Optimization of Bonus-Malus Systems

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1 Introduction

In this dissertation, I present the research on the optimization of the Bonus-Malus System (BMS), which I did with my supervisor, Kolos Csaba Ágoston. Our contribution to the topic was published in Gyetvai and Ágoston (2018); Ágoston and Gyetvai (2020) and Ágoston and Gyetvai (2021),also, in a Hungarian article, Ágoston and Gyetvai (2019). We also had a study with László Kovács that was published in Ágoston et al. (2019). In this dissertation, I present these articles' results with extensions of these studies.

BMS is a risk managing method mostly used in liability insurances. However, the most general application of the BMS is in the Motor third-party liability (MTPL) insurance. Hence in our research, we assume that the optimized BMS will be applied for MTPL insurance. However, the results of the models would presumably be similar for other types of insurance. Because the assumptions of the MTPL insurance may not be entirely valid in other insurances, we only considered the liability insurance for vehicles in this study.

In MTPL insurance, the policyholders create insurance if they cause damage to another individual. The research assumes that a policyholder is the insured vehicle's driver, although it is not always the case in practice. Whenever the policyholders cause any damage, they claim to the insurance company. Then the insurance company pays the damage that the policyholder has caused to the other individual. In every developed country, having MTPL insurance is compulsory for every actively used vehicle.

The insurance company pays the claim from a financial fund. This fund was created from all of the policyholders' payments of this specific insurance. Therefore, a policyholder with more claims during a period exploits more from this fund. This is why the riskiness of a policyholder should be close to his/her contribution to the fund. The policyholders contribute to the fund with their insurance premiums. Hence, the objective of the insurance company is to determine a "fair" premium for each policyholder.

A premium is "fair" if it is close to the policyholder's risk. Therefore, in insurance with "fair" premiums, the riskier policyholders generally pay more premiums. However, it is difficult to determine the exact riskiness of each policyholder.

It will be assumed that some unobservable parameters influence each policyholder's personal risk. Estimating these parameters is difficult with statistical methods, though with multi-period contracts, the insurance company can estimate the overall risk more accurately for each policyholder.

In other words, policyholders can be categorized by their risks. However, the insurance company cannot precisely determine with observable parameters (such as the age of the driver, age of the driving license, location of the vehicle owner, type of vehicle) which risk group a particular policyholder belongs to. These risk groups are usually called "types" in the literature. Therefore, this classification is prone to error, i.e., there is an underlying unobservable parameter that explains the risks of the policyholders.

For example, let us assume that the insurance company uses only a location parameter to estimate the premium. We may observe that policyholders living in a city are riskier than those living in rural areas. In general, it is true, but some deviation may also exist among the policyholders. Hence, some policyholders from a city have fewer claims than those from a rural area during the contract period.

The cause of this deviation can be some underlying parameter that is not described by the policyholder's location. For example, it can be the driver's carefulness or talent for driving or something else.

Even though the policyholder assumes how skillful driver he/she is, the insurance company cannot observe it. The policyholder's assumption may not be completely accurate, but presumably, it is closer to reality than the observation of the insurance company. Therefore this is an asymmetric informational problem. Asymmetric information causes welfare loss whose magnitude can be reduced, e.g., by applying a BMS.

In a BMS, there are finitely many classes, each having a different premium. At the start of the contract, each policyholder is assigned to the "initial class". Subsequently, suppose the policyholder has a claim in the following period. In that case, he/she moves to a worse class, so the policyholder's payment may increase in the subsequent period. If he/she does not have a claim in a particular period, then he/she moves to a better class; therefore, his/her payment may become less in the following period. The classification rule – how many classes the policyholder will move up or down in the system – is called the transition rule. Hence, a transition rule specifies where the policyholder will be reclassified in the subsequent period for each possible claim. Transition rules can be unified in the BMS, meaning every class has the same rule. Alternatively, it can be non-unified. Hence, the penalization of a claim may differ from class to class.

Without asymmetric information, each policyholder's premium (for each risk type) would equal the expected claim (in each class). Hence, the problem is setting the premiums to approximate the "ideal situation" (i.e., the case without asymmetric information) as closely as possible (this is not the same as adjusting expected premium levels to expected claims). A perfect match is impossible in real situations. Thus a natural goal is to minimize the difference from the "ideal" solution.

To achieve this, we strive at setting a "good" premium scale and "good" transition rules. The first possibility is widely studied in the actuarial literature. However, there is less emphasis on the second one.

The dissertation is organized as follows: In section 2, we give an overview of the relevant literature on BMS with particular emphasis on its optimization. Section 2.1 presents some BMSs that are used in practice. Finally, we searched for the BMS regulations in some European countries that are used nowadays and in the past.

In the BMS, the policyholders are classified in each period. The process of this classification has a property called Markov property. This property means that the classification only depends on the previous period. Furthermore, the classification of a BMS that we consider in our research can be considered a Markov chain. In section 3, we present some theories of the Markov chains. In addition, we introduce fundamental assumptions of our models in this section.

In section 4, we present optimization models of the BMS. In the BMS, the policyholders are classified, and the premium of the BMS depends only on their classification. Hence in optimization, we may optimize either the system's premiums or the rules of the classification. This section presents a linear programming (LP) model for optimizing the premiums, introduced in Heras et al. (2004). We also introduce a mixed-integer linear programming (MILP) model, where the classifications' rules are in the scope of the optimization. We introduced this model in Gyetvai and Ágoston (2018).

In section 5 we present an extension of this MILP model, where both premiums and the classification rule can be optimized simultaneously. We initially introduced this model in Ágoston and Gyetvai (2019) and Ágoston and Gyetvai (2020). In the optimization models, we consider stationary probabilities in the classification because of the Markov property. This section also introduces a modified MILP model, where instead of the stationary probabilities, we consider multiple periods in the model. Furthermore, we present a model modification where we may change the number of classes in the optimization. This section also presents a case study of the introduced models with real-world parameters. We obtained the real-world parameters from a dataset of an insurance company that operates in Hungary.

Usually, in the models, it is assumed that policyholders' payment is only the BMS premium. However, in practice, insurance companies can use other methods besides the BMS. Hence the premium of the policyholders does not only depend on the class they belong to. Thus, we considered a model with observable parameters and other aspects that affect the risk and are unobservable. Therefore the insurance company may calculate a premium for the observable parameters with a statistical method and use the BMS for the unobservable ones. In section 6, we compare methods to mix the statistical method and the BMS premium. We present a case study as well, with the data of an insurance company.

In the MTPL insurance's BMS, only the number of claims matters in the policyholders' classification, not the size, mostly due to practical reasons. In section 7, we present research where instead of the claims' number, the claims' amount determines the policyholders' classification. We introduced some parts of this research in Ágoston et al. (2019). We also extended this research with a more realistic model and a case study on real data. In section 8, we summarize the results of this dissertation.

Our contribution to the literature of the optimization of the BMS can be summarized as:

- We investigate a model that was introduced in (Heras et al., 2004) but with a modified objective function. We proved that an optimal premium-scale always exists with this objective function in which all premiums equal one of the risk groups expected claim.

- We considered the same model with a profit constraint. In this case, we proved that an optimal premium-scale always exists in which there is only one type of premium that is unequal to any risk group's expected claim.

- We introduced a MILP model for the optimization of transition rules with fixed premiums. We considered unified and non-unified transition rules optimization. In the case of unified transition rules, we gave the rule to exclude those transition rules that would lead to a non-irreducible Markov chain.

- We introduced a MILP model for the joint optimization of transition rules and premiums. We can determine the exact solution with the investigated objective function when we do not consider the profit constraint. However, we can only approximate it otherwise.

- We introduced an extended version of the model, where instead of the stationary probabilities, we use multi-period optimization.

- We introduced modeling approaches to consider the BMS premium with other statistical estimations in the final premium. Finally, we compared the methods with numerical experiments on realistic data.

- We introduced an optimization model for a BMS where the classification depends on the claim amount.

2 Literature overview

BMSs have a wide range of literature in both economics and mathematics. In economics, the research usually focuses on non-homogenous risks. It is crucial in any insurance to consider not just one risk but multiple risks in calculating the insurance premium. When numerous risks are considered together, some heterogeneity appears. Risk classification is the most used technique in the field of insurance to reduce heterogeneity. In risk classification, the policyholders are separated into risk groups using observable variables, such as age, type of vehicle, etc. For more about the risk classification see Crocker and Snow (1986) and Crocker and Snow (2000).

However, the risk-classification cannot eliminate the risk-heterogeneity completely. No matter how many risk group we create from how many observable variables, it is very likely that more than one type of policyholder remains within any risk group. Hence we cannot distinguish these different types of policyholders from each other with their observable parameters. In economic literature, this effect is called *adverse selection*.

In insurance mathematics, the adverse selection was first studied by Rothschild and Stiglitz (1976). They found that adverse selection causes social welfare loss, and the market equilibrium does not always exist. However, various methods can reduce social welfare loss caused by adverse selection.

Cooper and Hayes (1987) investigated multi-period contracts and found that if there is a contract where the premiums (and indemnities) depend on the claim history of the previous year(s), social welfare loss can considerably be reduced. The result is quite similar to the BMS. However, in their theoretical result, it is not worthy of considering every policyholders' claim history. However, applying BMS for only a fragment of policyholders in general not be possible in practice. Therefore according to their theoretical result, every risk group should have its own BMS to achieve the best reduction of the social welfare loss.

Another method to decrease the social welfare loss caused by adverse selection is compulsory insurance. In the MTPL, both methods are there since it is compulsory in most of the countries, and the claim history is also considered in general (see section 2.1.).

In general, in real-life situations, adverse selection is accompanied by moral hazard. In the models of moral hazard, the probability of claims also depends on the effort of the policyholder. However, the insurance company does not know the policyholder's actual effort to reduce the risk of the damage. The insurer can merely estimate it from the claim amount (Shavell (1979)). With the use of a BMS, the insurer motivates the policyholders to reduce risk. This is so because if someone has a claim, then the following period he/she will be assigned to a worse class (if there is any). Hence, his/her premium will increase.

However, Vanderbroek (1993) shows that the insurance's partial coverage is more effective in the motivation to induce a better level of care than the BMS.

The typical appearance of moral hazard in the MTPL insurances is the so-called "bonus-hunger", meaning that policyholders choose self-financing the damage rather than claiming to the insurance company. If the claim amount is lower than the premium increase of the following periods, then it is not worth reporting a claim to the insurer (De Prill (1979); Sundt (1989)).

Adverse selection and moral hazard are usually present simultaneously (as in the case of BMS). Holton (2001) investigated the moral hazard and adverse selection with an expected utility model. The author found that the BMS is only Pareto optimal (optimal for both the insurance company and the policyholders) if the adverse selection and the moral hazard or the insurance costs are considered in the analysis. There is broader literature about adverse selection and moral hazard in the field of *Contract theory* (see for example Bolton and Dewatripont (2005)).

In the literature, there is less emphasis on the empirical tests of adverse selection and moral hazard. The empirical findings are not entirely straightforward: see, e.g., Dahlby (1983) and Puelz and Snow (1994) who argue for the existence of adverse selection while Chiappori and Salanié (2000) are against it.

The existence of moral hazard in automobile third-party liability insurance has been empirically studied in some articles: Lee and Kim (2016) analyzed the Korean BMS, Dionne et al. (2013) analyzed the French system, Vukina and Nestić (2015) researched on Croatian data, and Abbring et al. (2008) analyzed the Dutch BMS.

In the mathematical literature, the BMS appears in the applications of Markov Processes. For example, Molnar and Rockwell (1966) introduces the BMS as an application of a Markov chain.

A BMS's efficiency is typically measured by an indicator called "elasticity" introduced in Loimaranta (1972). Elasticity shows how expected payment will increase if risk increases by 1%. A good BMS's elasticity is over zero, ideally one, but can be over one as well.

Lemaire (1995) conducted an empirical study on some BMSs that were used in practice. If the BMS's premiums do not vary that much, the elasticity is usually under 1 in practice. De Prill (1978) generalized the Loimaranta's efficiency.

Loimaranta (1972) also shows that the elasticity for measuring a BMS's effectiveness is not a perfect indicator. In MTPL insurance, the claim probabilities have relatively low values. Hence it is not impossible to construct a BMS with elasticity equal to one. With the optimization of the premiums, a very steep premium-scale would be the result to reach the ideal elasticity. However, Loimaranta (1972) argues that in that case, the insurance would be pointless. A too steep premium-scale results in a very volatile payment for the policyholders. Hence the insurance would not provide economic security to the policyholder.

The economic literature calls this as the risk-averse decision-maker (in this case, the policyholder), not only consider the premium but its fluctuation. Hence the policyholder may be satisfied with a slightly higher expected premium on the condition that the variation of premiums is decreased significantly is somewhat disregarded (or partially regarded). Nonetheless, in Loimaranta (1972) a model minimizes the variance of the premium scale assuming a fixed level of elasticity.

The transition rules define the classification of a BMS. Transition rules tell us how many classes the policyholders should go down in the following period if they have claims. Additionally, there should be a claim-free transition rule which sends the policyholder one or more classes up in the subsequent period. Thus, designing a BMS requires choosing the transition rules between the classes and determining the number of classes, the scale of premiums, and the initial class.

