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THESIS BOOK

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Optimization of Bonus-Malus Systems

Ph.D. Dissertation

Supervisor:
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Associate Professor

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Department of Operations Research and Actuarial Sciences

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1 Research literature and the justification of the topic

1.1 Objective of optimization

BMS is a risk managing method mostly used in liability insurances. However, the most general application of the BMS is in the Motor third-party liability (MTPL) insurance. Hence in our research, we assume that the optimized BMS will be applied for MTPL insurance. However, the results of the models would presumably be similar for other types of insurance. Because the assumptions of the MTPL insurance may not be entirely valid in other insurances, we only considered the liability insurance for vehicles in this study.

In MTPL insurance, the policyholders create insurance if they cause damage to another individual. The research assumes that a policyholder is the insured vehicle's driver, although it is not always the case in practice. Whenever the policyholders cause any damage, they claim to the insurance company. Then the insurance company pays the damage that the policyholder has caused to the other individual. In every developed country, having MTPL insurance is compulsory for every actively used vehicle.

The insurance company pays the claim from a financial fund. This fund was created from all of the policyholders' payments of this specific insurance. Therefore, a policyholder with more claims during a period exploits more from this fund. This is why the riskiness of a policyholder should be close to his/her contribution to the fund. The policyholders contribute to the fund with their insurance premiums. Hence, the objective of the insurance company is to determine a "fair" premium for each policyholder.

A premium is "fair" if it is close to the policyholder's risk. Therefore, in insurance with "fair" premiums, the riskier policyholders generally pay more premiums. However, it is difficult to determine the exact riskiness of each policyholder.

It will be assumed that some unobservable parameters influence each policyholder's personal risk. Estimating these parameters is difficult with statistical methods, though with multi-period contracts, the insurance company can estimate the overall risk more accurately for each policyholder.

In other words, policyholders can be categorized by their risks. However, the insurance company cannot precisely determine with observable parameters (such as the age of the driver, age of the driving license, location of the vehicle owner, type of vehicle) which risk group a particular policyholder belongs to. These risk groups are usually called "types" in the literature. Therefore, this classification is prone to error, i.e., there is an underlying unobservable parameter that explains the risks of the policyholders. For example, let us assume that the insurance company uses only a location parameter to estimate the premium. We may observe that policyholders living in a city are riskier than those living in rural areas. In general, it is true, but some deviation may also exist among the policyholders. Hence, some policyholders from a city have fewer claims than those from a rural area during the contract period.

The cause of this deviation can be some underlying parameter that is not described by the policyholder's location. For example, it can be the driver's carefulness or talent for driving or something else.

Even though the policyholder assumes how skillful driver he/she is, the insurance company cannot observe it. The policyholder's assumption may not be completely accurate, but presumably, it is closer to reality than the observation of the insurance company. Therefore this is an asymmetric informational problem. Asymmetric information causes welfare loss whose magnitude can be reduced, e.g., by applying a BMS.

In a BMS, there are finitely many classes, each having a different premium. At the start of the contract, each policyholder is assigned to the "initial class". Subsequently, suppose the policyholder has a claim in the following period. In that case, he/she moves to a worse class, so the policyholder's payment may increase in the subsequent period. If he/she does not have a claim in a particular period, then he/she moves to a better class; therefore, his/her payment may become less in the following period. The classification rule – how many classes the policyholder will move up or down in the system – is called the transition rule. Hence, a transition rule specifies where the policyholder will be reclassified in the subsequent period for each possible claim. Transition rules can be unified in the BMS, meaning every class has the same rule. Alternatively, it can be non-unified. Hence, the penalization of a claim may differ from class to class.

Without asymmetric information, each policyholder's premium (for each risk type) would equal the expected claim (in each class). Hence, the problem is setting the premiums to approximate the "ideal situation" (i.e., the case without asymmetric information) as closely as possible (this is not the same as adjusting expected premium levels to expected claims). A perfect match is impossible in real situations. Thus a natural goal is to minimize the difference from the "ideal" solution.

To achieve this, we strive at setting a "good" premium scale and "good" transition rules. The first possibility is widely studied in the actuarial literature. However, there is less emphasis on the second one.

