Doctorate School in
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THESIS EXTRACT

Szabolcs András Kopányi

Dynamic Estimation of the Term Structure

Ph.D. dissertation

Supervisor:

Gábor Kőrösi CSc
senior research fellow (IE, HAS)

Budapest, 2009

Investments and Corporate Finance Department

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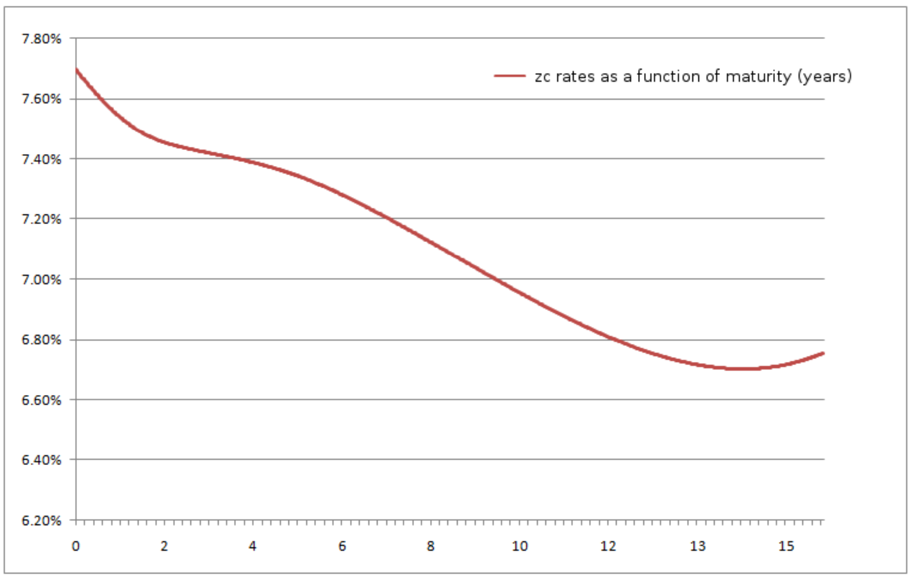
1. Introduction to Topic and Choice of Research

Bonds represent claims for future cash flows, show the time value of money. The term structure provides all the information required for expressing cash flows of different dates in a common language. Despite its high importance[1] the term structure is not directly observable.

Term structure estimation evolved into two distinct though still related problems of finance. The first tries to produce a continuous yield curve on the back of some traded prices: this is the *static approach*. The curve is a snapshot of a given market, just as shown on the example below of the Hungarian government bond market. The data source is the Hungarian Government Debt Management Agency (GDMA).

Figure 1

GDMA zero coupon yield curve on 2 Jan 2008



Source: Government Debt Management Agency (GDMA)

Let us assume, that I would like to get a zero coupon yield curve. I would like to get a curve which is a continuous function of time, but I come across difficulties. Firstly, the market

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[1] Extensive knowledge of the term structure is key in the following areas:

1. forecasting future interest rates, decision making support for economic actors (investment decisions of corporates, saving decisions of individuals),
2. monetary policy and its mechanisms,
3. debt management of treasuries (e.g. maturity profile),
4. pricing and hedging of interest rate derivatives (e.g. the value of both the most complicated interest rate derivatives and plain vanilla bonds (see: Arrow-Debreu prices) depend on rates).

trades coupon bearing bonds with YTM[2] or net price quotation and secondly maturities are scarce even in the case of the most liquid markets (i.e. there is no continuity at all). On a market scale, the number of cash flows is higher that the number of bonds (i.e. prices) and on the top of that prices or yields contain observation errors as a consequence of market conventions (e.g. bid-ask spread, rounding, the difference between on-the-run[3] and off-the-run[4] issues, taxation related distortions, etc.). Curve estimation is feasible via bootstrap method, Ordinary Least Squares (OLS) or Generalized Least Squares (GLS), yield curve fitting techniques (e.g. cubic splines).

The second problem in finance, which this thesis is devoted to, is related to the panel study approach and *focuses on dynamics of rates and the term structure*. The question is: how can we describe the evolution of interest rates over time? The concept is similar to how the evolution of a share price or a foreign exchange rate is estimated over time. Only similar, because the term structure – unlike share prices and FX rates – is not a scalar quantity. Different points (i.e. maturities) of the term structure cannot relate to each other in arbitrary ways, one must ensure that no-arbitrage rules apply. This viewpoint is shown on the chart below, displaying the evolution of the zero coupon yield curve as shown in Figure 1. over the period 2 January 2008 to 3 March 2008. The data source is again the Hungarian Government Debt Management Agency (GDMA).