There are many papers about the optimization of BMS's (e.g., Cooper and Hayes (1987); Lemaire (1995); Denuit et al. (2007); Heras et al. (2004); Brouhns et al. (2003); Denuit and Dhaene (2001); Mert and Saykan (2005); Najafabadi and Sakizadeh (2017)). Typically, in these works, the number of classes, the transition rules, and the initial class are fixed while the scale of premiums is determined in the optimization process. In these studies, the policyholder is usually not represented with a utility function. A rare exception is Lemaire (1995), where there is a model where the policyholder has an exponential utility function.

Recently, Tan et al. (2015), have incorporated in their model the effect of the transition rule changes.

In our research, we focused on the optimization models of the transition rules. The fundamentals of the model come from two directions. We used a similar approach to the LP model of Heras et al. (2004) for the optimization of the premiums.

For the modeling of the transition rules, we also considered the cash-balance model introduced in Ghellinck and Eppen (1967); Eppen and Fama (1968). The objective of this problem is to minimize the costs of a firm to keep sufficient cash (or deposit) for day-to-day transactions. The task is to find an operating policy over a predetermined period. Then, on each day, the company can decide how much cash they would keep. Keeping more cash than necessary has a holding cost because the cash can have alternative uses (such as investment). On the other hand, if the firm keeps less cash than necessary, it entails penalty costs for delaying the demand. The company can increase or decrease its capital, but there is a transaction cost as well. In general, the daily inflows and outflows of cash are not deterministic in this problem.

Eppen and Fama (1968) introduced an LP model for this problem. In the model, each day's possible cash is classified into a finite number of classes. For each period, transition probability variables were introduced. These variables denote the probability of being in cash-level and then moving to another (or staying in the same) cash-level in the subsequent period. The model had an alternative form, where stationary probabilities are considered. This approach is very similar to the modeling of the transition rules of BMS. However, in this case, instead of policyholders, the cash levels are determined in the classification.

2.1 Bonus-Malus in practice

Lemaire (1995) compared some BMSs that are used in practice. We presumed since 1995, perhaps there have been some changes in these systems. Therefore, we tried to find out how the regulations changed in the past few years in some of the countries Lemaire presented.

In many countries, such as Belgium and Portugal, the regulations were liberated. Hence the insurance companies can create their merit system to consider the claim history. Some of the countries already had liberated regulations in 1995, such as the United Kingdom and Sweden. However, there are still some countries where the regulations are strict, such as Hungary or Luxembourg. Hence the operating insurance companies have to use the same transition rules in the BMS. Also, there are countries, like Germany and the Netherlands, where the insurance companies can change some parameters of the centrally determined BMS. However, we did not find any countries where the rules stricken from liberated status since 1995. Therefore, in general, the regulations on the MTPL insurance regarding the BMS became less strict. However, even in countries with minimal regulations, the insurance companies have to use a merit system to consider the policyholders' claim history. Therefore, we found that the Bonus-Malus System is generally the most used method in these countries.

Therefore the need for the optimization of the insurance company's BMS can be relevant in these countries. However, for insurance companies, other aspects can be more important than adverse selection. For example, bonus protection is usually advertised in those countries where the rules enable individual BMSs for insurance companies. It means that the policyholder can insure their earned bonus; hence they would stay in the same class despite their claim. Because of this, the classifications of the policyholders in the system may differ from their actual riskiness. On the other hand, the bonus protection does not help in handling the moral hazard problem. Assumably will not incentivize the policyholders to drive more carefully as much as it would without protection.

In this section, we present some countries' current regulations regarding the BMS in MTPL insurance. We compared the systems in Lemaire (1995) and the current regulations that we found via internet search (for the sources, see Appendix 9.4). We were particularly interested in penalizing the claims and the periods that the policyholders have to spend to earn the highest discount from the beginning of the contract. Because the classes' enumerations differ from country to country, we use the phrase number of bonus classes for those classes that are better than the initial class—furthermore, number of malus classes for those classes, which have a higher premium.

Austria

The regulations are fully liberated. However, the insurance companies have to consider the claim history of the policyholder. Therefore if there is a claim, the premium of the policyholder has to increase. Usually, the insurance companies use BMS for the consideration of the claim history. However, the rules of the system may vary from insurance company to insurance company.

Belgium

BMS has been used for MTPL insurance in Belgium since 1971. In the first regulation, all insurance companies had to use the same rules. There were 18 classes, with two types of initial class: one for the regular drivers' vehicle and one for the vehicles used for business. The regular initial class had five bonus classes, while the business had 9. The transition rules were unified in each class. The policyholders moved one class upward if they were claimless in a period. The first claim resulted in a two-class decrease. For any further claims in that period, the down steps increased to three per claim. Those policyholders who had four consecutive periods without any claim were reclassified into the initial business class if they were in a worse position.

In 1992, the number of classes was increased to 23. Furthermore, transition rules became more strict. The first claim's punishment increased to a 4 class downgrade, and each further claim resulted in a 5 class reduction. The separated initial classes were kept. However, the regular initial class had 11 bonus classes, while the business had 14.

From the beginning of 2004, insurance companies can determine their system. Insurance companies, in general, use BMSs. In addition, all of the insurance companies use the so-called Joker, which is a Bonus-protection. The use of the Joker differs in each insurance company, but in general, it is applied after several claim-free periods. Gaining a Joker also depends on the insurance company: some give it for free if the policyholder drove without damage for a certain period. In other insurance companies, the policyholders can buy it. Finally, in some insurance companies, even multiple claims are not considered because of the Joker.

Finland

In 1995, the Finnish BMS had 17 classes with two malus classes and 14 bonus classes. It had non-unified transition rules; therefore, the rule of reclassification of a policyholder based on the class where he/she was assigned. In the initial class, the claim-free period resulted in upwards of two steps, while in every other class, it was only one. The penalty for the claims varied between 4 and 3. An interesting feature of this system was that the policyholders could not return to the initial class after the first period.

In 2017 there was a change in the legislation. Therefore the insurance companies have more freedom to consider the claims history in determining the premiums of the MTPL insurance. However, those insurance companies that we found in our search still use similar BMSs as Lemaire (1995) presented. In addition, however, the number of classes increased to 22, with more bonus classes.

France

In France, there is a multiplication method. Each policyholder starts with a coefficient of 1. If there is not any claim in a period, then the coefficient is multiplied by 0.95. If a claim occurs, the coefficient is multiplied by 1.25. Then the final premium of the insurance is multiplied by the coefficient. The best possible coefficient is set to 0.5. Therefore within 13 years, the policyholder can reach the highest possible discount. The maximum coefficient is maxed at 3.5.

Germany

Lemaire (1995) presented an old German system used in the early 80s and a newer one. The old one had 18 classes, and it was extended to 22 in the newer one. Both systems had non-unified transition rules. A period without claim resulted in one positive step in every class for both BMSs. However, the downgrades of the claims depended on the class where the policyholders were assigned. The interesting points of these systems were that there were two initial classes. The class where the new contracts started was determined by the experience and the number of vehicles the policyholder owned.

Nowadays, the regulations are a bit more relaxed. There are 54 classes, with two initial classes, where the policyholders cannot return. The initial class depends on the experience: if the policyholder has a driving license for a longer period, he/she starts in the better initial class. Otherwise, he/she starts in the other class for novice drivers. After a policyholder leaves the initial class, he/she cannot return to it. The claim-free periods always result in one upgrade in the system. The insurance companies independently specify the downgrades of the claims. The majority of the classes are bonus classes; only two malus classes are considered. One of them is between the two initial classes, and the other malus class is the worst-class.

Hungary

In Hungary, from 1982, a unique system was introduced (*25/1982. (IV. 9.) PM rendelet*) for the MTPL insurance. In this system, the premium of the insurance was calculated into the Gasoline price.

It was better than the previously used tax-based insurance because it considered the drivers' vehicle-usage time. In a tax-based insurance case, the vehicle owners had to pay this tax without considering vehicle usage. Therefore in the tax-based insurance system, those drivers who drove more did not get a higher premium. Presumably, those drivers who use their vehicles less have a fewer chance to cause collisions. Therefore a tax-based system was not that truthful in this sense as the gasoline-price insurance. However, the gasoline-price insurance could not handle the Moral hazard problem. Also, the experience of the driver was not considered.

Therefore this system was changed to a BMS in 1991. It was better than the gasoline-price insurance because the BMS could handle the Moral hazard. Also, those who drive more and thus have a higher chance of causing a collision may pay a higher premium as well. This system can also handle the drivers' experience level, which was not considered in the previous insurances. Usually, beginner contracts start from a relatively high premium class. However, if the driver becomes more experienced, their risks decrease, they may go upward in the system.

The system of 1991 is still in use in Hungary. There are 15 classes, where the best 10 class (*B1, . . . , B10*) are called as bonus classes. The worst 4 (*M1, . . . , M4*) are called as the malus classes. Between the Bonus- and the malus classes, there is the *A0* initial-class, where all beginner policyholder starts.

In Hungary, all of the insurance companies have to use this Bonus-Malus System. They cannot change any parameter of the transition rules. The transition rule is the following: If the policyholder had a claim-free period, then he/she moves one class upward, and there are two class reductions per claim. However, if a policyholder has at least four claims, in the next period, he/she will be classified into the worst class (*M4*).

The regulation allows the policyholders with claims to pay the claim size by themself. In this case, the claim is not considered in the classification. So the Bonus-hunger is observable for the smaller-sized claims.

Italy

Before 1991, the insurance companies were obliged to use a 13-class BMS. It had four malus classes and eight bonus classes. The policyholders' class was downgraded by one class if they had a claim. Their classification was upgraded to one class if there was no claim in the period. However, the recovery was faster from a malus class: the policyholders without claim moved two classes upward from these classes.

In 1991, the previous system was changed to an 18 class BMS. In this system, there are four malus classes and 13 bonus classes. The no-claim period results in one class upgrade in the system and a two-class downgrade per claim. This system is still in use.

Luxembourg

Lemaire (1995) presented an 'old' and 'new' BMS as well. In the older system, there were 22 classes and had a unified transition rule. The claim-free period resulted in one class improvement, while a claim reduced the classification with two classes. In addition, there were ten bonus classes and 11 malus classes. There was one special rule: a policyholder with four consecutive years without any claims was reclassified into the initial class if he/she was in a malus class.

In the 'new' system, the number of classes increased by two bonus classes. Also, the penalty for a claim is increased to three from two.

Nowadays, insurance companies still have to use centrally regulated rules. Currently, there are 26 classes. In addition, there are 14 bonus classes and 11 malus classes. The transition rules did not change since 1995, the claim-free period results in one step upward, and 3 class reductions penalize the claims. The special rule is still in use as well.

Netherlands

Lemaire (1995) presented a Dutch BMS that had to be used by each insurance company. In this system, there were 14 classes. The transition rules were non-unified. The claim-free period always resulted in one improvement, while the first claim decreased the class by 5,4, or 3, depending on the current classification. The reduction increased if the policyholder was classified into a higher bonus class. The second claim resulted in 4 additional class reductions, uniformly. In this BMS, there were four initial classes. The new policyholders' age determined the starting initial class.

Nowadays, some parts of the BMS are regulated. The insurance companies use at least 21 classes. The transition rules are unified and centrally regulated. A claim-free year results in one positive step in the ladder, and a claim results in five negative steps. The premiums of the BMS differ from insurance company to insurance company. The insurance company also determines the initial class.

Norway

In Norway, the regulations are not strict; the insurance companies can use their own BMS. Up to 1987, almost every insurance company used the same system. There were seven bonus classes; however, the number of malus classes was theoretically infinite. There were 14 normal malus classes, but there existed even more: if a policyholder had a claim and stepped over to the 14th malus class, a class is created. The transition rules were non-unified. The first claim was punished more in the top three bonus classes, with three class reductions. In every other class, the policyholders step two classes downward in the event of one claim. The second claim always resulted in two additional class reductions. When there was no claim in a period, it resulted in one class improvement. When the policyholder was under the 14th malus class, he/she was reclassified into the malus class 14 after a period without claim.

In 1987, the leading insurance company changed its system, described in Lemaire (1995). This system was more straightforward than the previously used: If a policyholder did not claim, he/she received a 13% bonus on the premium. For each claim, there was a fixed amount of extra payment. In addition, the insured vehicle's age determined the initial premium level: vehicles older than 25 years had a 20% increase in the initial premium.

Nowadays, several different BMSs exist in Norway. However, we found some insurance companies with similar systems. The policyholders receive a 10% bonus after they have a claim-free year. After that, it can go up to a 70% discount, and after that, the rules change from insurer to insurer. The best discount, 75%, can be reached from 70%, after 3 or 5 years without claims. After that, the claims are usually punished by a 10% decrease in the discount.

Portugal

Lemaire (1995) presented a Portuguese system that had only six classes. They improved one class if there were two consecutive years without claims. In the case of claims, the policyholders moved downward in the system.

Nowadays, insurance companies can have their merit system. As a result, insurance companies typically use BMS. However, the regulations only state to use a method where a claim shall increase the insurance premium.

Romania

There are 17 classes, eight bonus, and eight malus classes. If there is no claim in a period, the policyholders improve their classification by one. Thus, they fall two classes per claim. Before 2017 there were 14 bonus classes, but it was reduced to eight recently.