1.2 Economic literature

BMSs have a wide range of literature in both economics and mathematics. In economics, the research usually focuses on non-homogenous risks. It is crucial in any insurance to consider not just one risk but multiple risks in calculating the insurance premium. When numerous risks are considered together, some heterogeneity appears. Risk classification is the most used technique in the field of insurance to reduce heterogeneity. In risk classification, the policyholders are separated into risk groups using observable variables, such as age, type of vehicle, etc.

However, the risk-classification cannot eliminate the risk-heterogeneity completely. No matter how many risk group we create from how many observable variables, it is very likely that more than one type of policyholder remains within any risk group. Hence we cannot distinguish these different types of policyholders from each other with their observable parameters. In economic literature, this effect is called *adverse selection*.

In insurance mathematics, the adverse selection was first studied by Rothschild and Stiglitz (1976). They found that adverse selection causes social welfare loss, and the market equilibrium does not always exist. However, various methods can reduce social welfare loss caused by adverse selection.

Cooper and Hayes (1987) investigated multi-period contracts and found that if there is a contract where the premiums (and indemnities) depend on the claim history of the previous year(s), social welfare loss can considerably be reduced. The result is quite similar to the BMS. However, in their theoretical result, it is not worthy of considering every policyholders' claim history. However, applying BMS for only a fragment of policyholders in general not be possible in practice. Therefore according to their theoretical result, every risk group should have its own BMS to achieve the best reduction of the social welfare loss.

Another method to decrease the social welfare loss caused by adverse selection is compulsory insurance. In the MTPL, both methods are there since it is compulsory in most of the countries, and the claim history is also considered in general.

In general, in real-life situations, adverse selection is accompanied by moral hazard. In the models of moral hazard, the probability of claims also depends on the effort of the policyholder. However, the insurance company does not know the policyholder's actual effort to reduce the risk of the damage. The insurer can merely estimate it from the claim amount (Shavell (1979)). With the use of a BMS, the insurer motivates the policyholders to reduce risk. This is so because if someone has a claim, then the following period he/she will be assigned to a worse class (if there is any). Hence, his/her premium will increase.

Adverse selection and moral hazard are usually present simultaneously (as in the case of BMS). Holton (2001) investigated the moral hazard and adverse selection with an expected utility model. The author found that the BMS is only Pareto optimal (optimal for both the insurance company and the policyholders) if the adverse selection and the moral hazard or the insurance costs are considered in the analysis.

The typical appearance of moral hazard in the MTPL insurances is the so-called "bonus-hunger", meaning that policyholders choose self-financing the damage rather than claiming to the insurance company. If the claim amount is lower than the premium increase of the following periods, then it is not worth reporting a claim to the insurer (De Prill (1979); Sundt (1989)).

A BMS's efficiency is typically measured by an indicator called "elasticity" introduced in Loimaranta (1972). Elasticity shows how expected payment will increase if risk increases by 1%. A good BMS's elasticity is over zero, ideally one, but can be over one as well.

Lemaire (1995) conducted an empirical study on some BMSs that were used in practice. If the BMS's premiums do not vary that much, the elasticity is usually under 1 in practice. De Prill (1978) generalized the Loimaranta's efficiency.

Loimaranta (1972) also shows that the elasticity for measuring a BMS's effectiveness is not a perfect indicator. In MTPL insurance, the claim probabilities have relatively low values. Hence it is not impossible to construct a BMS with elasticity equal to one. With the optimization of the premiums, a very steep premium-scale would be the result to reach the ideal elasticity. However, Loimaranta (1972) argues that in that case, the insurance would be pointless. A too steep premium-scale results in a very volatile payment for the policyholders. Hence the insurance would not provide economic security to the policyholder.

The economic literature calls this as the risk-averse decision-maker (in this case, the policyholder), not only consider the premium but its fluctuation. Hence the policyholder may be satisfied with a slightly higher expected premium on the condition that the variation of premiums is decreased significantly is somewhat disregarded (or partially regarded). Nonetheless, in Loimaranta (1972) a model minimizes the variance of the premium scale assuming a fixed level of elasticity.

