $u'(0) \leq p_1$*, then* $x_1^* = 0$ *and* $x_2^* = 0$*;*
2. *if* $u'(0) > p_1$ *and* $\overline{m} \leq p_1 q_1$*, then* $x_1^* = \min\{(u')^{-1}(p_1),\ \overline{m}/p_1\}$ *and* $x_2^* = 0$*;*
3. *if* $u'(0) > p_1$*,* $\overline{m} > p_1 q_1$ *and* $u'(q_1 + \frac{\overline{m} - p_1 q_1}{p_2}) > p_2$*, then* $x_1^* = q_1$ *and* $x_2^* = \frac{\overline{m} - p_1 q_1}{p_2}$*;*
4. *if* $u'(0) > p_1$*,* $\overline{m} > p_1 q_1$*,* $u'(q_1) > p_1$ *and* $u'(q_1 + \frac{\overline{m} - p_1 q_1}{p_2}) \leq p_2$*, then* $x_1^* = q_1$ *and* $x_2^* = \max\{(u')^{-1}(p_2) - q_1, 0\}$*.*
5. *if* $u'(0) > p_1$*,* $\overline{m} > p_1 q_1$*,* $u'(q_1) \leq p_1$ *and* $u'(q_1 + \frac{\overline{m} - p_1 q_1}{p_2}) \leq p_2$*, then* $x_1^* = (u')^{-1}(p_1)$ *and* $x_2^* = 0$*.*

Proof. 1. Let us assume that in spite contrast with our assumption x_1^* or x_2^* is positive. Hence $p_2 > p_1 \geq u'(0) > u'(x_1^* + x_2^*)$. Therefore, the first two conditions in (11) could not been satisfied, which is a contradiction.

2. Obviously $\overline{m} \leq p_1 q_1$ implies $x_1^* \leq q_1$. If we suppose that $(u')^{-1}(p_1) \geq \frac{\overline{m}}{p_1}$, then we can verify that $x_1^* = \overline{m}/p_1$, $x_2^* = 0$, $\lambda_1^* = 0$ and $\lambda_2^* = \frac{u'(\overline{m}/p_1) - p_1}{p_1}$ is a solution of problem (11).

Otherwise $x_1^* = (u')^{-1}(p_1)$, $x_2^* = 0$, $\lambda_1^* = 0$ and $\lambda_2^* = 0$ is a solution of problem (11).

3. We have to verify that if $u'(0) > p_1$, $\overline{m} > p_1 q_1$ and $u'(q_1 + \frac{\overline{m} - p_1 q_1}{p_2}) > p_2$, then $x_1^* = q_1$ and $x_2^* = \frac{\overline{m} - p_1 q_1}{p_2}$ will be a solution of (11). We immediately see that the last two conditions in (11) hold as equalities for x_1^* and x_2^*. Therefore we have to show that there are existing adequate nonnegative λ_1^* and λ_2^*, which are together with x_1^* and x_2^* a solution of (11). But regarding our assumptions we have already shown this in the proof of the previous proposition.

4. We have to consider two cases. In the first case let us assume that $u'(q_1) > p_2$. This implies that there exists a value $\hat{x}_2 \in (0, \frac{\overline{m} - p_1 q_1}{p_2})$ such that $u'(q_1 + \hat{x}_2) = p_2$. Now applying the second part of proposition 3.3 we obtain what has to be proved.

In the second case we now assume the opposite, particularly $u'(q_1) \le p_2$. We now show that the solution is $x_1^* = q_1$ and $x_2^* = 0$. The last two conditions in (11) are clearly satisfied. From the last one we further get $\lambda_2 = 0$. The second condition can now be written as $u'(q_1) \le p_2$ which is now fulfiled by assumption. The first condition takes the $u'(q_1) = p_1 + \lambda_1$ form because of the positivity of x_1^*. This equation is solvable for nonnegative λ_1 because $p_1 < p_2$ and in point 4 we already assumed that $u'(q_1) > p_1$.

5. We have only to verify that $x_1^* = (u')^{-1}(p_1)$, $x_2^* = 0$, $\lambda_1^* = 0$ and $\lambda_2^* = 0$ is a solution to problem (11). But this is obvious. $\square$

Remark 3.5. The last condition in point 5 of the proposition above is redundant. It has only be included because with it, it is easy to see that none of the possibilities have been neglected in the solution of problem (10).

4 Summary

We have investigated the residual demand of a single consumer in a duopoly market. We have deduced the behaviour of a single consumer in a duopolistic market in cases of quasilinear and Cobb-Douglas utility functions. Finally, we have compared the obtained results with the values suggested by the two most frequently used rationing rules.