3 Preliminaries

There is usually an assumption that the policyholders pay only the system's premium in the MTPL insurance in the BMS optimization models. Therefore there are not any other factors that determine the payment of the policyholders. Hence the policyholders' payment in a period only depends on where they are classified in that period. Generally, the classifications in a BMS happen in each period, which is the assumption in the optimization models. Getting into a class in the subsequent period only depends on the currently assigned class. Therefore, knowledge of the previous periods' classifications is not necessary.

Let X_t denote the class, where the policyholder is classified in the period t. Moreover, let us consider any ρ statement that is already known before period t.

$$\mathbb{P}(X_t = k | X_{t-1} = l, \rho) = \mathbb{P}(X_t = k | X_{t-1} = l) \qquad \text{(Markov property)}$$

Because the policyholders' classification only depends on the number of claims and the previous period's classification, the classification process has the Markov property. Furthermore, the process holds the Markov property and can be considered a Markov chain because of the BMS classification rules.

Definition 1. *A discrete-time stochastic process is called a* ***Markov chain*** *if*

$$\mathbb{P}(X_{t+1} = k_{t+1} | X_t = k_t, X_{t-1} = k_{t-1}, \dots, X_1 = k_1, X_0 = k_0) = \mathbb{P}(X_{t+1} = k_{t+1} | X_t = k_t) \tag{1}$$

holds for every period t.

In our research, we assume that this condition holds for all of the considered BMSs. A BMS in practice may differ in a way that does not strictly follow this definition. For example, suppose there is a BMS, where the policyholders move upward only when they have two consecutive years without a claim (such as in the system used in Portugal). In this case, not only the last period determines the probability. However, in this case, only the previous two periods' classification is needed to calculate the exact probabilities.

Hence, for simplicity, we assume that in every considered BMSs, the classification only depends on the previous period. Therefore in every case, we assume the classification of the policyholders is a Markov chain. In this section, we present some theorems of the Markov chains that are essential for optimizing a BMS. We used the proofs for these theorems that were introduced in Kemeny and Snell (1976).

Definition 2. *The **t-th step transition probabilities** for a Markov chain are*

$$p_{k,l}(t) = \mathbb{P}(X_{t+1} = k | X_t = l)$$

Definition 3. *A Markov chain is homogenous if the t-th step transition probabilities $p_{k,l}(t)$ do not depend on t.*

Therefore, in a homogenous Markov chain, we denote the transition probabilities with $p_{k,l}$.

In realistic situations, the policyholders become more experienced over time, and thereby their claim probabilities may decrease in the future. The effect of the young, inexperienced policyholders' risk exceeding the average risk is usually called the "duration effect".

We will show in this section that a homogenous Markov chain (with other conditions) has a unique stationary distribution. Therefore, even though it is a very unrealistic assumption for the analysis of BMSs, it is generally assumed in the literature (e.g., Arató and Martinek (2014); Bonsdorff (1992); De Prill (1978); Heras et al. (2004); Lemaire (1995); Loimaranta (1972)). Therefore in our study, we also consider this condition. For the sake of simplicity, in the following, when we mention a Markov chain, we will always refer to the homogeneous case.

Definition 4. *The transition probabilities formulated in a matrix form is called as the **transition probability matrix** (P).*

In a Markov chain, a simple formulation is used to assign probability to a particular class in a period. For the optimization of the BMS, the statement of Theory 1 is essential to calculate the probabilities of being in a class.

Theorem 1.

$$\mathbb{P}(X_t = k) = \sum_h \mathbb{P}(X_{t-1} = h) p_{h,k}$$

Proof. Let $k_0, k_1, \ldots, h$ denote a possible sequence of classes that a policyholder can be classified from the period 0 to t. The probability of belonging to the class k at the period t is the sum of the probabilities of all possible sequences stepping into this class:

$$\mathbb{P}(X_t = k) = \sum_{(k_0, k_1, \ldots, h)} \mathbb{P}(X_0 = k_0, X_1 = k_1, \ldots, X_{t-1} = h, X_t = k)$$

The t-th period's probability can be reformulated into conditional probability:

$$\mathbb{P}(X_t = k) =$$
$$\sum_{(k_0,k_1,\ldots,h)} \mathbb{P}(X_0 = k_0, \ldots, X_{t-1} = h)\mathbb{P}(X_t = k | X_0 = k_0, \ldots, X_{t-1} = h) =$$
$$\sum_{(k_0,k_1,\ldots,h)} \mathbb{P}(X_0 = k_0, \ldots, X_{t-1} = h)P_{h,k}(t)$$

If we fix the h in the last sum, we may get

$$\mathbb{P}(X_t = k) = \sum_{(h)} \mathbb{P}(X_{t-1} = h)P_{h,k}(t)$$

In a Markov chain, the transition probabilities ($p_{h,k}(t)$) do not depend on the period (t). Hence, the last equality completes the proof.

□

For the sake of simpler notation let $c_{k,t}$ denote the probability that the policyholder in period t is classified into class k. Hence $c_{k,t} = \mathbb{P}(X_t = k)$. We denote C_t the row vector form of the $c_{k,t}$ variables. Because of Theorem 1, the next equation holds, for $t \geq 1$:

$$C_t = C_{t-1}P(t)$$

Where $P(t)$ denotes the matrix with entries $p_{h,k}(t)$. We may substitute the $C_{t-1} = C_{t-2}P(t-1)$ into this equation. Then with continuing the substitutions, we may get

$$C_t = C_0 \cdot P(1) \cdot P(2) \cdot_{\ldots} \cdot P(t)$$

And because in a Markov chain, the $p_{h,k}(t)$ probabilities are not depends on the period, equation 2 is held.

Theorem 2. *For each period for the probabilities $c_{k,t}$, organised in C_t row vector, the following equation should hold:*

$$C_t = C_0 P^t \tag{2}$$

The states of a Markov chain (in our case, the BMS classes) can be classified into two sets according to its movement from one class to another. The two sets are called the *ergodic set* and the *transition set*. Class l can be reached from class k if a policyholder in class k can be classified into class l later on (not necessarily in the consecutive period). We use the notation $k \mapsto l$ if the class l can be reached from the class k.

Definition 5. *Let E denote the **ergodic set**. Class $k \in E \Leftrightarrow$ if $k \mapsto l$ then $l \mapsto k$, for all $l \in E$*

A class k is an element of the ergodic set if class k can be reached from every class of the ergodic set, and from class k, every element of the ergodic class can be reached. Those classes that are not in an ergodic set are elements of a transition set.

In the general definition of the BMS, all of the classes are part of one ergodic set. Hence the classification of the policyholders is a Markov chain without a transition set. However, as section 2.1 presented, in some BMSs, the policyholders cannot return to the initial class. Hence these initial classes will be part of the transition set. In general, in a Markov chain, the process cannot return to the transition set once it leaves it. From the ergodic set perspective, once the process enters, it cannot leave this set. In the ergodic set, we may distinguish one particular class, which absorbs the policyholders.

Definition 6. *Class k is an **absorbing class**, if $P_{k,k} = 1$*

An absorbing class cannot be in a BMS. It would mean that once a policyholder gets into this class, it can never leave it. It would go against the basic ideas of the BMS: It would not motivate the drivers to drive more carefully and would not distinguish the "good" policyholders from the "bad" ones. In those countries where regulations are liberated, many insurance companies offer the so-called Bonus protection. In general, the bonus protection guarantees the policyholder to stay in a class (usually with the highest discount), with a 100% probability. However, it is a temporary option. Hence the class is only absorbing until a policyholder does not have a claim. In this case, the transition probabilities depend on the period.

The classification of the policyholders has only one ergodic set and does not have any transition set. Hence it means that the chain is irreducible.

Definition 7. *A Markov chain is said to be **irreducible**, if for every class k, j and period t there exists a period s, such that*

$$\mathbb{P}(X_{t+s} = k | X_t = j) > 0 \tag{3}$$

Hence, every class can be reached from any class. In the optimization of the BMS, we assume that the classification is irreducible. Hence we construct the transition rules accordingly. Furthermore, in the case of the BMS, the chain is even aperiodic.

Definition 8. *Let θ_k of class k be $\theta_k = gcd\{t \geq 1 : p_{k,k}(t) > 0\}$, where 'gcd' denotes the greatest common divisor. Let $\theta_k = \infty$ if $p_{k,k}(t) = 0$ for all $t \geq 1$.*
*Therefore class k is aperiodic if $\theta_k = 1$. And a Markov chain is **aperiodic** if every class is aperiodic.*

The Markov chain of all of the considered BMSs is aperiodic. If the policyholder is in the highest class, it is impossible to classify him/her into a higher class. Besides, from the lowest class, the policyholders cannot be classified lower. Therefore, the finite number of classes and the considered transition rules ensure the Markov chain's aperiodicity.

In a Markov chain with one ergodic set and the property of aperiodicity, no matter where the process starts, it can be in any class, after a sufficient number of periods. This chain is called a *regular Markov chain* in the literature. Hence for the transition probability matrix, the Theorem 3 should hold if the Markov chain is regular.

Theorem 3. *A transition probability matrix is regular, if and only if, for some t P^t does not have any zero elements.*

Hence, as we increase the number of periods (t), there will be a P^t matrix with only positive elements. Therefore, because of the irreducibility, every class can be reached from any class. Furthermore, over time, the change between the matrix P^t and P^{t+1} is becoming smaller. Hence P^t tends to a probability matrix, without zero elements, as the t increases.

Theorem 4. *If the transition probability matrix P is regular, then*

1. *As t increases, P^t approach a probability matrix A*
2. *All row of matrix A are the same probability vector: $C = \{c_1, \ldots, c_K\}$.*
3. *$A = \mathbb{1}C$, where $\mathbb{1}$ denotes a column vector, filled with only ones.*
4. *Every element of C is positive.*

Proof. Let us introduce a column vector y_j with K components. We set the j-th element of this vector to 1 and every other component to 0.

In the first part of the proof we only focus on transition probability matrices with no zero elements.

In this case, a connection between the minimal and the maximal elements of the vector $P^t y_j$ and $P^{t+1} y_j$ for each t period exists. Let γ_t denote the minimum of the elements in $P^t y_j$ and Γ_t the maximum. Also we introduce a vector $\widehat{P^t y_j}$, obtained from vector $P^t y_j$, with changing all elements to Γ_t, except one γ_t component. Hence $\widehat{P^t y_j} > P^t y_j$. All element of the $\widehat{P^t y_j}$ can be written as

$$\beta\gamma_t + (1-\beta)\Gamma_t \quad ,$$

where the β cannot be smaller than the minimal element of the P^t matrix, that we denote with ρ^t.

Therefore each element of the $P^t y_j$ cannot be larger than $\rho^t\gamma_t + (1-\rho^t)\Gamma_t = \Gamma_t + (\gamma_t - \Gamma_t)\rho^t$. And because $\widehat{P^t y_j} > P^t y_j$, and $P^{t+1}y_j = PP^t y_j$,

$$\Gamma_{t+1} \leq \Gamma_t + (\gamma_t - \Gamma_t)\rho^t \quad . \tag{4}$$

If we consider the $-\widehat{P^t y_j}$, then

$$-\gamma_{t+1} \leq -\gamma_t - (\Gamma_t - \gamma_t)\rho^t \quad . \tag{5}$$

And if we add (5) to (4), then we get:

$$\Gamma_{t+1} - \gamma_{t+1} \leq \Gamma_t - \gamma_t + (\gamma_t - \Gamma_t)\rho^t - (\Gamma_t - \gamma_t)\rho^t = (1-2\rho)(\Gamma_t - \gamma_t) \tag{6}$$

This results that for all t, $\gamma_t \leq \gamma_{t+1}$ and $\Gamma_t \geq \Gamma_{t+1}$.

Let us denote $s_t = \Gamma_t - \gamma_t$. Therefore the equation (6) can be written as

$$s_{t+1} \leq (1-2\rho)(s_t) = (1-2\rho)^{t+1} \quad . \tag{7}$$

Because

$$\lim_{t\to\infty}(1-2\rho)^{t+1} = 0 \quad ,$$

the s_{t+1} is tends to 0 as t increases. Hence the Γ_t and γ_t tends to a common value, that is positive and less than 1, that we denote by c_j.

Because y_j consists only one 1 value and every other is zero, the jth column of $P^t y_j$ has only the c_j values. Therefore as the t increases, the P^t approaches an A matrix, which all row has the $C = c_1, \ldots, c_K$ vector. Because each row's sum is equal to 1 in P^t, it will approach A as t increases.

If the transition probability matrix has zero elements, there is a period u, where P^u does not have any zero component because of the irreducibility property. And after period u, for any $t > 0$ P^{u+t} will not have any 0 element. Hence, it will also tend to an A probability matrix.

□

Theorem 5. *For any C_0 initial probability row-vector: $C_0 P^t$ is tends to C as t goes to infinity.*

Proof. If C_0 is a probability vector, then $C_0\mathbb{1} = 1$. Therefore $C_0 A = C_0\mathbb{1}C = C$. But due to $C_0 P^t$ tends to $C_0 A$ as the t increases, it has to tend to C.

□

Theorem 6. $PA = AP = A$

Proof. P^t tends to A, as the t increases. Furthermore, $P^{t+1} = P^t P$ is also tends to A, but in the mean time it is also tends to AP. Hence $AP = A$.

□

Theorem 7. *C is unique, therefore $CP = C$.*

Proof. Let D be another probability row-vector. And let us assume that $DP = D$. Because of Theorem 5 DP^t tends to C as t increases. However because $DP = D$, $DP^t = D$ too, which tends to C. Hence it is only possible if $C = D$, therefore C is unique.