Lemaire (1995) compared some BMSs that are used in practice. Since 1995, there were some changes in these systems.

In many countries, such as Belgium and Portugal, the regulations were liberated. Hence the insurance companies can create their merit system to consider the claim history. Some of the countries already had liberated regulations in 1995, such as the United Kingdom and Sweden. However, there are still some countries where the regulations are strict, such as Hungary or Luxembourg. Hence the operating insurance companies have to use the same transition rules in the BMS. Also, there are countries, like Germany and the Netherlands, where the insurance companies can change some parameters of the centrally determined BMS. However, we did not find any countries where the rules stricken from liberated status since 1995. Therefore, in general, the regulations on the MTPL insurance regarding the BMS became less strict. However, even in countries with minimal regulations, the insurance companies have to use a merit system to consider the policyholders' claim history. Therefore, we found that the Bonus-Malus System is generally the most used method in these countries.

1.3 Mathematical literature

In the mathematical literature, the BMS appears in the applications of Markov Processes. For example, Molnar and Rockwell (1966) introduces the BMS as an application of a Markov chain.

There is usually an assumption that the policyholders pay only the system's premium in the MTPL insurance in the BMS optimization models. Therefore there are not any other factors that determine the payment of the policyholders. Hence the policyholders' payment in a period only depends on where they are classified in that period. Generally, the classifications in a BMS happen in each period, which is the assumption in the optimization models. Getting into a class in the subsequent period only depends on the currently assigned class. Therefore, knowledge of the previous periods' classifications is not necessary.

Let X_t denote the class, where the policyholder is classified in the period t. Moreover, let us consider any ρ statement that is already known before period t.

$$\mathbb{P}(X_t = k | X_{t-1} = l, \rho) = \mathbb{P}(X_t = k | X_{t-1} = l) \qquad \text{(Markov property)}$$

Because the policyholders' classification only depends on the number of claims and the previous period's classification, the classification process has the Markov property. Furthermore, the process holds the Markov property and can be considered a Markov chain because of the BMS classification rules.

Definition 1. *A discrete-time stochastic process is called a **Markov chain** if*

$$\begin{aligned}\mathbb{P}(X_{t+1} = k_{t+1} | X_t = k_t, X_{t-1} = k_{t-1}, \dots, X_1 = k_1, X_0 = k_0) = \\ \mathbb{P}(X_{t+1} = k_{t+1} | X_t = k_t)\end{aligned} \tag{1}$$

holds for every period t.

In our research, we assume that this condition holds for all of the considered BMSs. A BMS in practice may differ in a way that does not strictly follow this definition. For example, suppose there is a BMS, where the policyholders move upward only when they have two consecutive years without a claim. In this case, not only the last period determines the probability. However, in this case, only the previous two periods' classification is needed to calculate the exact probabilities.

Hence, for simplicity, we assume that in every considered BMSs, the classification only depends on the previous period. Therefore in every case, we assume the classification of the policyholders is a Markov chain.

Definition 2. *The **t-th step transition probabilities** for a Markov chain are*

$$p_{k,l}(t) = \mathbb{P}(X_{t+1} = k | X_t = l)$$

Definition 3. *A Markov chain is homogenous if the t-th step transition probabilities $p_{k,l}(t)$ do not depend on t.*

Therefore, in a homogenous Markov chain, we denote the transition probabilities with $p_{k,l}$.

In realistic situations, the policyholders become more experienced over time, and thereby their claim probabilities may decrease in the future. The effect of the young, inexperienced policyholders' risk exceeding the average risk is usually called the "duration effect".

The states of a Markov chain (in our case, the BMS classes) can be classified into two sets according to its movement from one class to another. The two sets are called the *ergodic set* and the *transition set*. Class l can be reached from class k if a policyholder in class k can be classified into class l later on (not necessarily in the consecutive period).

A class k is an element of the ergodic set if class k can be reached from every class of the ergodic set, and from class k, every element of the ergodic class can be reached. Those classes that are not in an ergodic set are elements of a transition set.