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IV. Existence of pure strategy Nash equilibrium in Bertrand-Edgeworth oligopolies *

Abstract:

This article is searching for necessary and sufficient conditions which are to be imposed on the demand curve to guarantee the existence of pure strategy Nash equilibrium in a Bertrand-Edgeworth game with capacity constraints.

Keywords: Duopoly; Oligopoly.

JEL classification: D43; L13.

1 Introduction

We will investigate Bertrand-Edgeworth oligopoly with capacity constraints. We assume that the oligopolists' products are homogeneous. Furthermore we assume that there is no advertising, no possibility of outside entry into the market, and that the oligopolists possess complete information. In the Bertrand-Edgeworth game quantities and prices are both decision variables.

For a full specification of the model we need a so-called rationing rule. The aggregate demand function and the rationing rule together contain enough information on the determination of the sales of the oligopolists. We will only consider the two most frequently used rationing rules in the literature: the efficient and the proportional rationing rules. For a description of these rationing rules see for example Tirole (1988).

It has been shown for linear demand curves that when capacities are either small or large, then the Bertrand-Edgeworth duopoly with capacity constraints has an equilibrium in pure strategies (see Tirole (1988) or Wolfstetter (1993)). However, for capacities in an intermediate range, the model only has an equilibrium in mixed strategies. The mixed strategy equilibrium was computed in closed form by Beckmann (1965) for proportional rationing and by Levitan and Shubik (1972) for efficient rationing. Dasgupta and Maskin (1986) demonstrated the existence of mixed strategy equilibrium in the case of proportional rationing for demand curves which intersect both axes.

In section 2 we will show that if we impose assumptions on the elasticity of the demand curve, then pure strategy equilibrium will exist at all capacity levels in a Bertrand-Edgeworth duopoly. So for a certain class of demand curves a nondegenerate mixed strategy equilibrium will never arise.

In section 3 we will consider the oligopolistic case. We will show that as the number of firms increases, the Nash equilibrium price approaches the oligopolists' marginal costs. Similar convergence results have been obtained by Vives (1986) for efficient rationing and by Allen and Hellwig (1986) for proportional rationing. Due to the assumptions imposed on the demand curve our proof will be very simple.

2 Duopoly

First we need to specify the class of demand functions we will investigate.

Assumption 2.1. $\forall p > 0 : D(p) > 0$ and $D'(p) < 0$.

We denote by $\epsilon(p)$ the price elasticity of the demand curve. Regarding the oligopolists we make the following assumptions:

Assumption 2.2. There are N oligopolists on the market with zero marginal costs and $0 < k_i$ capacity constraints ($i \in [1..N]$). Each of them can set his price (p_i) and quantity (q_i) simultaneously.

In this section we will only consider duopolies.

The following proposition formulates a necessary and sufficient condition, which has to be imposed on the demand curve to guarantee the existence of pure strategy equilibrium in the case of efficient rationing.

Proposition 2.3. *Under the assumptions of 2.1, 2.2 and efficient rationing we can formulate the statements below about the corresponding Bertrand-Edgeworth duopoly game:*

1. *If*
$$\forall p > 0 : \epsilon(p) \leq -1, \tag{1}$$
then there exists a unique pure strategy Nash equilibrium for all $k_1 > 0$ and $k_2 > 0$. The equilibrium is given by
$$q_i^* = k_i \quad \textit{and} \quad p_1^* = p_2^* = D^{-1}(k_1 + k_2). \tag{2}$$

2. *If D' is continuous and $\exists p > 0 : \epsilon(p) > -1$ then there are positive k_1 and k_2 capacity constraints, such that pure strategy Nash equilibrium does not exist.*

Proof. 1. First we check that (1) implies $\lim_{p\to 0} D(p) = \infty$. Assume not; then $\lim_{p\to 0} D(p) < \infty$ because D is decreasing, and so $\lim_{p\to 0} pD(p) = 0$ would follow. From (1) we obtain, that $pD(p)$ is nonincreasing on $(0, \infty)$. Hence we get $\forall p > 0 : pD(p) \leq 0$, which contradicts the obviously true $\forall p > 0 : pD(p) > 0$ statement. So we can conclude that a demand curve satisfying (1) does not cut the horizontal axis. Hence, $D^{-1}(k_1 + k_2)$ is well defined and $p_i^* > 0$.

Now we will show that only (2) can be an equilibrium. No equilibrium can exist with $p_1 < p_2$ because, if $D(p_1) > k_1$, firm 1 will want to increase its price, and if $D(p_1) \leq k_1$, firm 2 will wish to reduce its price below p_2. Similarly, no equilibrium is possible with $p_2 > p_1$. There cannot be an equilibrium with $p_1 = p_2 > p_i^*$, since both firms have the incentive to lower their prices slightly. It is obvious that a price below p_i^* cannot be rational for any firm. Hence, we can rule out prices below p_i^*.