□

Therefore in the BMS, the c_k denotes the probability that the policyholders are classified into class k in the long run. We will refer to these probabilities as the stationary probabilities of the BMS. From Theorem 7, the vector of these probabilities is unique. Hence there is only one stationary probability for each class. Moreover, because of Theorem 5, these probabilities are independent of the process's start. Hence, determining the initial class is not essential in the BMS optimization models, which consider the stationary probabilities. No matter which class is where the policyholders start, the stationary probabilities will be the same.

3.1 Preliminaries of the optimization models

We assume that there are I different risk groups (types) among the policyholders. Each type has a different risk that does not change over time.

With this assumption, the classification of the policyholders is a regular Markov chain. Therefore the stationary probabilities can be used in the optimization. Although the condition that the transition probabilities cannot change over time and such changes are unrealistic, as we mentioned earlier, it is often assumed in the analyses of the BMS.

An exception is Borgan et al. (1981), in which the authors use weights on the periods to analyze the impact of the duration effect. Another exception is Niemiec (2007), where the claim probabilities depend on the time, and the BMS was analyzed with ergodic Markov set-chains.

However, for simplicity, we assume the transition probabilities independent in time; hence, the theory of regular Markov chains can be applied in the optimization.

Furthermore, it is necessary for the regular Markov chains that all policyholders stay in the system. Hence the ratios of the risk groups are also independent in time. It is also a general assumption for the analysis of the BMS. However, this condition is not met in reality. In practice, the policyholders may change insurance companies (which may have different transition rules). In addition, the age of the policyholders determines the risk group. For example, younger drivers are generally riskier, and we may consider them in a risk group, but they will eventually become part of another risk group over time.

For the sake of simplicity, we used these two unrealistic assumptions in our models. Although we did not investigate this case, in section 5.2 we present a multi-period model in which the conditions of the regular Markov chains are not necessary. Hence, this model can be extended for those cases where the transition probabilities and the number of policyholders may depend on time. However, we did not research the models with realistic conditions because the computational time of the multi-period model was very lengthy, even with the fundamental assumptions.

In the practice of BMS, transition rules are based only on claim numbers, and the claim amount is ignored. This is reasonable since the risk groups can be distinguished more accurately by the number of claims than the (conditional) claim amount. We use the same assumption in the MILP model. Therefore we only consider the number of claims. For the sake of simplicity, we assume that the claim amount is the same for each type of policyholder (in every model, we assume it is one for each risk group).

Let $M > 0$ be the highest number of possible claims in a period and let λ^i_m be the probability of the occurrence of m claims for the policyholders of type i ($i = 1, \dots, I$, $\sum_{m=0}^{M} \lambda^i_m = 1$). We denote the risk-parameters (expected claim amount) for risk group i with λ^i, ($\lambda^i = \sum_{m=0}^{M} m\lambda^i_m$). The types are indexed in an increasing risk order to keep notation simple. The expected claim amount is the least for type 1 and the highest for type I. Let ϕ^i be the proportion of the type i policyholders among all of the policyholders ($\sum_{i=1}^{I} \phi^i = 1$). In BMS there are $K+1$ classes indexed from 0 to K. The premium of class k is denoted by π_k. In a BMS the premiums should be monotonic, hence we assume that $\pi_{k-1} \geq \pi_k$ ($k = 1, \dots, K$).

We assume that the payment for the insurance for all policyholders depends only on classification. However, in practice, the insurance company may also use other methods to determine the "fair" premium besides the BMS. We investigate this case in section 6. Also, each insurance contract has several costs that the insurance company must consider in determining the insurance premium. We assumed that each policyholder's contract had the same cost. Therefore we excluded the costs from the optimization.

Transition rules determine how the policyholders will be reclassified after a certain number of claims in a period. Hence, there is a transition rule for all m claims, and we can write the rules in matrices T^m. Each T^m is a binary matrix that means if the element of row k_1 and column k_2 equals to one ($T^m_{k_1,k_2} = 1$), then a policyholder with m claims and currently in the class k_1 will be reclassified into the class k_2 in the next period.

As section 3. presents, the policyholders' classification in a BMS can be considered as a regular Markov chain. In the model, we consider multiple types of policyholders with different claim probabilities. Hence for each type i, there is a transition probability matrix, depending on the λ^i and the common transition matrix. Theorem 7 shows that the regular Markov chain has a unique stationary probability distribution. In the model, there is a different regular Markov chain for each type. Let c^i_k be the probability that the type i policyholders is classified into class k after spending sufficiently enough time in the BMS. We will refer to the c^i_k as the stationary probabilities of the type i in class k. Because of Theorem 7, the distribution of these stationary probabilities can be calculated by solving the following system of equations, separately for each type:

$$c^i_k = \sum_{j=0}^{K} c^i_j p^i_{j,k} \quad k = 0, \ldots, K \tag{8}$$

$$\sum_{k=0}^{K} c^i_k = 1 \ . \tag{9}$$

Where the $p^i_{j,k}$ denotes the transition probability of the type i policyholders from the class j to k. The stationary probabilities can be written using the transition matrix' components as:

$$p^i_{j,k} = \sum_{m=0}^{M} \lambda^i_m T^m_{j,k} \tag{10}$$

Hence substituting (10) into (8) the first part of the system of equations can be written as:

$$c^i_k = \sum_{j=0}^{K} \sum_{m=0}^{M} c^i_j \lambda^i_m T^m_{j,k} \quad k = 0, \ldots, K \ . \tag{11}$$

And because there are multiple types of policyholders, we weigh the probabilities according to the risk group's ratio. Therefore we change equation (9) into:

$$\sum_{k=0}^{K} c^i_k = \phi^i \ . \tag{12}$$

4 Optimization models of Bonus-Malus Systems

In this section, we present a linear programming model for optimizing the premium scale and transition rules. In the literature of BMS's, using LP (or MILP) technique is not very common. Only one known LP model exists, introduced in Heras et al. (2004). Therefore, we adopted the assumptions made in that article.

Optimizing a BMS means looking for an appropriate premium scale and transition rules that minimize the difference between the expected claim amount and the premiums in some norm:

$$\min_{(\pi_0,\pi_1,\ldots,\pi_K,T^0_{0,0},T^1_{0,0},\ldots,T^m_{k_1,k_2},\ldots,T^M_{K,K})} \sum_{k=0}^{K}\sum_{i=1}^{I} \phi^i c^i_k(T^0_{0,k}, T^1_{0,k}, \ldots, T^M_{K,k}, \lambda^i_0, \lambda^i_1, \ldots, \lambda^i_M) d(\pi_k, \lambda^i)$$

subject to

$$\text{constraints on the decision variables,}$$

where $d(.,.)$ is usually the ℓ^2 or ℓ^1 norm (see, Norberg (1976); Heras et al. (2002)). The most used constraints are the profitability constraint and constraints on the premium scale: e.g., the difference between the premiums of two consecutive classes cannot be more than 20%. Certainly, besides these most used constraints, we can give other constraints for special purposes. The above minimization problem is nonlinear. In section 4.1-5 we describe how the problem can be linearized.

4.1 Optimizing the premium scale when the transition rules are fixed

Optimizing the premium scale means that we seek the appropriate premiums for a BMS with fixed transition rules. Since transition rules are external parameters, the stationary probabilities are parameters as well. The following LP can obtain the premiums:

$$\min \sum_{i=1}^{I} \sum_{k=0}^{K} \phi^i g_k^i \tag{LP1.obj}$$

Subject to

$$\pi_k c_k^i + g_k^i \geq \lambda^i c_k^i \qquad \forall i, k \tag{LP1.1}$$

$$\pi_k c_k^i - g_k^i \leq \lambda^i c_k^i \qquad \forall i, k \tag{LP1.2}$$

$$\pi_{k-1} \geq \pi_k \qquad k = 1, \ldots, K \tag{LP1.3}$$

$$\pi_k \geq 0 \qquad \forall k$$
$$g_k^i \geq 0 \qquad \forall k, i$$

Because the transition rules are fixed, the stationary probabilities can be calculated. Thus for each class (k) and group (i), the c_k^i are known parameters. Besides, each group's expected number of claims (λ^i) and their ratios (ϕ^i) are also outer parameters.

The variables of the model are the premiums of the classes (π_k). The objective is to find a premium scale where each groups' expected payment is as close to their expected number of claims as possible. Thus, in this LP model, we minimize the absolute deviation. Hence, we introduce g_k^i auxiliary decision variables that denote the absolute deviations of the group i in class k. Constraints (LP1.1) and (LP1.2) define the deviation of group i and class k. Constraints (LP1.3) set the premium scale to be monotonic. In the model's objective function, the absolute deviation variables (g_k^i) are weighted by the ratio of the groups (ϕ^i).

Moreover, it is worth remarking that an approximation of a quadratic (and many other) loss functions can be used in an LP model instead of the absolute deviation.

The first concept of this LP model was introduced in Heras et al. (2002). In this article, the authors argue that using absolute deviation is reasonable since it is better than the quadratic deviation to distinguish the 'bad' and 'good' policyholders. The quadratic loss function consider the overpayments and underpayments equally around the “fair” premium.

Also, Box and Tiao (1973) presents practical situations where it is not reasonable to use the quadratic loss function. Furthermore, Smith (1988) argues that in simple decision problems using piecewise linear functions for the loss function is viable. Hence, an absolute loss function is reasonable in these cases since it is the simplest piecewise linear loss function.

If the operator of a Bonus-Malus System insists on quadratic loss function, it can be approximated with piecewise linear functions. Therefore we can solve its approximation as a linear model. In later, we will extend this model to optimize the premiums, and the transition rules jointly. We use binary variables for this. Hence the model is a Mixed Integer LP (MILP). However, without linear approximation, the joint MILP model leads to a quadratic mixed integer programming problem, which is unsolvable in practical situations.

Although quadratic loss function and absolute deviation do not result in the same optimum, we think that (in not extreme cases) minimization of absolute loss function will decrease the squared deviation as well.

Suppose the operator of a Bonus-Malus System insists on quadratic loss function in the joint optimization. In that case, we suggest a two-stage optimization process: In the first stage, optimize the joint model with absolute deviation objective function, then fix the transition rules and optimize the premium scale using quadratic loss function.

The premium optimization LP problem first appeared in Heras et al. (2002), but the above LP is different. In the original model the difference between the *expected* premium and expected claim is minimized, so the constraints (LP1.1) and (LP1.2) looked as:

$$\sum_{k=0}^{K} \pi_k c_k^i + g_k \geq \lambda^i \qquad \forall i \tag{13}$$

and

$$\sum_{k=0}^{K} \pi_k c_k^i - g_k \leq \lambda^i \qquad \forall i \tag{14}$$

and the objective function was: $\sum_{i=1}^{I} \phi^i g_k$.

If the number of types is less than the number of BMS classes, this model's optimal objective function value will be 0. In other words, the expected premium equals the expected claim for all risk groups. Merely considering the overall expected deviation would result in high dispersion among the premiums of the classes. In some numerical experiments, we encountered cases where the highest premium was more than 1 million times higher than the lowest one. Such a substantial difference between premiums is undoubtedly not adequate in an insurance contract because it would not reduce the risk for the policyholder. The most standard way to handle this problem is the manual limitation of the dispersion of the premiums. However, we applied a different approach for managing the risk aversion of policyholders.

In our objective function (LP1.obj) we minimize the absolute deviations of each type's expected payment from the expected claims weighted with the proportion of the types (this expression appears in many studies (Norberg (1976); Tan et al. (2015)) with the difference that ℓ^1 norm is used instead of ℓ^2). The zero value for the objective function in this model would mean that each risk group's premium is constant (i.e., does not change from class to class), and it is equal to the expected claim for each type. In actual circumstances, this is impossible. Heras et al. (2002) set other constraints to limit the Lomaintra-efficiency. These constraints can be inserted into our models. Still, we think a smaller objective value would be preferable to the policyholder than a higher objective value with a better efficiency measure.

Theorem 8. *There is an optimal solution of LP1, where for all k there is a risk group i, where $\pi_k = \lambda^i$.*

Proof. Assume on the contrary that there is a class k where the premium differs from each type's expected claim ($\pi_k^{'} \neq \lambda^i, \forall i$).

Set $\mathcal{K}_0$ contains classes where premium equals to $\pi_k^{'}$ ($\mathcal{K}_0 := \left\{k|\pi_k = \pi_k^{'}\right\}$). Furthermore, set $\mathcal{I}_p$ and $\mathcal{I}_n$ contain risk groups where the expected claims are greater/less than $\pi_k^{'}$:

$$\mathcal{I}_p := \left\{i|\lambda^i > \pi_k^{'}\right\}; \qquad \mathcal{I}_n := \left\{i|\lambda^i < \pi_k^{'}\right\} .$$

If we start to increase $\pi_k^{'}$ with a value ε, then the objective of the model will change with εg where

$$g = \left(\sum_{i\in\mathcal{I}_n}\sum_{k\in\mathcal{K}_0}\phi^i c_k^i - \sum_{i\in\mathcal{I}_p}\sum_{k\in\mathcal{K}_0}\phi^i c_k^i\right)$$

If $\pi_k^{'} < \lambda^1$, then g is negative which means that with the increase of the $\pi_k^{'}$, the value of the objective function can be better. The situation is similar if $\pi_k^{'} > \lambda^I$. In this case, g is positive, which means that decreasing $\pi_k^{'}$ leads to a smaller objective function value.