In the general definition of the BMS, all of the classes are part of one ergodic set. Hence the classification of the policyholders is a Markov chain without a transition set. Furthermore, it means that the chain is irreducible.

Definition 4. *A Markov chain is said to be **irreducible**, if for every class k, j and period t there exists a period s, such that*

$$\mathbb{P}(X_{t+s} = k | X_t = j) > 0 \tag{2}$$

Hence, every class can be reached from any class. In the optimization of the BMS, we assume that the classification is irreducible. Hence we construct the transition rules accordingly. Furthermore, in the case of the BMS, the chain is even aperiodic.

Definition 5. *Let θ_k of class k be $\theta_k = gcd\{t \geq 1 : p_{k,k}(t) > 0\}$, where 'gcd' denotes the greatest common divisor. Let $\theta_k = \infty$ if $p_{k,k}(t) = 0$ for all $t \geq 1$.*
Therefore class k is aperiodic if $\theta_k = 1$. And a Markov chain is ***aperiodic*** *if every class is aperiodic.*

The Markov chain of all of the considered BMSs is aperiodic. If the policyholder is in the highest class, it is impossible to classify him/her into a higher class. Besides, from the lowest class, the policyholders cannot be classified lower. Therefore, the finite number of classes and the considered transition rules ensure the Markov chain's aperiodicity.

In a Markov chain with one ergodic set and the property of aperiodicity, no matter where the process starts, it can be in any class, after a sufficient number of periods. This chain is called a *regular Markov chain* in the literature.

In a regular Markov chain, the classification tends to a unique stationary probability distribution. These stationary probabilities are usually considered in the optimization models of the BMS.

2 Applied methods

2.1 Optimization models

We assume that there are I different risk groups (types) among the policyholders. Each type has a different risk that does not change over time. In the practice of BMS, transition rules are based only on claim numbers, and the claim amount is ignored.

Let $M > 0$ be the highest number of possible claims in a period and let λ_m^i be the probability of the occurrence of m claims for the policyholders of type i ($i = 1, \ldots, I$, $\sum_{m=0}^{M} \lambda_m^i = 1$). We denote the risk-parameters (expected claim amount) for risk group i with λ^i, ($\lambda^i = \sum_{m=0}^{M} m\lambda_m^i$). The types are indexed in an increasing risk order to keep notation simple. The expected claim amount is the least for type 1 and the highest for type I. Let ϕ^i be the proportion of the type i policyholders among all of the policyholders ($\sum_{i=1}^{I} \phi^i = 1$). In BMS there are $K+1$ classes indexed from 0 to K. The premium of class k is denoted by π_k. In a BMS the premiums should be monotonic, hence we assume that $\pi_{k-1} \geq \pi_k$ ($k = 1, \ldots, K$).

The regular Markov chain has a unique stationary probability distribution. In the model, there is a different regular Markov chain for each type. Let c_k^i be the probability that the type i policyholders is classified into class k after spending sufficiently enough time in the BMS. We will refer to the c_k^i as the stationary probabilities of the type i in class k.

2.1.1 Optimizing the premium scale when the transition rules are fixed

We present a linear programming (LP) model for optimizing the premiums, originally introduced in (Heras et al., 2004).