Finally, we have to show that raising prices unilaterally above p_i^* will not increase firm i's profit. We will show this for firm 1. Therefore we can establish that the residual profit function for firm 1 using the residual demand function does not increase in price. Under efficient rationing the residual profit function is: $\pi^r(p) = pD^r(p) = p(D(p) - k_2)$ for $p > p_2^*$. The nonpositivity of the first derivative is a sufficient condition, or formally

$$\frac{\mathrm{d}\pi^r}{\mathrm{d}p}(p) = pD'(p) + D(p) - k_2 \leq 0 \quad \Leftrightarrow \quad \epsilon(p) \leq -1 + \frac{k_2}{D(p)}. \tag{3}$$

This inequality is satisfied because of assumption (1).

2. Define the function $F(p) := pD'(p) + D(p)$. We can pick an open interval $I = (a, b)$ from $F^{-1}((0, \infty))$ and fix any $\tilde{p} \in I$. Obviously we can choose a capacity $0 < k_1 < D(b)$ for firm 1 such that $F(\tilde{p}) - k_1 > 0$. We can verify that $D(\tilde{p}) > k_1$ also holds. So we can set the capacity for firm 2 as $k_2 = D(\tilde{p}) - k_1$.

Reasoning similar to that in point 1 shows that only $p_i := \tilde{p}$ could be an equilibrium price. But in the case of $p_i = \tilde{p}$ firm 2 has an incentive to raise its price because $\tilde{p}D'(\tilde{p}) + D(\tilde{p}) - k_1 = F(\tilde{p}) - k_1 > 0$. □

For example the demand function $D(p) = p^{-\frac{1}{\alpha}}$, where $p \geq 0$ and $0 < \alpha \leq 1$, satisfies the assumptions of point 1 of proposition 2.3. So for these the Bertrand-Edgeworth game will have a pure strategy equilibrium.

Assumption 2.1 can be replaced in proposition 2.3 with $\forall \overline{p} > p > 0 : D(p) > 0$, $D'(p) < 0$ and $\forall p \geq \overline{p} : D(p) = 0$. We have only to consider that both firms will set a price below $\overline{p}$. We can check that if $\lim_{p \to \overline{p}-0} D'(p)$ is bounded, then $\lim_{p \to \overline{p}-0} \epsilon(p) = -\infty$. Hence, there are demand curves that are price elastic at all price levels and cut the vertical axis. As we already showed in the proof of proposition 2.3 they cannot cut the horizontal axis.

Dasgupta and Maskin (1986) proved in their third annotation that in case of random rationing a pure strategy Nash equilibrium may not exist, if the demand function is price inelastic at the $D^{-1}(k_1 + k_2)$ price. So an identical proposition to 2.3 holds for proportional rationing. This can be demonstrated as proposition 2.3. One difference is that we have to use the

$$\pi^r(p) = pD^r(p) = pD(p)\left(1 - \frac{k_2}{D(p_2^*)}\right) \tag{4}$$

residual profit function. The other difference is that the nonexistence part can be shown more easily. Particularly for $k_1 = k_2 = D(\tilde{p})/2$ capacity constraints, there does not exist a pure strategy equilibrium. Allen and Hellwig (1986) gave also in their proposition 3.1 a necessary and sufficient condition for the existence of pure strategy equilibrium for proportional rationing.

Considering the proof of proposition 2.3 we can recognize that the selection of such capacity levels for which the Bertrand-Edgeworth game has no pure strategy equilibria can require the selection of a very small capacity constraint for firm 1. Therefore we can say more in the case of efficient rationing.

Proposition 2.4. *Under the assumptions of 2.1, 2.2, efficient rationing and assuming that the set of admissible capacities is*

$$\mathrm{K}_\alpha := \{(k_1, k_2) \in \mathbb{R}^2 \; |k_i > 0, \; \frac{k_i}{k_1 + k_2} \geq \alpha, \; i = 1, 2\} \tag{5}$$

for some $0 < \alpha \leq \frac{1}{2}$*, we can make the following statements about the corresponding Bertrand-Edgeworth duopoly game:*

1. *If*

$$\forall p > 0 : \epsilon(p) \leq -1 + \alpha \tag{6}$$

then there exists a unique pure strategy Nash equilibrium for all $(k_1, k_2) \in \mathrm{K}_\alpha$*. The equilibrium is given by (2).*

2. *If* D' *is continuous and*

$$\exists p > 0 : \epsilon(p) > -1 + \alpha \tag{7}$$

then there are $(k_1, k_2) \in \mathrm{K}_\alpha$ *so that pure strategy equilibrium does not exist.*

Proof. 1. As in the proof of proposition 2.3 we have to show that (6) implies $\lim_{p \to 0} D(p) = \infty$. First we have to prove that $\lim_{p \to 0} D^r(p) = \infty$. By using now the residual demand function we can do this similarly to the proof of proposition 2.3. From that $\lim_{p \to 0} D(p) = \infty$ follows immediately.