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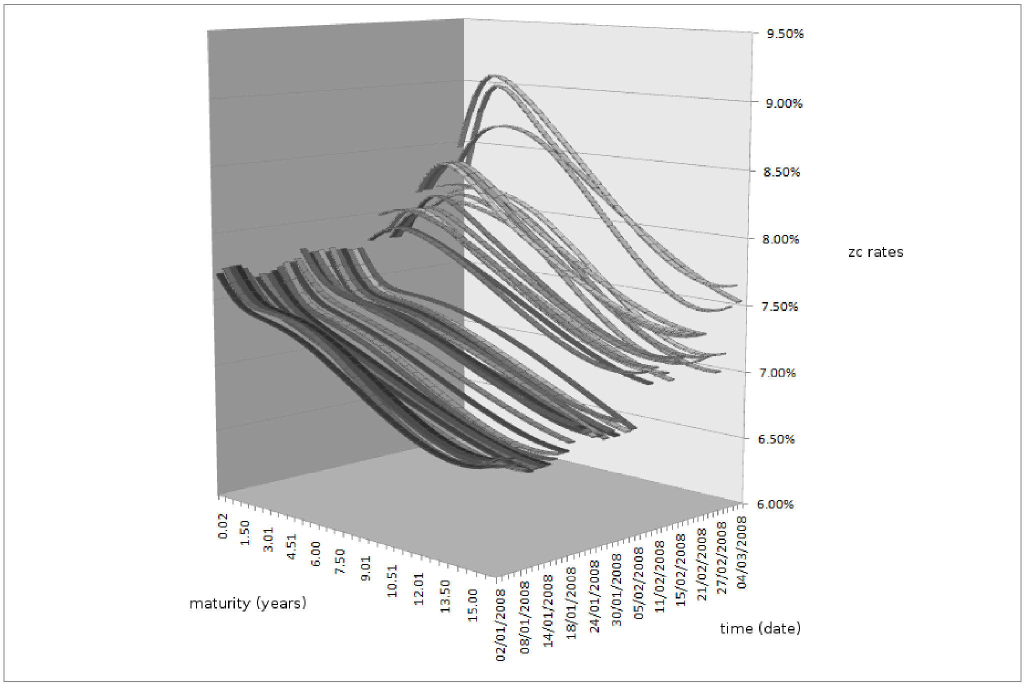
[2] yield to maturity
[3] active bonds with upcoming taps
[4] bonds with no future taps

Figure 2

GDMA zero coupon term structure between 2 Jan and 3 Mar 2008



Source: Government Debt Management Agency (GDMA)

Relying on time series in the estimation procedure might take many forms: we can apply a pure time series and a panel study approach. When finding the appropriate interest rate model we shall ensure that the sampling frequency is consistent with the underlying stochastic process of the IR model. Besides the panel study approach, dynamic term structure estimation also refers to time series analysis, where not the whole term structure, but only chosen maturities of it are modelled. Next to the handling of the time dimension the second key issue is the choice of the interest rate (IR) model. To find an appropriate interest rate model is in itself a highly complicated issue, since there are dozens of frequently quoted models in literature. There is no universal IR model, therefore researchers often find their models as part of the estimation procedure (non-parametric estimation). Estimation with structured models focuses on obtaining the distribution of the underlying stochastic variable(s) in the IR model. Shall this be infeasible (the pricing stochastic differential equation (SDE) has no analytic solution) various moments of these distributions are estimated. The underlying stochastic variable is often not observable (e.g. volatility in models with several factors), therefore it has to be estimated as part of the estimation procedure. Only the sky and the lack of creativity limits the scope for empirical models.

Once the estimation is done it is still too early to be popping Champagne corks, since the underlying model has to be examined from both statistical and economic perspective. In the former it has to be checked whether estimation residuals are consistent with our a-priori assumptions (e.g. zero expected value). Regarding economic viability one must examine if the model offers reasonable explanation for term structure dynamics over time in the sample. In the case of a mismatch researchers might have two different conclusions: either we might have chosen an inappropriate model or efficiency of the observed market can be doubted.

Yield curve estimation based on structured models[5], which approach is taken by this thesis, starts with the selection of an IR model[6] the parameters of which are then estimated. Structured models apply restrictions regarding relative moves along the yield curve, thus assuring no-arbitrage principles and allowing distributions of interest rates different than normal. These restrictions appear in the dynamics of state variables and in the functional form of the market price of risk in the model. They have a highly important role: they keep interest rate dynamics consistent, and enable the split between expected interest rates and risk premiums[7]. The early article of Sargent (1979) concludes at the expectations hypothesis, where investors of long bonds cannot expect systemic extra profits. More recent studies (e.g. Bekaert and Hodrick (2001)) led to different conclusions: holders of longer maturities can expect systemic extra profits, the scale of which is not constant over time though. Restrictions stemming from the consistency of the yield curve model this risk premium, too.