$$\min \sum_{i=1}^{I} \sum_{k=0}^{K} \phi^i g_k^i \tag{LP1.obj}$$

Subject to

$$\pi_k c_k^i + g_k^i \geq \lambda^i c_k^i \qquad \forall i, k \tag{LP1.1}$$

$$\pi_k c_k^i - g_k^i \leq \lambda^i c_k^i \qquad \forall i, k \tag{LP1.2}$$

$$\pi_{k-1} \geq \pi_k \qquad k = 1, \ldots, K \tag{LP1.3}$$

$$\pi_k \geq 0 \qquad \forall k$$
$$g_k^i \geq 0 \qquad \forall k, i$$

2.1.2 Optimizing transition rules when the premium scale is fixed

We introduce a mixed-integer linear programming (MILP) model, where the transition rules are in the scope of the optimization. Transition rules are typically defined by transition matrices. For this model we introduce binary variables $T_{j,m,k}$ for each entry of the transition matrices. If $T_{j,m,k} = 1$, then the policyholders with m claims are moved from class k, j classes upward (downward if $j < 0$) in the following period. Denote the domain of j by $J_k = [\underline{J_k} : \overline{J_k}]$ for class k where $-k = \underline{J_k} < 0$ and $K - k = \overline{J_k} > 0$ are the two extremes. If a binary variable $T_{j,m,k} = 1$ and index j is positive, then the policyholders with m claims are put upward in the system. Put differently, they move to a class with a lower premium if it is possible. In the case of $j < 0$, the policyholders move downward if they have m claims. Index j can be 0 as well, meaning they stay in the same class in the subsequent period.

The model aims to find the best transition rule that evenly separates the risk-groups' expected payment. Thus, we want to minimize each class's deviation of the payment and expected claims (in some norm).

Often in practice, the transition rules do not differ from class to class, which means a unified transition rule for each claim m. This means that instead of binary variables $T_{j,m,k}$ we can simply use binary variables $T_{j,m}$. In this case, J_k is the same for all k; therefore, it is sufficient to set only one upper ($\overline{J} = K$) and lower limit ($\underline{J} = -K$).

Some numerical experiments were presented for each model. In general, an optimal transition rule can be significantly better for distinguishing the policyholders. With more classes, the BMS performs better with optimized transition rules.

In general, the non-unified(NU) type of transition rules could significantly improve the sorting ability of a BMS compared to the unified (U) transition rules. Unfortunately, computing an NU model with more than ten classes within a reasonable time was not possible. Hence we introduced two approximation methods to find a good solution for these models. We found that even the approximated results were generally better than the solution of the models with unified transition rules.

2.1.3 Joint optimization of transition rules and premiums

Furthermore, we present another MILP model, where both premiums and the classification rule can be optimized simultaneously. In this case, if we use

π_k ($\forall k$) as non-negative variables, we will get a quadratic constraint problem (MIQCP). Because solving a MILP usually takes less computational time than the corresponding MIQCP, we linearize the quadratic constraints. To this end, we start with default premiums for each class that can be increased if needed. We set each default premium to the expected claims of types with the lowest risk $\pi_k = \lambda^1$, $\forall k$. We then introduce ε as a value for changing the default premium and also consider various layers of these modifications. ε^ℓ denotes how much the premium changes in layer ℓ compared to the default premium.

Binary variable O_k^ℓ indicates whether we increase the premium in class k by ε^ℓ, i.e., if $O_k^\ell = 1$, then the final premium of class k is $\lambda^1 + \varepsilon^\ell = \lambda^\ell$.

All of the previously introduced MILP models are based on the stationary distribution. However, in some cases, for the probabilities to reach the stationary level, more time periods are needed than the duration for which policyholders may remain in the system. In such cases, instead of the stationary distribution, using the probabilities in each period of the insurance contract would be more appropriate for the optimization. We introduced a modification of the model, where we do not use stationary probabilities.

Because this model does not require stationary probabilities, the classification process of the policyholders does not have to be a regular Markov chain. Hence, it is possible to optimize realistic situations where the claim probabilities of the policyholders or the ratios of the risk groups depend on time. However, for simplicity, we only present a model that considers the same assumptions in the stationary case. Although the model can be formulated to consider time-dependent transition probabilities, we did not investigate this prospect because the time required for finding the optimal solution was extremely long, even in the simplest case.

To optimize a multi-period BMS, we take the first Θ periods of the insurance contract. The index of time is denoted by t ($t = 0, \ldots, \Theta$) where $t = 0$ indicates the beginning of the contract, and Θ is the end of it. We introduce a binary variables B_k for all classes to determine the initial class.

When the B_k variable takes the value 1, then class k is the initial class.

The variables that was depending on the stationary probabilities now have a time index as well in this case.

Moreover, we introduced a modification of this model to optimize the number of classes as well.