For the case when $\lambda^1 < \pi_k^{'} < \lambda^I$ notice that the value of g depends on the values of $\mathcal{I}_n$ and $\mathcal{I}_p$. This means that if there is a premium $\pi_k^{'} > \lambda^i$ and we increase this premium with ε, then g changes only if $\pi_k^{'} + \varepsilon > \lambda^{i+1}$. Put differently, if g of the $\pi_k^{'} + \varepsilon$ is zero, then λ^{i+1} should also be optimal.

Since this holds for each k, there should be an optimal premium scale where the premiums are all equal to a λ.

□

Theorem 8 suggests a quite unusual solution. The remarks below are meant to explain the motivation.

- If there is only one risk group, then each class premiums are equal ($\pi_k = \pi, \forall k$); this fact corresponds to the statement of Theorem 8 for $I = 1$. In this context, it is just a generalization that when there are two types of policyholders, there are two premium values. For further motivation see example 4.1.

- If we have only a few types of policyholders, the premium can be the same in many classes. At first sight, it seems that we can get the same results with a less spread BM system. However, the stationary probabilities of a larger BM system may differ from a smaller one's. For instance, in example 4.1, a policyholder with 0.9 expected claim pays the high premium with a probability of 0.99998 and the low premium with a probability of 0.00002. These probabilities cannot be reproduced in a two-class BMS.

- If there are many risk groups, then every class may have a separated premium value.

- If the designer of the BM system prefers to have distinct premium values in each class, he/she can easily prescribe it with additional constraints.

Example 4.1. *Let us consider a BMS with the most straightforward transition rule: in any claims, the policyholder moves downward, otherwise upward a class. Given this transition rule, there is a relatively easy relation amongst the stationary probabilities:*

$$c_k^i = \left(\frac{1-\lambda^i}{\lambda^i}\right)^k c_0^i \ , \tag{15}$$

where the k outside the parenthesis is power and not an index. After applying the expression for the sum of this geometric sequence:

$$c_0^i = \frac{\frac{1-\lambda^i}{\lambda^i} - 1}{\left(\frac{1-\lambda^i}{\lambda^i}\right)^{K+1} - 1} \tag{16}$$

Let $K = 2h+1$, then:

$$\sum_{k=0}^{h} c_k^i = \frac{\frac{1-\lambda^i}{\lambda^i} - 1}{\left(\frac{1-\lambda^i}{\lambda^i}\right)^{2h+2} - 1} \frac{\left(\frac{1-\lambda^i}{\lambda^i}\right)^{h+1} - 1}{\frac{1-\lambda^i}{\lambda^i} - 1} = \frac{\left(\frac{1-\lambda^i}{\lambda^i}\right)^{h+1} - 1}{\left(\frac{1-\lambda^i}{\lambda^i}\right)^{2h+2} - 1} \tag{17}$$

If $\lambda^i > 0.5$ then expression (16) *tends to 1 (as h tends to infinity), otherwise it tends to 0. Hence, a policyholder with a claim probability higher than 50% will be almost surely in the lower half of the BM system. Similarly, any policyholder with less than 50% claim probability would be most likely in the upper half.*

Let us assume that there are two types of policyholders: $\lambda^1 < 0.5$ and $\lambda^2 > 0.5$. The premium is λ^2 for classes $0, \dots, h$ and λ^1 for the other classes. Asymptotically both types will pay the same amount as their risks. This statement holds for quadratic (and any other meaningful) loss function. If the premium increases gradually, we cannot get the same result. In certain cases, quite small h is enough to approximate the asymptotic result: let $\lambda^1 = 0.1, \lambda_2 = 0.9$ and $h = 4$. Table 1 shows *both risk groups' stationary probabilities. If the premium is* 0.1 *for classes 5-9 and* 0.9 *for classes 0-4, then both types of policyholders would pay their fair premium with* 0.99998 *probability.*

class	$\lambda^i = 0.1$	$\lambda^i = 0.9$
9	0.8889	2.29×10^{-9}
8	0.0988	2.06×10^{-8}
7	0.0110	1.86×10^{-7}
6	0.0012	1.67×10^{-6}
5	0.0001	1.51×10^{-5}
4	1.51×10^{-5}	0.0001
3	1.67×10^{-6}	0.0012
2	1.86×10^{-7}	0.0110
1	2.07×10^{-8}	0.0988
0	2.29×10^{-9}	0.8889

Table 1: Stationary probabilities of a 10 class BMS

4.1.1 Profit constraint

A crucial question is the financial balance of the BMS. In the long run, it is not worth designing an unprofitable BMS. In the model LP1, if the objective value is as close to zero, it is financially balanced. Besides that, each model that operates with the Lomaintra-efficiency ensures some balance.

In the relevant literature, there are studies where profitability is explicitly prescribed (Coene and Doray (1996)) and articles where it is not (Heras et al. (2004); Tan et al. (2015)). With our notation the profit constraint takes the form

$$\sum_{i=1}^{I} \sum_{k=0}^{K} \left(\pi_k c_k^i\right) \geq \sum_{i=1}^{I} \phi^i \lambda^i. \tag{LP1.4}$$

The financial balance of the BMS is a crucial requirement; however, in this case, Theorem 8 does not hold anymore.

Theorem 9. *There is an optimal solution of LP1 with constraint* (LP1.4)*, where there is only one type of premium that is unequal to any risk group's expected claim.*

Proof. By way of contradiction, let us assume that the optimal solution involves two premium values ($\pi_{k_1} < \pi_{k_2}$) that differ from any type's expected claim. Sets $\mathcal{K}_1$ and $\mathcal{K}_2$ contain classes where the premium equals to π_{k_1} and π_{k_2} ($\mathcal{K}_1 := \{\kappa | \pi_\kappa = \pi_{k_1}\}$, $\mathcal{K}_2 := \{\kappa | \pi_\kappa = \pi_{k_2}\}$). Furthermore, sets $\mathcal{I}_{p_1}$, $\mathcal{I}_{p_2}$ and $\mathcal{I}_{n_1}$, $\mathcal{I}_{n_2}$ are defined as

$$\mathcal{I}_{p_1} := \{i | \lambda^i > \pi_{k_1}\}; \qquad \mathcal{I}_{p_2} := \{i | \lambda^i > \pi_{k_2}\};$$
$$\mathcal{I}_{n_1} := \{i | \lambda^i < \pi_{k_1}\}; \qquad \mathcal{I}_{n_2} := \{i | \lambda^i < \pi_{k_2}\}.$$

If $\pi_{k_1} < \lambda^1$, then the premium in classes $k \in \mathcal{K}_1$ shall be increased by ε. This change does not violate the profit constraint (LP1.4) but reduces the objective function by $g\varepsilon$;

$$g = \sum_{i \in \mathcal{I}_{p_1}} \phi^i \sum_{k \in \mathcal{K}_1} c_k^i \ ,$$

which means the premium scale cannot be optimal.

If $\lambda^1 < \pi_{k_1}$, then decreasing premiums in classes $k \in \mathcal{K}_1$ by ε produce premiums that are equal to the increase $\delta(\varepsilon)$ in classes $k \in \mathcal{K}_2$. To preserve the financial balance of the system we must have

$$\delta(\varepsilon) = \varepsilon \frac{\sum_{i=1}^{I} \sum_{k \in \mathcal{K}_1} c_k^i}{\sum_{i=1}^{I} \sum_{k \in \mathcal{K}_2} c_k^i}.$$

Decreasing premium in classes $k \in \mathcal{K}_1$ by ε (and increasing it by $\delta(\varepsilon)$ in classes $k \in \mathcal{K}_2$) will change the objective function by $g\varepsilon$;

$$g = \sum_{i \in \mathcal{I}_{p_1}} \phi^i \sum_{k \in \mathcal{K}_1} c_k^i - \sum_{i \in \mathcal{I}_{n_1}} \phi^i \sum_{k \in \mathcal{K}_1} c_k^i + \frac{\sum_{i=1}^{I} \sum_{k \in \mathcal{K}_1} c_k^i}{\sum_{i=1}^{I} \sum_{k \in \mathcal{K}_2} c_k^i} \left(\sum_{i \in \mathcal{I}_{n_2}} \phi^i \sum_{k \in \mathcal{K}_2} c_k^i - \sum_{i \in \mathcal{I}_{p_2}} \phi^i \sum_{k \in \mathcal{K}_2} c_k^i \right) .$$

If g is negative, then the increase of the premium in classes $k \in \mathcal{K}_1$ will result in a better value for the objective function; if it is positive, then the value of the objective function will be worse.

Let $\pi_{k_1} > \lambda^{i_1}$ for $k_1 \in \mathcal{K}_1$ and $\pi_{k_2} < \lambda^{i_2}$ for $k_2 \in \mathcal{K}_2$. If the premium decreases in classes $k_1 \in \mathcal{K}_1$, then the g changes only if $\pi_{k_1} - \varepsilon < \lambda^{i_1}$ for $k_1 \in \mathcal{K}_1$ or $\pi_{k_2} + \delta(\varepsilon) > \lambda^{i_2}$ for $k_2 \in \mathcal{K}_2$. This means that if $g = 0$, then at least one premium can be replaced with a λ which is the assertion of the theorem. □

4.2 Optimizing non-unified transition rules when the premium scale is fixed

Transition rules are typically defined by transition matrices as described in section 3.1. To build a MILP model, we introduce binary variables $T_{j,m,k}$ for each entry of the transition matrices. If $T_{j,m,k} = 1$, then the policyholders with m claims are moved from class k, j classes upward (downward if $j < 0$) in the following period. Denote the domain of j by $J_k = [\underline{J_k} : \overline{J_k}]$ for class k where $-k = \underline{J_k} < 0$ and $K - k = \overline{J_k} > 0$ are the two extremes. If a binary variable $T_{j,m,k} = 1$ and index j is positive, then the policyholders with m claims are put upward in the system. Put differently, they move to a class with a lower premium if it is possible. In the case of $j < 0$, the policyholders move downward if they have m claims. Index j can be 0 as well, meaning they stay in the same class in the subsequent period.

The model aims to find the best transition rule that evenly separates the risk groups' expected payment. Thus, we want to minimize each class's deviation of the payment and expected claims (in some norm).

4.2.1 Defining reasonable transition rules

There should be a strict rule for each possible m accident. Therefore in each period, one transition rule should describe how much class reduction or increase should affect the policyholder with $m \geq 0$ claim. The constraints (MILP1.1) ensure a transition rule for each possible claim and class.

$$\sum_{j=\underline{J_k}}^{\overline{J_k}} T_{j,m,k} = 1 \ , \qquad \forall m, k \tag{MILP1.1}$$

In practice, only the claimless period result in positive steps in the system. Therefore constraints (MILP1.2) ensure that the policyholder without claims should move upward, and in-class K, he/she should stay in the class.

$$\sum_{j=\min(\overline{J_k},1)}^{\overline{J_k}} T_{j,0,k} = 1 \ , \qquad \forall k \tag{MILP1.2}$$

When any claim occurs in a period, there should be a class reduction, or the policyholder may remain in the same class. A claimless period should always result in a positive step in the system, but it is possible to stay in the same class for the subsequent period for claims. There should be at least one case where there is a downward classification. We only prescribe the negativity for the largest number of claims:

$$\sum_{j=\underline{J_k}}^{\max(\underline{J_k},-1)} T_{j,M,k} = 1 \ , \qquad \forall k \tag{MILP1.3}$$

More accidents should cause a more considerable (not less) decrease in the classes. Constraint (MILP1.4) guarantees the transition rule to be stricter if the number of claims gets higher.

$$\sum_{\ell=j}^{\overline{J_k}} T_{\ell,m,k} \geq T_{j,m+1,k} \qquad \forall j, k, \ m = 0, \ldots, M-1 \tag{MILP1.4}$$

4.2.2 Obtaining the stationary distribution

A BM system is generally modeled as an irreducible Markov chain, which converges to a stationary distribution. Denote stationary distributions by non-negative variables c_k^i. They represent the probability of a type i policyholder being in BM class k in the stationary distribution. Surely, these variables sum up to the ratio of the type i policyholders:

$$\sum_{k=0}^{K} c_k^i = \phi^i \qquad \forall i \tag{MILP1.5}$$

We also have to take care of connecting the stationary probabilities to the transition rules (see (11)). The following quadratic constraints accomplish this:

$$c_k^i = \sum_{j=-(K-k)}^{k} \sum_{m=0}^{M} \lambda_m^i T_{j,m,k} c_{k-j}^i \quad k = 1, \ldots, K-1, \forall i. \tag{18}$$

A linearization is possible for a multiplication of a binary and a nonnegative continuous variable, with an introduction of another variable, if we know an upper bound of the continuous variable. For the sake of simplicity, let us consider the multiplication of a nonnegative continuous variable c and a binary variable T. Let us assume that c is bounded, therefore $c \leq M$. For the linearization, we introduce a nonnegative variable d. With the following constraints, the value of the d will be equivalent to Tc multiplication:

$$\begin{aligned} &MT \geq d \\ &d \geq c - (1-T)M \\ &c \geq d \end{aligned}$$

Therefore if $T = 0$, then $d = 0$, otherwise $d = c$. In (18) c is c_{k-j}^i, T is $T_{j,m,k}$. We also know that $c_{k-j}^i \leq \phi^i$.