[5] First steps were taken by Sargent (1979), which examined the expectations hypothesis with a VAR study. Pearson and Sun (1994) build their model with the short rate (SR) and inflation as a latent factor; Litterman and Scheinkman (1991) use three latent factors in their widely known article, level slope and curvature to explain 97 percent of curve changes in their sample; Dai and Singleton (2000) concludes to use yield level, steepness and a so called butterfly factor which has a very similar meaning to curvature.

[6] In non-parametric modeling there is no a-priori model selection, but the model is selected within the estimation procedure.

[7] Restrictions on curve dynamics pose a challenge in estimation. Already affine models – where state variables follow affine diffusions – have nonlinear parameters. The lack of linearity makes OLS estimation impossible. Maximum likelihood (ML) estimation is not a solution either, because interest rates' density is not known in closed form.

IR models can be split into groups among several trade-off's:

1. According to modeling time: continuous and discrete time models,
2. According to primary modeling objective: equilibrium and no-arbitrage models,
3. According to the number of model variables: models with *1*, *2*, *…*, *N* variables,
4. According to allowed functional forms of interdependencies within the model[8]: affine, quadratic, regime shift and jump-diffusion models.

Continuous time models are supported by the following arguments: Ad 1) Term structure studies have no ideal time interval, continuous time models steer clear of the dilemma of choosing the optimal time interval. Ad 2) Continuous time models are widely used in the literature. In a few but more important cases bond and IR derivative prices are attainable via analytic formulae. Ad 3) Even if there are no analytic solutions, there are several estimation methods and numerical techniques. *Discrete time models* have the advantages: Ad 1) Real life does not happen in continuous time, prices change from one point of time to another (time related transaction costs even have a theoretical minimum). Ad 2) Discrete models are often easier to understand (e.g. binomial models). Ad 3) What is the point in starting with continuous time models, if either way we end up estimating them via their discretized counterparts (numerical techniques)?

Equilibrium models[9] share the primary modeling objective of forecasting the term structure of interest rates and the evolution of bond trading strategies[10]. Pioneering studies belonged into this model family, therefore equilibrium models are often referred to as classical models. The most important modeling blocks are assumptions on the stochastic dynamics of the SR and on investors' preferences (e.g. risk premiums, market price of risk). These models end up with an endogenous yield curve, which often fails to match the observed term structure. One of their biggest advantage is internal consistency, so that model parameters are relatively stable in time.

[8] Only selected models.

[9] e.g. Vasicek (1977), Cox et al. (1985) and Brennan and Schwartz (1979)

[10] see: Tuckman (1995)

No-arbitrage models[11] fit the observed market yield curve per definition. No-arbitrage rules have a strong argument: IR derivatives do not depend on investors' preferences. Exact fit in the sample has a drawback, too, as modeling parameters are not necessarily stable in time.

Arguments for *few model variables and relatively simple functional parametrization* are that so modeling is simpler and practitioners have a better chance to be able to use analytic pricing formulae. *More model variables and more complicated functional forms* are mainly applied to increase the model's complexity and flexibility in order to capture more from the observed market.

Affine models (see: Duffie and Kan (1996) and Dai and Singleton (2000)) have linear relationship between their variables, quadratic models (see: Ahn et al. (2002), Ahn et al. (2003) and Leippold and Wu (2002)) offer complexity beyond linearity in order to – at least according to authors of quadratic models – increase their modeling capacity. Models with regime changes (see: Bansal and Zhou (2002) and Bansal et al. (2004)) or jump diffusions (see: Duffie et al. (2000)) introduce shocks into the world of ordinary diffusions, thus moving models closer to real life.

[11] e.g.: Heath et al. (1992) and Ho and Lee (1986)

2. Methods Applied

The structure of the thesis is as follows. A short introduction comes first into available methods for estimation. This is followed by starting points and assumptions of structured models (assumpions on stochastic processes, fundamental basics of bond pricing, the market price of risk, etc.).

As a next logical link the dilemma of choosing the appropriate IR model is shown. This study focuses on continuous time models and offers in depth analysis for affine models. Affine models are shown according to Dai and Singleton (2000), as this article is one of the most thorough ones (including modeling possibilities and challanges) regarding classification of this model class. After this I make the reader familiar with other than affine modeling possibilities.