When realistic parameters are considered in the joint optimization model, there are a considerable number of binary variables. Hence the running time can be extremely long. The optimization models for the premiums and the transition rules can be calculated much faster separately than the joint optimization model. Hence, we may use an iterative method to approximate the optimal solution. First, we calculate the optimal transition rules with a fixed premium. Then, we find the optimal premiums to these transition rules, which we consider as parameters. Next, we use the optimal premiums of this model as parameters and re-optimize the transition rules. We continue this until we cannot improve the objective function further. We compared the computation of the MILP model and the iterative heuristic. We found that the running time of the iterative heuristic, in general, was much faster than the computation of the exact solution, while the results were not much worse.

2.2 BMS comparison with other methods

In the practice of MTPL insurances, insurance companies often use other techniques besides the BMS. Usually, the policyholders' risk-groups are calculated via some statistical methods, then part of the premium is determined based on this classification. The other part of the premium comes from the BMS.

A better estimation is essential for the insurance company, but errors are unavoidable in practice. With some observable parameters (such as the driver's age, location of the vehicle owner, age of the driving license, type of vehicle, etc.), we may estimate risk groups. However, there may be other unobservable parameters that influence the risk pertaining to each policyholder. Hence, the existence of unobservable parameters may result in some deviations from the estimated risk groups. For decreasing the error related to the estimated and "fair" premium, some tools such as the BMS can be used. We investigate how the BMS can be optimized if we consider the statistical model as well. We compare approaches for determining the premium with a statistical model and an optimized BMS:

- ***Scaling***: The insurance company optimizes the BMS and SM separately. Then it finds an appropriate scale parameter ($0 \leq \alpha \leq 1$) between them to determine the final premium.
- ***Merging***: In this method, the expected premium comes from the BMS alone. Therefore, we consider the types, and consequently the SM, in the premiums of the classes. Therefore, the payment by two policyholders belonging to the same class may differ if their observable parameters are not the same.
- ***Independent***: In this case, we optimize a BMS independently for each observable risk group.

2.3 Optimization model for a BMS where the transition rules based on the size of claims

In practice – most often – the transition rules are based on the number of claims. Therefore, the size of the claims does not affect the transition rules.

This is reasonable since empirical studies suggest that 'good' and 'bad' policyholders differ more in terms of the probability of the number of claims than in amount (assuming there is at least one claim). Despite the fact that differences in the number of claims are more significant than differences in the claim amounts, we can observe deviations in the (conditional) amount of claims as well.

We consider a BM system with $K + 1$ classes, but the transition rules depend on the claim amount instead of the number.

The (aggregate) claim amount is described with random variable L^i for group i, which differs in each risk-group.

To consider the transition rules based on the claim amount, we introduce K breakpoint variables for every class k: $\ell_1^k > \ell_2^k > \cdots > \ell_K^k$. The transition rules are based on these breakpoints: if a policyholder is assigned to class k and its claim amount is between ℓ_h^k and ℓ_{h+1}^k, the policyholder will be transitioned to class h in the next period. If the policyholder's claim amount is higher than ℓ_1^k, he/she gets into class 0; if it is less than ℓ_K^k, then the policyholder gets into class K.

In this case, for the transition rules, we have to find $K^2 + K$ optimal breakpoints.

We can reduce the number of breakpoints if they do not differ in the classes. Accordingly, we define $2K + 1$ breakpoints $\ell_{-K} > \ell_{-(K-1)} > \cdots > \ell^{-1} > \ell_0 > \ell_1 > \cdots > \ell_K$.

In this case, if the claim amount is between ℓ_h and ℓ_{h+1}, the policyholder moves h classes upward (if $h < 0$, it will be a downward move).

In the second approach, we can further reduce the number of breakpoints. We can consider breakpoints $\ell_{-D} > \ell_{-(D-1)} > \cdots > \ell^{-1} > \ell_0 > \ell_1 > \cdots > \ell_U$ with $U, D < K$. Thus, the policyholder can move downward at most D classes and upward at most U classes.

3 New results of the dissertation

3.1 Optimizing the premium scale

We presented a model that was introduced in (Heras et al., 2004), but we modified the objective. With the objective function, the fluctuation of the premiums is also considered.