Hence, for the linearization, we introduce the variables $d_{k,j,m}^i$, as the probabilities that an individual from risk group i and class k moves to class $k+j$ in the next period. We define these variables with the constraints (MILP1.7). Constraints (MILP1.6), (MILP1.7), (19), (20) are meant to linearize the quadratic constraints:

$$c_k^i = \sum_{j=-(K-k)}^{k} \sum_{m=0}^{M} d_{k-j,j,m}^i \qquad \forall i, k \tag{MILP1.6}$$

$$d_{k,j,m}^i \geq \lambda_m^i c_k^i - (1 - T_{j,m,k})\phi^i \qquad \forall i, j, k, m \tag{MILP1.7}$$

$$d^i_{k,j,m} \leq \lambda^i_m c^i_k \qquad \forall i, j, k, m \tag{19}$$

and

$$d^i_{k,j,m} \leq \phi^i T_{j,m,k} \qquad \forall i, j, k, m \ . \tag{20}$$

Theorem 10. *If $T_{j,m,k} = 1$, then $d^i_{k,j,m} = \lambda^i_m c^i_k$, otherwise $d^i_{k,j,m} = 0$, provided that* (MILP1.1), (MILP1.5), (MILP1.7) *and* (MILP1.6) *hold.*

Proof.

$$\phi^i = \sum_{k=0}^{K} c^i_k \geq \sum_{k=0}^{K} \sum_{m=0}^{M} \sum_{(j,m)|T_{j,m,k}=1} d^i_{k,j,m} \geq \sum_{k=0}^{K} \left(\sum_{m=0}^{M} \lambda^i_m c^i_k \right) = \sum_{k=0}^{K} c^i_k = \phi^i \ ,$$

The first equality holds because of constraints (MILP1.5), the second inequality holds because of (MILP1.6). The third inequality comes from (MILP1.7) and the last equality is valid since the sum of parameters λ^i_m equals 1 for each type i. This means that all relations are equalities implying that variables d that are not present in $\sum_{k=0}^{K} \sum_{m=0}^{M} \sum_{(j,m)|T_{j,m,k}=1} d^i_{k,j,m}$ have to be 0 while all other d's have to be equal to $\lambda^i_m c^i_k$. □

Because of Theorem 10 the constraints (19) and (20) can be omitted from the *MILP1* model.

4.2.3 Ensuring irreducibility of Markov chains

Previous constraints (MILP1.2), (MILP1.3), (MILP1.4), and (MILP1.1) on transition rules do not necessarily result in an irreducible Markov chain.

For example, if we have a BMS with three classes (i.e., $K = 2$), and the transition rule is if somebody causes an accident (or more) moves two classes down, if he/she does not have any claims, then he/she moves two classes up. In this case, the policyholder will be in the middle class (in the stationary distribution) with 0 probability. With this transition rule, the Markov chain is not irreducible. Nevertheless, we can interpret the situation as if we would have a two-class BMS, where the policyholder moves downward if he/she has claims and moves upward in the claim-free case.

Because in the optimization models, there are stationary probabilities, it is sufficient to assume that each stationary probability (for each k) be positive. In MILP models, we cannot use strict inequalities, but with a parameter $\tau > 0$ and $\tau \approx 0$, we can prescribe that each stationary probability be positive. This is an eligible condition for an irreducible Markov chain. However, if τ is unnecessarily high, we may exclude some transition rules that give irreducible Markov chains.

$$\sum_{i=1}^{I} c_k^i \geq \tau \qquad \forall k \tag{MILP1.8}$$

There are alternative solutions for the irreducibility constraint as well. For example, we can set the transition rule of the claim-free case to exactly one upward (as it is ordinarily done in practice). Also, it is possible to iteratively determine all transition rules that result in an irreducible Markov chain and exclude these rules with additional constraints. However, checking all the transition rules can be time-consuming.

4.2.4 Profit constraint and Objective function

In this model, we may also prescribe that the BM system would not result in a loss for the insurance company. The same constraint of (LP1.4) can be used in this MILP model as well. However, in this case, every π_k is a parameter, and the c_k^i is the decision variable.

$$\sum_{i=1}^{I} \sum_{k=0}^{K} \left(\pi_k c_k^i\right) \geq \sum_{i=1}^{I} \phi^i \lambda^i. \tag{MILP1.9}$$

This model also needs constraints to define the g_k^i variables to be the absolute deviation from the "fair" premiums:

$$\pi_k c_k^i + g_k^i \geq \lambda^i c_k^i \qquad \forall i, k \tag{MILP1.10}$$

$$\pi_k c_k^i - g_k^i \leq \lambda^i c_k^i \qquad \forall i, k \tag{MILP1.11}$$

Overall, we use the next variables in the model:

$$\begin{aligned} T_{j,m,k} &\in (0,1) && \forall j, m, k \\ g_k^i \geq 0; c_k^i &\geq 0 && \forall k, i \\ d_{k,j,m}^i &\geq 0 && \forall k, j, m, i. \end{aligned}$$

The objective function of the *MILP1* model is similar to the objective of the *LP1*. We want to minimize the absolute deviation from the expected claims of each risk group in each class:

$$\min \sum_{i=1}^{I} \sum_{k=0}^{K} \phi^i g_k^i \tag{MILP1.obj}$$

We minimize (MILP1.obj), subject to constraints (MILP1.1), (MILP1.2), (MILP1.3), (MILP1.4), (MILP1.5), (MILP1.6), (MILP1.7), (MILP1.10) and

(MILP1.11). Additionally we may consider constraints (MILP1.8) for the irreducibility. Furthermore the profit constraint (MILP1.9) also can be included.

4.3 Optimizing unified transition rules when the premium scale is fixed

Often in practice, the transition rules do not differ from class to class, which means a unified transition rule for each claim m. This means that instead of binary variables $T_{j,m,k}$ we can simply use binary variables $T_{j,m}$. In this case, J_k is the same for all k; therefore, it is sufficient to set only one upper ($\overline{J} = K$) and lower limit ($\underline{J} = -K$).

We may simply rewrite the constraints (MILP1.1)-(MILP1.4) without the parameter k:

$$\sum_{j=\underline{J}}^{\overline{J}} T_{j,m} = 1 \qquad \forall m \tag{MILP2.1}$$

$$\sum_{j=1}^{\overline{J}} T_{j,0} = 1 \tag{MILP2.2}$$

$$\sum_{j=\underline{J}}^{-1} T_{j,M} = 1 \tag{MILP2.3}$$

$$\sum_{\ell=j}^{\overline{J}} T_{\ell,m} \geq T_{j,m+1} \qquad \forall j,\ m = 0, ..., M-1 \tag{MILP2.4}$$

However, the constraints (MILP1.6) should be different because we have to omit reclassifications leading to non-existent classes. For example, suppose a policyholder is in class 0 and causes an accident, then he/she would not decrease (say) two classes since class 2 does not exist. Therefore, he/she will simply remain in class 0. The case of the top BM class has to be treated similarly.

$$\begin{aligned}
c_k^i &= \sum_{j=\underline{J}}^{0} \sum_{\ell=j}^{0} \sum_{m=0}^{M} d_{k-\ell,j,m}^i && k = 0, \forall i \\
c_k^i &= \sum_{j=\max(\underline{J},-(K-k))}^{\min(\overline{J},k)} \sum_{m=0}^{M} d_{k-j,j,m}^i && k = 1, \dots, K-1, \forall i \\
c_k^i &= \sum_{j=0}^{\overline{J}} \sum_{\ell=0}^{j} \sum_{m=0}^{M} d_{k-\ell,j,m}^i && k = K, \forall i
\end{aligned} \tag{MILP2.6}$$

Due to the fewer possibilities of the unified transition rules, we may exclude those transition rules that would not lead to an irreducible Markov chain. Hence we may exclude the irreducibility constraint (MILP1.8) to give a not only eligible condition.

Theorem 11 presents a rule which applies to those transition rules when there can be up most one claim per period. For this, let j_0 denote the transition rule for the claim-free case. Furthermore, let j_1 denote the transition rule when there is a claim. We assume that $j_0 > 0$ and $j_1 < 0$.

Theorem 11. *Let $gcd(j_0, |j_1|)$ denote the greatest common divisor between j_0 and $|j_1|$. Let $j_0 > 1$ and $|j_1| > 1$. A Bonus-Malus System, with $(j_0; j_1)$ transition rule is not irreducible, if at least one of these conditions is met:*

$$\begin{cases} j_0 + |j_1| > K + 1 \\ gcd(j_0, |j_1|) > s \qquad \& \quad j_0 + |j_1| \leq K + 1 \end{cases}$$

Where $s = 1$ if $K + 1$ is odd, otherwise $s = 2$.

Proof.

In every case, if either j_0 or $|j_1|$ is equal to 1, the transition rule results in an irreducible Markov chain.

If $j_0 + |j_1| > K + 1$, the lowest class can be reached from class 0 is class j_0. If $j_0 > |j_1|$, then the highest class can be reached from a downgrade is class $K - |j_1|$. Because $j_0 > K - |j_1|$, at least one class cannot be reached above or below. Therefore, if $j_0 + |j_1| > K + 1$, the BMS with this transition rule cannot be irreducible.

Hence we only have to focus on those transition rules $(j_0; j_1)$, where $j_0 + |j_1| \leq K + 1$.

For the rest of the cases, we present an illustration in Figure 1, about a method, how can we reach all of the classes from class 0.

We start from class 0. Then we step j_0 classes until it is possible to take a complete step (hence we stop in class k if $k + j_0 > K$). Then we step $|j_1|$ classes down from each explored class, which are greater than $|j_1|$. After this, we increase all of the freshly explored classes with j_0 until it is possible. Then, we continue it until we can find more unexplored classes.

If we find K with this method, then the system is irreducible if all of the classes k satisfy the following equation:

$$k = \alpha_k j_0 + \beta_k j_1 \qquad \forall k = 0, \dots, K \tag{21}$$

Where α_k and β_k denotes nonnegative integers. In other words, all classes can be reached with a combination of periods with claims and without claims from class 0. Thus, α_k denotes the number of periods without claims, and β_k is the number of periods with claims.

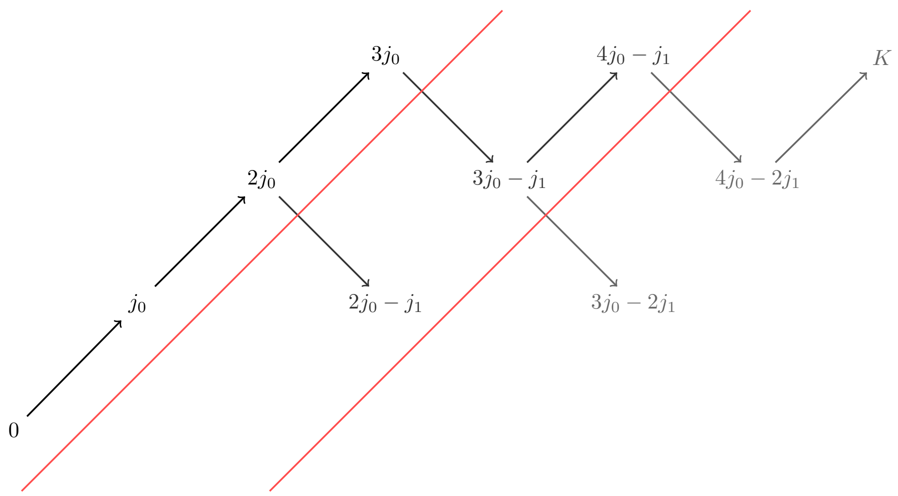

Figure 1: Reaching every class from class 0

Let us focus on the first class' equation:

$$1 = \alpha_1 j_0 + \beta_1 j_1$$

α_1 and β_1 are nonnegative integers, $j_0 > 1$ and $j_1 < -1$. Therefore is not exist appropriate α_1 and β_1, if $gcd(j_0, |j_1|) > 1$. In other words if j_0 and $|j_1|$ are not relatively prime and exists a $K = \alpha_K j_0 + \beta_K j_1$ equation, then the Markov chain is not irreducible.

Let us now consider the case when there is no α_K and β_K that fulfill the equation of class K in (21). If we invert the previously used method, hence we start from class K instead of 0, then we can determine similar equations to (21) :

$$k = K + \gamma_k j_0 + \delta_k j_1 \qquad \forall k = 1, \ldots, K-1 \tag{22}$$

That means, starting from class K, each class can be reached with δ_k periods with claims and γ_k periods without claims.

If there are not exist α_K and β_K, then for each class k either (21) or (22) holds, but cannot be both, since:

$$k = K + \gamma_k j_0 + \delta_k j_1 = \alpha_k j_0 + \beta_k j_1$$

$$K = (\alpha_k - \gamma_k) j_0 + (\beta_k - \delta_k) j_1$$

Because $\alpha_k - \gamma_k$ and $\beta_k - \delta_k$ are integers, it can only occur if K can be reached from 0 with the previously introduced method. Hence, if it cannot, all classes can be reached from only class 0 or only class K. It can only happen, if the $gcd(j_0, |j_1|) > 1$.

It is easy to see if $gcd(j_0, |j_1|) = 2$ and the number of classes is even, then there does not exist any α_K and β_K. In this case, we do not discover the same classes starting from the class 0 and from the class K. However, we find every class; consequently, the Markov chain is irreducible. If the number of classes is odd, then always exist an α_K and β_K if $gcd(j_0, |j_1|) = 2$.