Model selection aspects are followed by a short introduction into technical problems of estimation. This chapter is less detailled than the modeling one simply because this thesis has a finance focus. Econometric methods are only used, their discussion solely relates to applicability issues.

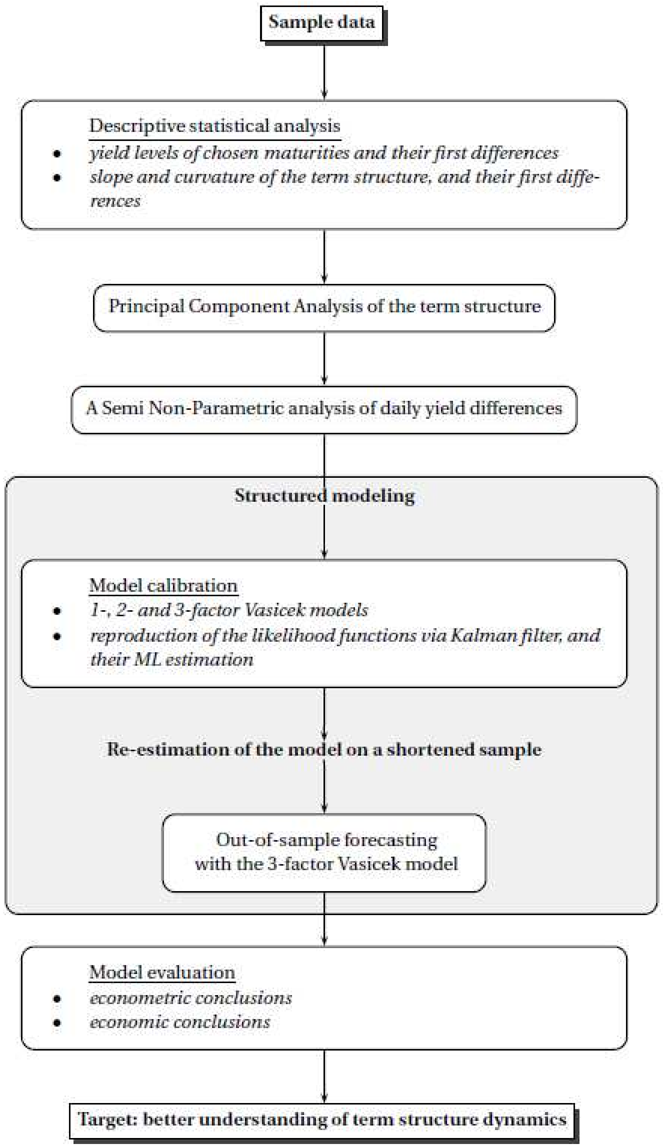
The review of modeling literature and the technical introduction into estimation issues is followed by empirical considerations of the author including research objectives, methodology, hypothesis setting and research plan. On the course of my empirical research, after a short descriptive analysis and Principal Factor Analysis (PCA), I implement a semi non-parametric (SNP) test on my zero coupon data set, then I calibrate Vasicek[12] type affine models with the help of the Kalman filter algorithm. I then continue with analysing out-of-sample forecasting skills of the models.

[12] see: Vasicek (1977), in a single factor case: $dr_t = \kappa(\theta - r_t)dt + \sigma dW_t$

The flowchart of my empirical research is shown on Figure 3.

Figure 3

The flowchart of my empirical research



Source: own study

3. Key Results of the Dissertation

My primarily deductive oriented research delivered the most important results in estimation and out-of-sample forecast of the Hungarian term structure of interest rates. As the dissertation shows, bookshelves groan under the studies carried out on US markets, many of them reaching the same conclusions but some of them leading to a stark contrast between authors' opinions. The most thorough analyses[13], regarding the Hungarian term structure have been carried out by the National Bank of Hungary (NBH) staff, but all these studies implement a static philosophy (the yield curve is calculated from observations of a single given date) and curve dynamics are rarely paid attention to. Therefore it is of the highest importance to implement the models shown before for the Hungarian term structure. With my model tailor-made for the Hungarian market one can analyse the term structure and its dynamics over time by quantitative methods. The NBH and the GDMA could also both benefit from my econometric model with which they could model the future evolution of the term structure. Last but not least, the simulation based econometric model could be applied for risk management and oversight purposes. By simulating future trajectories one could create a measure similar to VaR.

Next I show my hypotheses and their validity.