Theorem 1. *There is an optimal solution of LP1, where for all k there is a risk group i, where $\pi_k = \lambda^i$.*

We considered the optimization of the premium-scale of the BMS with a profit constraint:

$$\sum_{i=1}^{I} \sum_{k=0}^{K} \left(\pi_k c_k^i\right) \geq \sum_{i=1}^{I} \phi^i \lambda^i. \tag{LP1.4}$$

Theorem 2. *There is an optimal solution of LP1 with constraint* (LP1.4)*, where there is only one type of premium that is unequal to any risk group's expected claim.*

3.2 Optimizing the transition rules

In the optimization of the BMS, we considered two types of transition rules. Often in practice, the transition rules do not differ from class to class, which means a unified transition rule for each claim m. We also considered the non-unified transition rules, where the transition rule depends on the class.

In the BMS we have to consider only those transition rules, which results in an irreducible Markov chain. Because in the optimization models, there are stationary probabilities, it is sufficient to assume that each stationary probability (for each k) be positive. In MILP models, we cannot use strict inequalities, but with a parameter $\tau > 0$ and $\tau \approx 0$, we can prescribe that each stationary probability be positive. This is an eligible condition for an irreducible Markov chain. However, if τ is unnecessarily high, we may exclude some transition rules that give irreducible Markov chains.

Due to the fewer possibilities of the unified transition rules, we may exclude those transition rules that would not lead to an irreducible Markov chain. Hence we may exclude the irreducibility constraint based on the τ value to give a not only eligible condition.

Theorem 3 presents a rule which applies to those transition rules when there can be up most one claim per period. For this, let j_0 denote the transition rule for the claim-free case. Furthermore, let j_1 denote the transition rule when there is a claim. We assume that $j_0 > 0$ and $j_1 < 0$.

Theorem 3. *Let $gcd(j_0, |j_1|)$ denote the greatest common divisor between j_0 and $|j_1|$. Let $j_0 > 1$ and $|j_1| > 1$. A Bonus-Malus System, with $(j_0; j_1)$ transition rule is not irreducible, if at least one of these conditions is met:*

$$\begin{cases} j_0 + |j_1| > K + 1 \\ gcd(j_0, |j_1|) > s \quad \& \quad j_0 + |j_1| \leq K + 1 \end{cases}$$

Where $s = 1$ if $K + 1$ is odd, otherwise $s = 2$.

Theorem 4 is an extension of theorem 4 for the case, when $M = 2$.

Theorem 4. *A Bonus-Malus System, with $(j_0; j_1, j_2)$ transition rule is not irreducible if at least one of these conditions is met:*

$$\begin{cases} j_0 + |j_1| > K + 1 \\ gcd(j_0, |j_1|, |j_2|) > s & \& \quad gcd(j_0, |j_1|) + |j_2| \leq K + 1 \\ gcd(j_0, |j_1|) > s & \& \quad gcd(j_0, |j_1|) + |j_2| = K + 2 \quad \& \quad \frac{K+1}{gcd(j_0, |j_1|)} \notin \mathbb{Z} \\ gcd(j_0, |j_1|) > s & \& \quad gcd(j_0, |j_1|) + |j_2| > K + 2 \end{cases}$$

Where $s = 1$ if $K + 1$ is odd, otherwise $s = 2$.

3.3 Joint optimization of the premium-scale and the transition rules

Because of Theorem 1 in the optimization of the premium-scale there are finite possibilities of the premiums in the optimal solution with the considered objective function. Hence, we introduced binary variables for each possibility in each class. That means we can find the exact solution of a BMS where we consider both transition rules and premiums in the optimization.

In the optimization model, we applied the theorems of the regular Markov chains. Hence there is a unique stationary probability distribution in the classification. We considered the stationary probabilities in the optimization model. However, we also introduce a modification with multi-period probabilities instead of the stationary ones.

We conducted numerical experiments for all of the presented optimization models. In these tests, the consideration of the transition rules in the optimization greatly improved the sorting capability of the BMS.

We conducted numerical tests of these models. We found that the BMS is the most effective if there are many BMS classes and the difference between the policyholders' parameters is large. However, using too many classes in a BMS may not be viable in practice due to the contract exists only for a finite number of periods. Hence, we investigated the multi-period model, where we got a slightly different result than the stationary model, which assumes the contract will never end. However, even in this case, the optimal result was to use more classes than usually used in practice.