If $gcd(j_0, |j_1|) > 2$ and no α_K and β_K exist, there is always at least one unreachable class between a class that can be reached from class 0 and a class that can be reached from class K.

$\square$

For this let j_m denote the step, with m claim in the transition rules ($j_0 > 0$ and $j_M \leq j_{M-1} \leq \cdots \leq j_1 < 0$). Hence, there is only step for both direction: j_0 and j_1.

Theorem 12 is an extension of theorem 12 for the case, when $M = 2$.

Theorem 12. *A Bonus-Malus System, with $(j_0; j_1, j_2)$ transition rule is not irreducible if at least one of these conditions is met:*

$$\begin{cases} j_0 + |j_1| > K + 1 & & \\ gcd(j_0, |j_1|, |j_2|) > s & \& \quad gcd(j_0, |j_1|) + |j_2| \leq K + 1 & \\ gcd(j_0, |j_1|) > s & \& \quad gcd(j_0, |j_1|) + |j_2| = K + 2 & \& \quad \frac{K+1}{gcd(j_0, |j_1|)} \notin \mathbb{Z} \\ gcd(j_0, |j_1|) > s & \& \quad gcd(j_0, |j_1|) + |j_2| > K + 2 & \end{cases}$$

Where $s = 1$ if $K + 1$ is odd, otherwise $s = 2$.

Proof. If the $(j_0; j_1)$ transition rule is irreducible, then of course, with any j_2 additional step, the Markov chain will remain irreducible. Hence, we are only interested in those transition rules where any of the two conditions of theorem 11 is met.

If $j_0 + |j_1| > K + 1$, then the transition rule will not be irreducible with any j_2, because $|j_1| \leq |j_2|$.

Hence, we only have to focus on transition rules, where $gcd(j_0, |j_1|) > s$ and $j_0 + |j_1| \leq K + 1$.

It is worth to note that if there exist an s for a $(j_0; j_1)$ transition rule, then starting from class 0 and K we can reach the same classes as with the $gcd(j_0, |j_1|); gcd(j_0, |j_1|)$ transition rule. Therefore the set of reached classes from class 0 and K is the same for the $(gcd(j_0, |j_1|); |j_2|)$ transition rule as for the $(j_0; j_1, j_2)$ if $gcd(j_0, |j_1|) + |j_2| \leq K + 1$.

Therefore if $gcd(j_0, |j_1|) + |j_2| \leq K + 1$ and $gcd(_2|) > s$, the transition rules are not irreducible, because of the second condition of theorem 11.

If $gcd(j_0, |j_1|) + |j_2| > K + 1$, then the $(gcd(j_0, |j_1|); |j_2|)$ transition rule is not irreducible because of the first condition of theorem 11. However in this case, because there are two downward steps (j_1 and j_2), it is not an eligible condition.

If we start the steps from class 0 for the $(gcd(j_0, |j_1|); j_2)$ transition rule, there will be at least one class between class 0 and class $gcd(j_0, |j_1|)$ that cannot be reached with $K - |j_2|$ step. However, in this case, j_1 is also a downward step, so $(j_0; j_1, j_2)$ can be irreducible. If $gcd(j_0, |j_1|) + |j_2| > K + 2$, then there will be more than one consecutive unreachable class from 0 and K, therefore it cannot be irreducible. When $gcd(j_0, |j_1|) + |j_2| = K + 2$ it can only be irreducible if $\frac{K+1}{gcd(j_0,|j_1|)} \in \mathbb{Z}$. In this case the pattern of the $(gcd(j_0, |j_1|); gcd(j_0, |j_1|))$ steps can find the unreached classes with a j_2 downstep. Otherwise the same occurs as in the first condition of theorem 11, there remains at least one class that cannot be reached from either class 0 or class K. □

Let J^* denote the set of all transition rules that result in a not irreducible Markov chain. Namely $(j_0, j_1, \ldots, j_M) \in J^*$ if the condition of Theorem 11 is not held for this transition rule. To exclude these transition rules, we need to use the following constraint.

$$\sum_{m=0}^{M} T_{j_m,m} \leq M \qquad \forall (j_0, j_1, \ldots, j_M) \in J^* \tag{MILP2.8}$$

4.4 Numerical experiments

We used an AMD Ryzen 5 2600 Six-Core CPU 3,40 GHz computer with 16 GB DDR4 RAM for calculations. We ran the program in Python 3.7.3. and used the Gurobi 8.1.0 solver for the optimization. To reduce the numerical problems caused by the many "big-M" constraints, we did concurrent optimization within Gurobi to solve the LP. The solver uses multiple algorithms simultaneously and returns the solution obtained first.

4.4.1 Optimization of the premiums, with fixed transition rules

First, we considered the model for the optimization of the premiums with fixed transition rules.

We considered two types of policyholders for the calculations: a "good" with lower risk and a "bad" with high risk. For the sake of simplicity, we assumed $M = 1$ in every case and the same proportion of types. Four alternative scenarios were investigated, two non-realistic with high claim-probabilities $(10\%; 20\%)$ and $(10\%; 50\%)$, and two scenarios with more realistic parameters $(1\%; 2\%)$ and $(1\%; 5\%)$, respectively.

In one of the considered rules, a claim would result in a two-class worsening (-2). In the other one, the policyholder gets reclassified into the worst class ($-K$) in the event of a claim. A period without a claim would result in one positive step in the system in both transition rules.

We calculated the optimal premiums with different-sized BM systems. For example, we calculated the premiums from a 3-class to a 120-class BMS.

To visualize the sorting capability of the BMS, we use an indicator that represents the overpayment for a type i policyholder that we denote by OP^i. Thus, OP^i shows the ratio of the paid and the ideal payment of type i policyholders.

$$OP^i = \frac{\sum_{k=0}^{K} \pi_k c_k^i}{\lambda^i} - 1 \tag{23}$$

Therefore, if OP^i is positive, then the expected payment of risk group i is more. On the other hand, if negative, it is less than the expected claim.

We also introduce the $\Omega = \sum_{i=1}^{I} |OP^i|$, which can be interpreted as the righteousness of the BMS. If it is close to zero, then every policyholder pays close to his/her ideal level.

Ω differs from the model's objective due to the risk aversion of the policyholders. Hence, it is possible to design a BMS where the Ω is smaller than our model's optimal result, though deviations of the premiums can be considerably higher in that case.

The following figure presents the OP^i values of these BM systems with the optimal premiums.

The transition rule of $(1; -2)$ had constant OP^is in the smaller probability cases. Therefore, with smaller probabilities, the number of the classes could not decrease the OP^i values of the risk groups. In these cases, the types with the lower probabilities paid almost the fair premiums while the other group paid much less. The $10 - 20\%$ case was similar to the small probability cases. However, the less risky group paid more than their fair value.

When the rule of claim changed with the number of classes $(1; -K)$, the OP^i values also changed. However, after a certain number of classes, the size of the BMS has not influenced the OP^i values. This is because when the probabilities of the groups were higher, it became constant with fewer classes.

4.4.2 Optimization of the Transition rules, with fixed premiums

To optimize the transition rule with fixed premiums, we have to define a premium-scale as an outer parameter. Therefore, we considered two types of premium scales in numerical experiments. Because of Theorem 10, we know that only the risk groups "fair" premium can appear in the optimal premium-scale with the used objective function. Hence, we considered a premium scale with only these premiums. We set

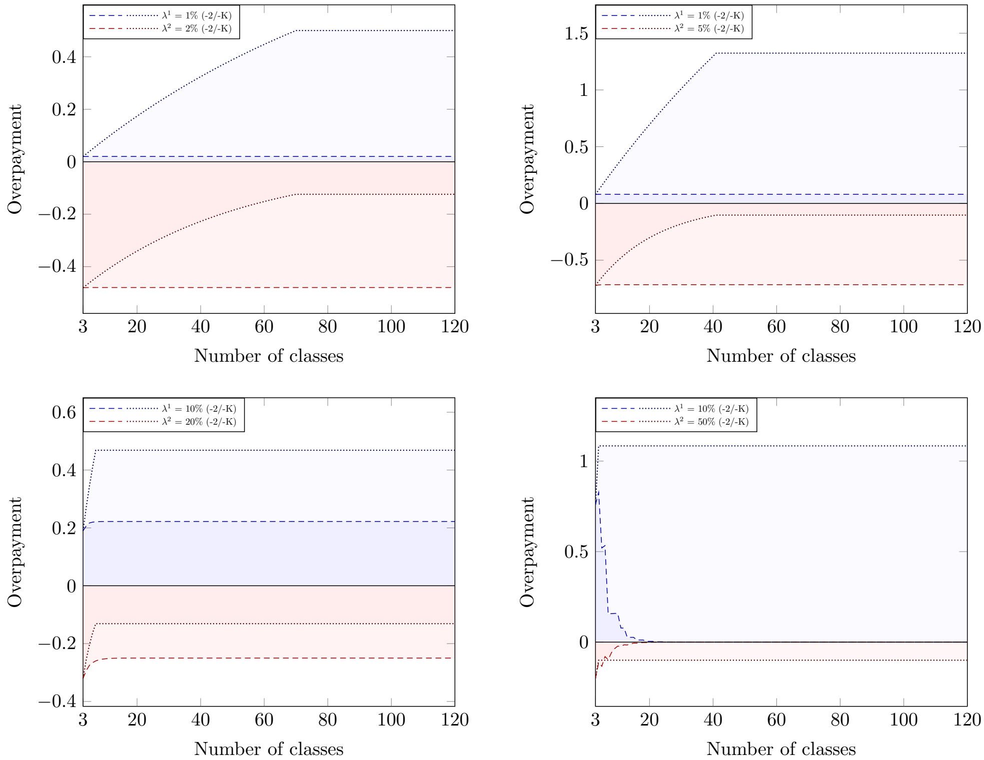

Figure 2: The Overpayments of the risk groups with premium optimization with transition rule $(1;-2)$ and $(1;-K)$.

this premium scale as a proportional representation of the policyholders. Therefore we introduce own classes for each risk groups. The own class means the class premium equals the type's fair premium. The risk groups have that many own classes that are proportional to their percentage of all policyholders. Of course, we organize it in decreasing order and round down if there are not enough classes to create a perfect proportional representation.

We also used a linear premium scale because, according to the elasticity, a linear premium scale is considered the best one.

We calculated the Ω values for every BMS from 3 to 120 classes for both types of premiums. First, we used the unified transition rules. Figure 3 presents the OP values for 3-class BMS to 120-class BMS in each case.

The OP values get closer to 0 as we increase the number of classes in both cases. If the probabilities are smaller, the effect is smaller as well.

The Proportional scale seems a bit better in every case. The Ω is much smaller than the Linear one. In the $10\% - 20\%$ and $10\% - 50\%$ case, the perfect distinction happens with a given number of classes. Hence, there is a BMS in these cases, with no cross-payment among the risk groups. For the $10\% - 20\%$ case, at least almost 100 classes are needed, while in the $10\% - 50\%$ case, a 40-class BMS was sufficient.

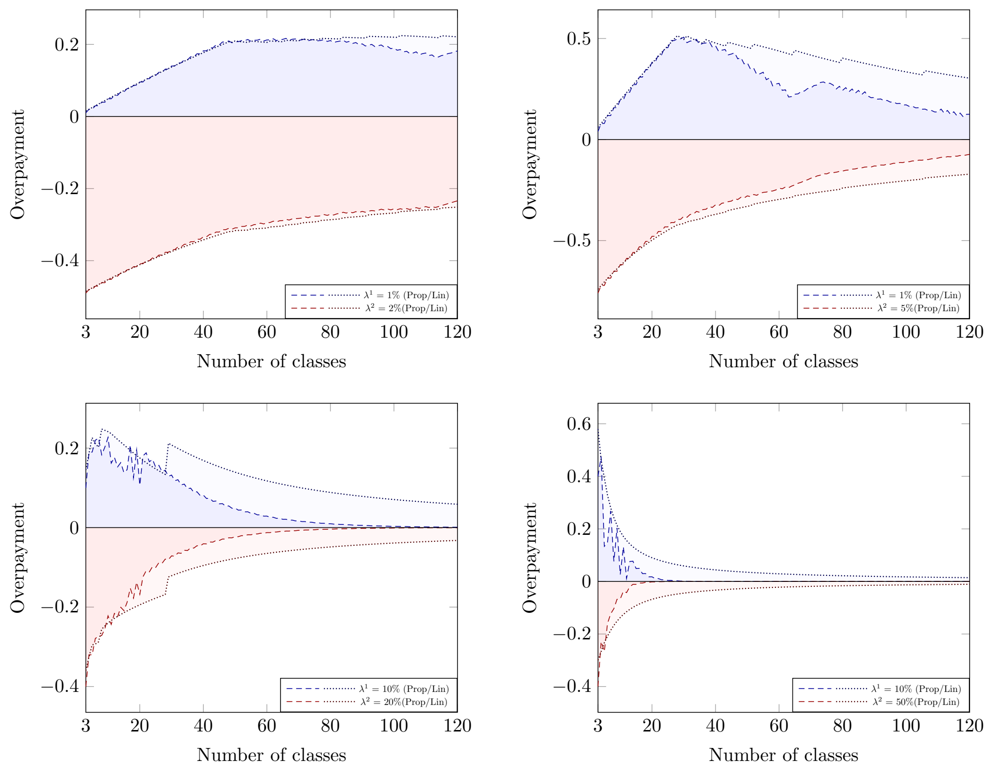

Figure 3: The Overpayments of the risk groups with unified transition rules optimization with Proportional *(Prop)* and Linear *(Linear)* premium-scales

The linear premium scale has a similar effect but could not decrease the Ω to zero. As we can see in the $10\% - 50\%$ case, some over-and underpayment still exists even with a 120 class BMS.