As we have already seen in the proof of proposition 2.3 the only candidate for an equilibrium price is $p^* := p_1^* = p_2^*$. We have to show that raising prices unilaterally above p_i^* does not increase firm i's profit. We will prove this for firm 1. Again, condition (3) has to be verified. But (3) is satisfied, because we have for all $p > p_2^* : D(p) < k_1 + k_2$ and in consideration of our assumption (6)

$$\epsilon(p) \leq -1 + \alpha \leq -1 + \frac{k_2}{k_1 + k_2} < -1 + \frac{k_2}{D(p)} \tag{8}$$

holds for all $(k_1, k_2) \in \mathrm{K}_\alpha$. So we can conclude that it is not worthwhile for firm 1 to set its price above p_1^*.

2. Define the function $G(p) := pD'(p) + (1 - \alpha)D(p)$. G is continuous. Therefore, we can pick an open interval $I = (a, b)$ from $G^{-1}((0, \infty))$, because we assumed (7). Fix any $\tilde{p} \in I$. Let $k_1 := \alpha D(\tilde{p})$ and $k_2 := (1 - \alpha)D(\tilde{p})$. It is obvious that $(k_1, k_2) \in \mathrm{K}_\alpha$ and $D(b) < k_1 + k_2 < D(a)$. But now in the case of $p_1 = p_2 = \tilde{p}$ firm 2 has an incentive to raise its price because $\tilde{p}D'(\tilde{p}) + D(\tilde{p}) - k_1 = G(\tilde{p}) > 0$. □

Restricting the capacities to K_α implies that one firm's capacity could not be arbitrarily small relative to the other firm's capacity. This restriction is quite acceptable for some α because if we want to model a duopoly, then we essentially will not be interested in a market in which one firm is relatively negligible with respect to the other firm.

The result of proposition 2.4 is that as long as the size of both firms is significant relative to each other, we can assure the existence of pure strategy equilibrium even if the demand curve has price elastic parts. This result is considerable because we now know that there are demand curves with price elastic parts and with corresponding ranges of capacities for which the Bertrand-Edgeworth game possesses equilibrium in pure strategies, such that even a monopoly without capacity constraint has a profitmaximizing price. Particularly when $\alpha = \frac{1}{2}$ the two firms have the same capacity.

One must also be aware that efficient rationing cannot be replaced by proportional rationing in proposition 2.4.

3 Oligopoly

We can state analogous propositions to those in section 2 for oligopolies.

Proposition 3.1. *Under the assumptions of 2.1, 2.2, the continuity of D' and efficient or proportional rationing the Bertrand-Edgeworth oligopoly game has a unique pure strategy Nash equilibrium for all $k_i > 0$ capacities, if and only if the demand curve satisfies (1). If (1) holds, then the equilibrium is* $\forall i \in [1..N]$:

$$q_i^* = k_i \quad \textit{and} \quad p_i^* = D^{-1}(\sum_{i=1}^{N} k_i). \tag{9}$$

Proof. The proof of sufficiency can be done similarly to the proof in proposition 2.3. We have to prove that if the firms' prices are not all identical, then we cannot have a pure strategy equilibrium. Furthermore we have to show that (9) is an equilibrium.

In order to prove the necessity we can choose for the first $N-1$ firms capacity constraints such that $F(\tilde{p}) - \sum_{i=1}^{N-1} k_i > 0$. Now we can set the capacity for firm N as $k_N = D(\tilde{p}) - \sum_{i=1}^{N-1} k_i$. It can be verified that firm N has an incentive to raise its price. □

Corollary 3.2. *If the aggregate capacity $\sum_i k_i$ tends to infinity as we increase the number of oligopolists to infinity, then the equilibrium price approaches zero, which is assumed to equal the oligopolists' marginal costs.*

Similar results have been obtained by Vives (1986) for efficient rationing and by Allen and Hellwig (1986) for proportional rationing. Our proof was very simple because due to our assumptions imposed on the demand curve we did not have to deal with mixed strategy equilibria.

According to proposition 2.4 we can state more in the case of efficient rationing and equal capacities.