We also conducted research on realistic parameters calculated from Hungarian data. We found that more classes and more rigid transition rules would separate the policyholders better than the currently used system.

3.4 Consideration of observable variables

We investigated how the optimization model can be used in practical situations. First,we assumed that the payment of the policyholders only depends on the BMS' premium. However, the insurance company may estimate part of the premium using statistical methods (SM) in practice. We compared some methods on how the BMS and SM can be applied together. We considered the *scaling* method, in which the insurance company optimizes the BMS and SM separately. The final premium is determined by the weighted sum of the two premiums. We also examined the *Merging* method. In this case, the transition rules differ for each observable risk groups. Moreover, we investigated the *independent* method, where the whole BMS differs for each observable risk group.

Overall, the *independent* method was the most efficient. However, in general, using the BMS and statistical method jointly almost always resulted in a better solution than considering only one.

We also presented a case study based on realistic data. We proposed two approaches for determining the unobservable parameters. We found that the optimized BMS's effectiveness depends on the risk groups' parameters. Hence, if the applied statistical method is accurate, the BMS cannot improve the solution greatly. However, in both realistic models, the BMS could improve the results.

3.5 The transition rules based on the size of claims

In practice, usually, the transition rules only depend on the number of claims, and the sizes of the claims does not influence the classification We also considered a BMS, where the classification only depends on the claim amount. Finding optimal classification rules and premiums require a non-linear optimization model. When transition rules depend on the number of claims, we could linearize the model by introducing binary variables for each possibility of the transition rule. If the claim amount determines the transition rules, this approach is inadequate since there are too many possibilities. In this case, we would need to introduce binary variables for each possible interval. Hence, we approximate the optimal solution.

We compared simulated annealing (SA) and biased random-key genetic algorithm (BRKGA) to approximate the optimal solution. We found that the SA is much faster than the BRKGA at larger instances; however, the BRKGA generally found better results.

We compared the claim size model with the original models – where the transition rules only depend on the number of claims. A case study on realistic data is also presented. We found that even though claim amounts can result in more flexible classifications, the efficiency of BMS (in terms of separating the risk groups) was very similar to the results of the claim number model.

4 Our publications on the BMS optimization

1. Ágoston, K. Cs. and Gyetvai, M. (2021). Comparison of an iterative heuristic and joint optimization in the optimization of bonus-malus systems. *In: Drobne S., Zadnik Stirn L., Kljajić Borštar M., Povh J. and Žerovnik J.(ed.) Proceedings of the 16th International Symposium on Operational Research : SOR '21 Online (2021)* pp. 55-60., 5 p.

2. Ágoston, K. Cs. and Gyetvai, M. (2020). Joint Optimization of Transition Rules and the Premium scale in a Bonus-Malus System *ASTIN Bulletin: The Journal of the IAA, Volume 50, Issue 3*, pp. 743-776, 34 p.
DOI: `https://doi.org/10.1017/asb.2020.27`

3. Ágoston, K. Cs. and Gyetvai, M. (2019). Átsorolási szabályok optimalizálása bónusz-málusz rendszerekben - *Szigma, Vol. 51, Issue 4* pp. 335-362., 28 p.

4. Ágoston, K. Cs., Gyetvai, M. and Kovács, L. (2019). Optimization of transition rules based on claim amounts in a bonus-malus system. *In: Zadnik Stirn, L; Kljajić Borštar, M; Žerovnik, J; Drobneand, S; Povh, J (ed.) Proceedings of the 15th International Symposium on Operational Research : SOR '19 Ljubljana, Slovenia : Slovenian Society Informatika, (2019)* pp. 375-380., 6 p.

5. Gyetvai, M. and Ágoston, K. Cs. (2018). Optimization of transition rules in a Bonus-Malus system. *Electronic Notes in Discrete Mathematics. Vol. 69.* pp. 5-12., 8 p.
DOI: `https://doi.org/10.1016/j.endm.2018.07.002`

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