- H1: In the Hungarian sample innovations are nonlinear using a time series approach: I accept the nullhypothesis, as innovations are governed by a 6^{th} order polynomial as shown in the SNP study.
- H2: In the Hungarian sample volatility is heteroscedastic using a time series approach: the nullhypothesis is valid, since the SNP auxiliary model is a GARCH process.
- H3: The Hungarian sample shows asymmetric volatility: I decline the nullhypothesis, because the model fit worsened by introducing asymmetric volatility (leverage effect) to the SNP auxiliary model.
- H4: In the Hungarian sample, panel study approach and time series approach lead to the same conclusions: I accept the nullhypothesis, as estimation using single dimensional time series and pure time series approach pointed out the same

[13] see: Csajbók (1999), Gyomai and Varsányi (2002) and Reppa (2008)

consequences as a panel study run on a multidimensional data set including interactions.

- H5: In structural modeling explanatory power of estimated models grows by increasing the number of model factors (from 1 to 2 and from 2 to 3): I accept the nullhypothesis, since model fit improved dramatically by appending a new factor to the model, i.e. model explanatory power has been increased.
- H6: By careful calibration of the 3-factor Vasicek model one can ensure that relative fit (after correction with average yield level) in the Hungarian sample is not significantly worse (a 25% threshold is used here) than that in the US case. Formalised in a different way, relative in-sample forecast inaccuracy is max. 25% more in the Hungarian market than in the US one: the nullhypothesis is true, as in the Hungarian sample the 3-factor Vasicek model implied average daily mismatch amounted to 8 basis points, whereas in the US case to 5 bp respectively. Correcting these measures with average yield levels (8.17% in the Hungarian and 4.64% in the US sample) reveals 0.98 bp estimation error for the Hungarian market and 1.08 bp mismatch for the US data set per 100 bp yield level, which means the model implied estimation error is 9% smaller in the Hungarian sample than in the US case. This is clearly within the 25% reference threshold and in fact indicates opposite relation.
- H7: The 3-factor Vasicek model offers acceptable out-of sample forecasts on a 6 month forecast horizon. This is quantified by average out-of-sample forecast errors being not more than 5 times more than in-sample forecast errors. I decline the nullhypothesis, since backtesting results show average (based on 10 thousand trajectories) out-of-sample forecast errors at almost 25 times in-sample forecast levels.

Considering the validity check of my hypotheses I reach the important conclusion that the 3-factor Vasicek model is the best choice among studied alternatives for dynamic models on the Hungarian term structure. This statement is supported by empirical evidence regarding the model's in-sample forecasting potential. The 8 basis point average estimation error is first negligible with relation to the Hungarian market (practically it amounts to one unit bid-ask spread) and second reveals better relative (as corrected with average yield level) in-sample fit in the Hungarian market than in the US one.

My other experiences with regards to structural model estimation are highlighted by the following points.

- It is sensible to choose an IR model for structural modeling which we have an efficient tool for to estimate. What is the point in having a too complicated model which has to be calibrated by a nonlinear estimation method which in turn cannot even recognize simple functional dependencies? Therefore I chose the affine model family and the Vasicek model.
- I carried out active empirical research with the CIR[14] model, which led to conclusions supported by Brigo and Mercurio (2006): the CIR model cannot handle the inverted Hungarian yield curve.
- Decisions regarding the number of modeling factors are best guided by PCA studies. For empirical research on the Hungarian term structure I recommend the use of 3-factor models.
- *Regarding estimation methods I had positive experience with the Kalman filter, in opposition I do not recommend the EMM for dynamic studies on the Hungarian yield curve.*

Considering out-of-sample forecasting potential of the 3-factor Vasicek model my arguments supported by extensive quantitative research are the followings

- *The model has limited potential for pure forecasting purposes. This does not mean that they are worthless, but results have to be interpreted by healthy cautiousness.* The model is not an oracle to "tell the winning lottery draw", but a tool to show a range of expected future interest rates.
- Less volatile maturities of the term structure should be used for forecasting purposes. I had the most accurate results with the 10 year tenor.
- Out-of-sample forecasting accuracy can be naturally improved by applying shorter forecast horizons. With a forecast horizon of 1 week I got 13 bp average forecast error for the 10 year tenor, based on 10 thousand trajectories. This amounts approximately to 1.5-fold market bid-ask spread, i.e. it is a relatively acceptable result. Given that,

[14] see: Cox et al. (1985)

the combined 2-hour runtime for the estimation and simulation algorithms is quite luring.

Considering the results detailled above the target audience of the presented methodology is rather the National Bank of Hungary, the Government Debt Management Agency and the Hungarian Financial Supervisory Authority. Term structure dynamics are indeed a priority for the mentioned actors. Commercial banks might find the methodologies useful, too; though their benefits are more likely to show up as more efficient risk management than hard proits of proprietary trading desks.

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