Proposition 3.3. *Under the assumptions of 2.1, 2.2, the continuity of D', efficient rationing, and equal capacities* (k) *our Bertrand-Edgeworth oligopoly* *game has a unique pure strategy Nash equilibrium for all $k > 0$, if and only if*

$$\forall p > 0 : \epsilon(p) \leq -1 + \frac{1}{N}. \tag{10}$$

If (10) holds, then the equilibrium is given by $\forall i \in [1..N]$:

$$q_i^* = k \quad \textit{and} \quad p_i^* = D^{-1}(Nk). \tag{11}$$

Proof. The proof is analogous to that of proposition 2.4. □

To guarantee the existence of a pure strategy Nash equilibrium at all capacity levels under the assumptions of proposition 3.3, we need not assume a price elastic demand curve. But the more oligopolists we have, the less demand curves secure equilibrium at all capacity levels.

4 Conclusions

We have shown that for a special class of demand functions the lack of pure strategy equilibrium does not arise in the Bertrand-Edgeworth game with capacity constraints. Furthermore for demand functions outside of this class there always can be found capacity constraints, such that pure strategy equilibrium does not exist. However demand functions in this special class do not intersect the horizontal axis. For efficient rationing and equal capacities they still can be price elastic.

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V. A two-stage Bertrand-Edgeworth game *

Abstract:

In our investigation we are expanding a Bertrand-Edgeworth duopoly into a two-stage game in which during the first stage the firms can select their rationing rule. We will show that under certain conditions the efficient rationing rule is an equilibrium action of the first stage.

JEL classification: D43; L13.

Keywords: Duopoly; Rationing.

1 Introduction

We will investigate a two-stage extension of the capacity constraint Bertrand-Edgeworth duopoly game. In stage one both firms simultaneously announce a rationing rule, according to which they will serve the consumers, if they become the low-price firm. In stage two they are engaged in a modified capacity constrained Bertrand-Edgeworth game. We will refer to this game as the *rationing game.*

Davidson and Deneckere (1986) already formulated a three-stage extension of the Bertrand-Edgeworth game in that each duopolist can select the way it will serve the consumers, if it becomes the low-price firm. In their model the firms compared to our rationing game additionally can select their capacity levels. They established that in a subgame perfect Nash equilibrium the duopolists will serve the consumers according to the random rationing rule. Their result assumes that the duopolists are risk-neutral. On that point our analyzes will differ.

For a full specification of the Bertrand-Edgeworth game we need a so-called rationing rule. The aggregate demand function and the rationing rule together contain enough information on the determination of the duopolists' sales. We will introduce the notion of combined rationing, which contains as special cases the two most frequently used rationing rules, the so-called efficient and random rationing rules. For a description of these rationing rules see for example Tirole (1988).

It has been shown for linear demand curves that when capacities are either small or large, then the Bertrand-Edgeworth duopoly with capacity constraints has an equilibrium in pure strategies (see Wolfstetter, 1993). However, for capacities in an intermediate range, the model only has an equilibrium in mixed strategies. The mixed strategy equilibrium was computed in closed form by Beckmann (1965) for random rationing and by Levitan and Shubik (1972) for efficient rationing. Dasgupta and Maskin (1986b) demonstrated the existence of mixed strategy equilibrium in the case of random rationing for demand curves which intersect both axes.

In Section 2 we will introduce the set of rationing rules from which the firms can choose their first stage action. In Section 3 we will determine the set of those capacity levels for which the Bertrand-Edgeworth game has a pure strategy equilibrium. In Section 4 we will establish that if the firms have special preferences above the set of expected profits and uncertainty, then in the first stage of the rationing game the efficient rationing rule is an equilibrium action.

2 Rationing rules

We impose the following assumptions on the demand curve.

Assumption 2.1. We shall consider demand curves that are strictly decreasing, continuously differentiable, and intersect both axis.

Assumption 2.2. The function $G(p) := pD'(p) + D(p)$ is strictly decreasing.

A monopolist facing a demand curve satisfying Assumptions 2.1 and 2.2 has a unique positive revenue maximizing price. Let us denote the set of demand curves fulfilling Assumptions 2.1 and 2.2 by $\mathcal{D}$. The demand facing firm $j \in$

$\{1, 2\}$ is given by a rationing rule. In our model we allow the duopolists only to choose from a special class of rationing rules. We call these combined rationing rules.

Definition 2.3. A function $\Delta : \mathcal{D} \times \mathbb{R}^2_+ \times \mathbb{R}^2_+ \to \mathbb{R}^2_+$ is called a *combined* rationing rule with parameter $\lambda \in [0, 1]$, if the demand firm $j \in \{1, 2\}$ faces is given by

$$\Delta_j(D, p_1, p_2, q_1, q_2) := \begin{cases} D(p_j) & \text{if} \quad p_j < p_i, \ i \neq j; \\ \frac{q_j}{q_1+q_2} D(p_j) & \text{if} \quad p_j = p_i, \ i \neq j; \\ \max\left(D(p_j) - \alpha(p_i, p_j) q_i, 0\right) & \text{if} \quad p_j > p_i, \ i \neq j; \end{cases}$$

where $\alpha(p_i, p_j) = (1 - \lambda) \frac{D(p_j)}{D(p_i)} + \lambda$.

The efficient and the random rationing rules are also combined rationing rules. We can see this by selecting for λ in Definition 2.3 the values 1 and 0 respectively.

We describe two different markets in which a combined rationing rule can be implemented. First, suppose that there are n consumers with identical individual demand functions $d(.)$, who are served by the low-price firm in order of their arrival. Let n be sufficiently large, so that the amount purchased by the marginal consumer, who still obtains a positive level of the product, can be neglected. Let $p_1 < p_2$ and $q_1 \leq D(p_1) = nd(p_1)$. The low-price firm can serve $m := \lfloor q_1/d(p_1) \rfloor$ consumers totally. Fix an arbitrary value $0 \leq \lambda \leq 1$. Assume that firm 1 serves $m_1 := \lfloor (1-\lambda) q_1/d(p_1) \rfloor$ consumers completely. Each remaining consumer obtains $\frac{q_1 - m_1 d(p_1)}{n - m_1}$ amount of the product. In the described case the residual demand is

$$D^r(p_2) \approx D(p_2) - (1 - \lambda) q_1 \frac{D(p_2)}{D(p_1)} - \lambda q_1,$$

if n is sufficiently large. The way how the low-price firm serves the consumers combines the two different methods, how on the market with identical consumers the efficient and the random rationing rule can be achieved (see Davidson and Deneckere, 1986).

Second, we assume that $D(p)$ is the summation of inelastic demands of heterogenous consumers, all of whom want to purchase one unit of the good, provided the price is below their reservation price. Suppose the low-price firm begins with selling $(1-\lambda)q_1$ output on a first-come-first-served basis. The consumers served in that way are a random sample of the consumer population. Hence, the demand of the so far unsatisfied consumers at price p_2 is $D(p_2) - (1-\lambda)q_1 D(p_2)/D(p_1)$. Thereafter, it sells the remaining λq_1 output to the consumers with the highest reservation values first. This leads to a combined rationing rule with parameter λ.

It is worthwhile to mention that if the demand side of the market can be described by a representative consumer having a Cobb-Douglas utility function $u(x,m) = Ax^{(1-\lambda)}m^{\lambda}$ where x is the amount purchased from the duopolists' product and m is the consumption from a composite commodity, then we obtain a combined rationing rule with parameter λ on the market (for details see Tasnádi, 1998).

3 Pure strategy equilibrium

For given λ_1 and λ_2 we determine the set of those capacity levels to which pure strategy equilibrium exists in the capacity constraint Bertrand-Edgeworth game. Let us remark that the existence of mixed strategy equilibrium follows easily from Dasgupta's and Maskin's Theorem 5 (1986a).

We assume without loss of generality that the marginal costs of the firms are zero. We consider the capacity constraints k_1 and k_2 of the two firms as given.

We restrict ourselves to capacities from the set

$$L := \{(k_1, k_2) \in \mathbb{R}^2_{++} | k_1 + k_2 \leq D(0)\}$$

because for capacities not in L the Bertrand-Edgeworth game reduces to the Bertrand duopoly, or it will not have a pure strategy equilibrium for any rationing rules. To any $\lambda_1, \lambda_2 \in [0,1]$ parameters describing the rationing rules of the firms, we introduce the set $K(\lambda_1, \lambda_2) \subset L$ containing those capacity levels for which the corresponding Bertrand-Edgeworth game possesses Nash equilibrium in pure strategies. Assumption 2.2 assures that $K(\lambda_1, \lambda_2)$ will not be empty.

Proposition 3.1. *The set $K(\lambda_1, \lambda_2)$ increases strictly if $\min\{\lambda_1, \lambda_2\}$ increases so far as $K(\lambda_1, \lambda_2) \neq L$. If $(k_1, k_2) \in K(\lambda_1, \lambda_2)$, then the pure strategy Nash equilibrium is given by*

$$q_i^* = k_i \quad \textit{and} \quad p^* = p_1^* = p_2^* = D^{-1}(k_1 + k_2). \tag{1}$$

Proof. First, we show that only (1) can be an equilibrium. No equilibrium can exist with $p_1 < p_2$ because, if $D(p_1) > k_1$, firm 1 will want to increase its price, and if $D(p_1) \leq k_1$, firm 2 will wish to reduce its price below p_2. Similarly, no equilibrium is possible with $p_2 > p_1$. There cannot be an equilibrium with $p_1 = p_2 > p^*$, since both firms have the incentive to lower their prices slightly. It is obvious that a price below p^* cannot be rational for any firm.

The price p^* is the only candidate for a pure strategy equilibrium price. The profit function of firm i for $p^* < p \leq \overline{p}$ is:

$$\pi_i(p) = pD^r(p) = p\left(D(p) - \lambda_j k_j - (1 - \lambda_j)k_j \frac{D(p)}{D(p_j^*)}\right),$$

where $j \neq i$ and $D^r(\overline{p}) = 0$. For prices greater than $\overline{p}$ the residual profit function is zero. Hence, setting prices unilaterally above $\overline{p}$ is not rational, because prices p^* yield positive profits. The profit function is nonincreasing for prices $p^* < p < \overline{p}$ because of Assumption 2.2, if

$$\frac{\mathrm{d}\pi_i}{\mathrm{d}p}(p^*) = (p^* D'(p^*) + D(p^*))\left(1 - (1 - \lambda_j)\frac{k_j}{k_i + k_j}\right) - \lambda_j k_j \leq 0 \tag{2}$$

holds. Rearranging (2) we obtain

$$G(D^{-1}(k_i + k_j)) \leq \frac{\lambda_j k_j}{1 - (1 - \lambda_j)\frac{k_j}{k_i + k_j}} = \frac{\lambda_j k_j (k_i + k_j)}{k_i + \lambda_j k_j}. \tag{3}$$

We have that $K(\lambda_1, \lambda_2)$ increases if $\min\{\lambda_1, \lambda_2\}$ increases, because (3) must hold for both firms and $\frac{\lambda_j k_j (k_i + k_j)}{k_i + \lambda_j k_j}$ is strictly increasing in λ_j.

It remains to show that the set $K(\lambda_1, \lambda_2)$ increases strictly. Let us introduce the following notations:

$$\begin{aligned} K^\alpha(\lambda_1, \lambda_2) &:= \{(k, \alpha k) \in L | (k, \alpha k) \in K(\lambda_1, \lambda_2)\}, \\ K^\alpha_*(\lambda_1, \lambda_2) &:= \{(k, \alpha k) \in L \mid k \in (0, D(0)/(1+\alpha)]\} \end{aligned}$$

for any $\alpha > 0$. Rearranging (3) and substituting equal capacities we obtain

$$\frac{G(D^{-1}((1+\alpha)k))}{k} \leq \frac{(1+\alpha)\lambda_j}{\alpha + \lambda_j}. \tag{4}$$

Obviously, an analogous condition to (4) must hold for firm j. Thus, $(k, \alpha k) \in K^\alpha(\lambda_1, \lambda_2)$ if and only if

$$\frac{G(D^{-1}((1+\alpha)k))}{k} \leq \min\left\{\frac{(1+\alpha)\lambda_1}{\alpha+\lambda_1}, \frac{(1+\alpha)\lambda_2}{\alpha+\lambda_2}\right\}. \tag{5}$$

Furthermore, the left side of (5) is continuous for all $k \in (0, \frac{D(0)}{1+\alpha}]$, therefore the set $K^\alpha(\lambda_1, \lambda_2)$ increases strictly if $\min\{\lambda_1, \lambda_2\}$ increases, as long as $K^\alpha(\lambda_1, \lambda_2) \neq K^\alpha_*(\lambda_1, \lambda_2)$, because the function $\frac{(1+\alpha)\lambda}{\alpha+\lambda}$ is strictly increasing in λ for $\lambda \in [0, 1]$.

If $K(\lambda_1, \lambda_2) \neq L$, then there is an $\alpha > 0$ so that $K^\alpha(\lambda_1, \lambda_2) \neq K^\alpha_*(\lambda_1, \lambda_2)$. Suppose that $\lambda_1 < \lambda_1' < \lambda_2$, then $K^\alpha(\lambda_1, \lambda_2)$ is a proper subset of $K^\alpha(\lambda_1', \lambda_2)$. Finally, since $K^\alpha(\lambda_1, \lambda_2) \subset K(\lambda_1, \lambda_2)$ and $K^\alpha(\lambda_1', \lambda_2) \setminus K^\alpha(\lambda_1, \lambda_2)$ is nonempty and disjoint from $K(\lambda_1, \lambda_2)$, therefore $K(\lambda_1, \lambda_2)$ is a proper subset of $K(\lambda_1', \lambda_2)$. We can argue similarly in the case of $\lambda_1 > \lambda_2$ and $\lambda_1 = \lambda_2$. □

If the demand curve is linear and if we restrict ourselves to symmetric capacities, then $K(\lambda_1, \lambda_2)$ has a simple structure, as we will establish in Proposition 3.2. We have to mention that in case of a linear demand curve the price and quantity units can be chosen so that the demand curve has the form $D(p) = 1 - p$. Let $H(\lambda_1, \lambda_2) := \{k \in (0, D(0)/2] \mid (k, k) \in K^1(\lambda_1, \lambda_2)\}$.

Proposition 3.2. *If the demand curve is $D(p) = 1 - p$, then*

$$H(\lambda_1, \lambda_2) = \left(0, \frac{1}{2}\min\left\{\frac{1+\lambda_1}{2+\lambda_1}, \frac{1+\lambda_2}{2+\lambda_2}\right\}\right]. \tag{6}$$

Proof. Regarding the proof of Proposition 3.1 we only have to determine those capacity constraints for which (2) holds for both firms in the case of equal capacities. Therefore, for firm $i \in \{1, 2\}$ the following inequality has to be satisfied.

$$(-p^* - 1 - p^*)(1 - \frac{1}{2}(1 - \lambda_i)) - \lambda_i k = (4k - 1)\frac{1}{2}(1 + \lambda_i) - \lambda_i k \leq 0 \qquad (7)$$

Rearranging (7) and regarding that it has to hold for both firms, we obtain (6). $\square$

If both firms are serving the consumers according to the efficient rationing rule, then by Proposition 3.2 we get $H(1, 1) = (0, 1/4]$. This well-known result can be found for instance in Wolfstetter (1993). Additionally, if both firms select the random rationing rule, then $H(0, 0) = (0, 1/3]$. This result can be found in Tirole (1988) for example.

4 The rationing game

In this section we only want to indicate that in the two-stage game the efficient rationing rule is under certain conditions an equilibrium first-stage action.

The action sets of both firms in stage one is $[0, 1]$ and in stage two it is the set of price distributions with finite variances above the set $[0, \hat{p}]$, where we denote by $\hat{p}$ the smallest price for that $D(\hat{p}) = 0$. A degenerated probability distribution corresponds to a pure strategy in stage two. Now we modify the payoff functions by assuming that the firms have preferences above the space of expected profits and profit variances, which can be determined by the chosen rationing rule and probability distributions.

Davidson and Deneckere (1986) found that random rationing is the equilibrium action of the appropriate stage in the case when both firms preferences depend only on their expected profits. This means that the firms are risk neutral. We investigate another extreme case in that both firms are extremely risk averse. Let the firms have the following lexicographic preferences $\succ \subset \mathbb{R}_+^2$

$$(e, v) \succ (e', v') \quad \Leftrightarrow \quad v < v' \text{ or } (v = v' \text{ and } e > e'),$$

where e, e' denote expected profits and v, v' denote variances.

We introduce the set valued function $\Lambda : L \to \mathcal{P}([0,1] \times [0,1])$ as follows

$$\Lambda(k_1, k_2) := \{(\lambda_1, \lambda_2) \mid \exists (k_1, k_2) \in K(\lambda_1, \lambda_2)\}.$$

Proposition 4.1. *If the two-stage game has a pure strategy subgame perfect Nash equilibrium, then choosing the efficient rationing rule in the first-stage is a subgame perfect Nash equilibrium action for both firms.*

Proof. If the two-stage game has a pure strategy subgame perfect Nash equilibrium, then $\Lambda(k_1, k_2) \neq \emptyset$. After any first stage action $(\lambda_1, \lambda_2) \in \Lambda(k_1, k_2)$ both firms will set their price to $p^* = D(k_1 + k_2)$ because of Proposition 3.1. Firms are indifferent between any rationing rule pair from set $\Lambda(k_1, k_2)$, because in equilibrium they all guarantee the same profits without uncertainty. The efficient rationing rule is always an equilibrium action of stage one because $(k_1, k_2) \in K(\lambda_1, \lambda_2)$ implies that $(k_1, k_2) \in K(1, 1)$ regarding Proposition 3.1. □

5 Concluding remarks

These results indicate that the equilibrium rationing rule may lie between the efficient and random rationing rule depending on the firms' preferences above expected profits and profit variances. This conjuncture deserves further analyzes, although in general the expected values and variances cannot be determined in closed form since in general the mixed strategy equilibrium cannot be expressed in closed form either.

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