The lower risky policyholders pay a bit over with too few classes, but mainly close to their fair premium. In general, there is an increase in the payments as the number of classes increases, then the trend changes. For example, the $1\% - 2\%$ case needs at least 40 classes to the change, but the payments are still increasing afterward. In the $1\% - 5\%$ case, after 30 classes, the BMS gets better sorting capacity as we increase the number of classes. In the $10\% - 50\%$ and $10\% - 50\%$ cases, the trend is decreasing after less than ten classes.

The non-unified transition rules may result in a better solution. However, because of the more possibilities, the computation for the optimal solution needs significantly more time. Therefore, we set the upper limit for the computational time to 1 hour. However, the bound was violated when we considered 15 classes in most of the cases.

Figure 4 presents how much percentage the objective value could improve with non-unified transition rules compared to the unified ones. In the high-probability cases, the improvement is significant. However, in the two small-probability cases

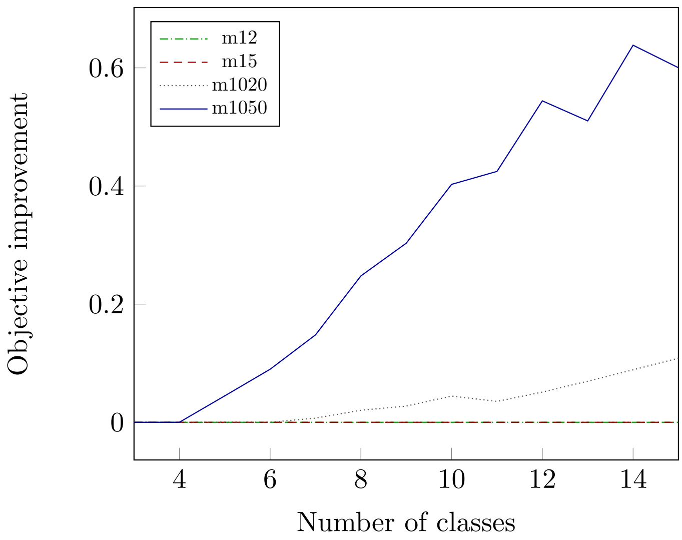

Figure 4: Improvement of the Objective value of the NU models over the U models.

using non-unified transition rules does not improve the objective value.

In the previous experiments, we found that the larger BMSs are more effective. Hence, as the number of classes increases, the decrease in overpayment and underpayment is more noticeable. We wanted to examine how the non-unified transition rules improve the results compared to the solution of the models with unified transition rules. However, we could not calculate the exact improvements because of the significant increase in the running time. Hence we decided to approximate the optimal solutions of the larger instances with heuristics.

Assumably the optimal unified and the optimal non-unified transition rules are not that different. Suppose the optimal unified transition rule is $(1; -3)$. The structure of the stationary probabilities of the non-unified transition rules should be closer to the $(1; -3)$ transition rule than the $(1; -2)$ or $(1; -4)$. Otherwise, the optimal unified transition rule would be different. Hence, for example, each class's optimal transition rules cannot be $(1, -2)$ in the non-unified model.

Therefore, we rephrase the MILP model to improve the results of the unified transition rule model. In this model, we set the transition rule in each class to the unified model's optimal solution. And we set the J values as to how much the transition rule may differ from the initial value.

Definition 9. *We call a **j-improved solution** to a result of a model, where starting from the optimal transition rule of the unified model, we may change the transition rule by* $[j : -j]$ *in each class.*

Hence, in the 1-improved solution of the previous example, each class's transition rules can be altered by plus or minus one, which means the

$(1;-2),(1,-3),(1,-4),\ldots,(2,-4)$. The K-improved solution is the same as the original non-uniformed MILP model's optimal solution. However, if we decrease the j, the result may get worse. Nevertheless, the computational time is shorter because of the fewer variables.

To find a better solution than the unified model, we calculated the 1-improved solution. Then we again calculated the 1-improved solution of the already improved result. Finally, we repeated it until we could get a better objective value.

If the unified transition rule can be improved, it can likely be found with this type of local improvement. However, in extreme cases, the 1-improvement may not improve the result. Hence only a higher j-improvement would find any better result. Suppose that the $(1,-3)$ transition rules remain in each class using the 1-improvement. Nevertheless, if we use a 2-improvement, then in one class, we get $(2,-3)$ and $(1,-5)$ in another one.

However, it can only happen when the positive step (a.k.a claim-free case) and the negative-step change because of the structure of the stationary probabilities. In our previous tests, with the used claim distributions, the 0-claim transition rule was always 1. Therefore assumably, our heuristic is very likely to find an improvement from the unified model.

This heuristic was much faster than solving the original MILP model. However, unfortunately, the running time exceeded the 1-hour time limit, around 23-25 classes in each case. Hence, we decided to further reduce the problem's complexity by splitting the optimization into smaller pieces for even larger instances. We describe this heuristic in Figure 5.

We considered the 1-improvement solution in this heuristic, but we only considered one class at each iteration. It results in a faster computational time because there are even fewer variables in each iteration.

Therefore, this heuristic also starts with the optimal solution of the unified model. Then we search for the 1-improved solution, but the change is possible for only one class's transition rules. Moreover, we iterate through the classes.

In each iteration, we compute the possible individual improvement of a class. It means how the objective value can be improved if every class's transition rules are fixed except this individual class. We consider the plus or minus one changes on the transition rules. Then we choose the class with the best result and change the transition rules of this class accordingly. After the change, we calculated each class's possible individual improvement, considering the modified transition rules. We repeat it until no class exists with possible individual improvement.

The difference between the first and the second heuristic is that we focus on only one class in the latter. Theoretically, the second heuristic may not result in as good solutions as the other one. It is much easier to find a local optimum and cannot

Calculate the model with Unified transition rules, return
Z = Objective value
T = Transition rule matrices for each claims
while *Optimal* = *False* **do**
  for *k in Classes* **do**
    Optimize the model, with class k transition rules,
     that can change with +1/-1
    return the objective: $Z'[k]$
    and the modified transition matrices: $T'[k]$
  end
  $Z^* = \min(Z'[0], \dots, Z'[K])$
  $T^* = T'[\text{argmin}(Z'[0], \dots, Z'[K])]$
  if $Z^* < Z$ **then**
    $Z = Z^*$
    $T = T^*$
    Optimal = False
  else
    Optimal = True return the found best solution Z.
  end
end

Figure 5: Second heuristic

improve it further because it would need more than one class's transition rules to change. However, this heuristic's running time was considerably faster because there were only a few variables in each iteration. We could calculate some improvements for even 100 class BMSs within the 1-hour time limit with this method.

This heuristic still assumably finds an improvement over the initial solution. It will not find a better solution if more than one class's transition rules need to be changed together. However, we assume it may happen only in some extreme situations. Furthermore, if it is the only possible improvement, the optimal non-uniformed objective value would be close to the uniformed model's objective value. However, if we only consider a half-fixed variation, where the transition rule of the $m = 0$ case is set to one, this heuristic indeed finds improvement if it exists.

Figure 6 presents the improvement of the objective value over the uniformed model.

On the left side, the two higher probability cases can be seen. In these cases, the non-unified models improved the objective value significantly. However, after around 55 classes, in the $10\% - 50\%$ case, the unified models' objective value reached zero. Hence it could not be improved further. With different lines, the exact method and both heuristics are presented. The solutions of the second heuristic are significantly worse than the other two techniques. Hence assumably, there may exist an even better solution for the BMSs with more than 25 classes.

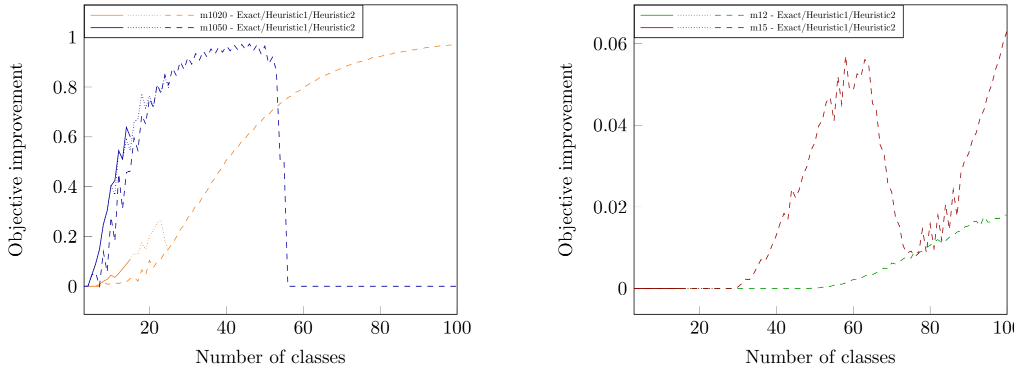

Figure 6: Improvement of the Objective value of the NU models over the U models, with the heuristics.

However, the first heuristic that we introduced resulted in the same solutions to the exact solutions, where we could calculate both (under 15 classes).

On the right side, the two smaller probabilities results are shown. Here, the improvements are not that significant. The $1\% - 2\%$ case barely exceeds the 1% improvement under 80 classes. Moreover, we could not find any solution that improved the uniformed model's objective value with 2% even with 100 classes BMSs.

The $1\% - 5\%$ has more noticeable effects than the $1\% - 2\%$. However, there is a relapse between the 75 class and 95 class BMSs. This setback is probably because the heuristic could not improve from a local optimum. In the smaller probability cases, the exact method and the two heuristics resulted in the same solutions. However, there may be better improvements with more classes, as the higher probability cases are shown.

Figure 7 presents the OP^i values of the four instances. We considered the solution with the best objective value amongst the exact method and the two heuristics for the non-unified model.

At the bottom of the figure, the two higher probability cases are presented. In these cases, both types of policyholders' payments are noticeably closer to their "fair" level than with unified transition rules. Also, the overall territory (Ω) is visibly smaller. Hence it needs fewer classes to reach zero overpayments than with unified transition rules.

In the smaller probability cases, the effect is a bit different. A little increase in both types of policyholders' payments is noticeable. Hence, the more risky policyholders' underpayment decreases, and the less risky policyholders' overpayment increases. As Figure 6 presents, the unified transition rule's objective value is worse than the non-unified one. Nevertheless, the effect is barely noticeable and not beneficial to the less risky policyholders in this case.

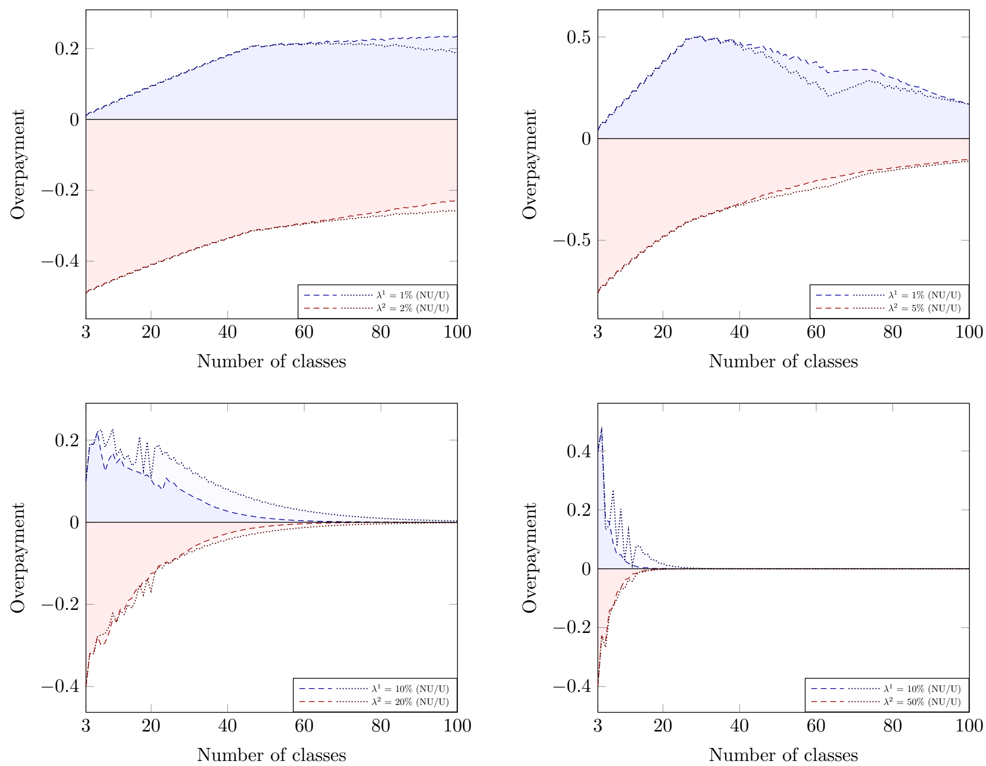

Figure 7: The Overpayments of the U and the NU models

4.5 Summary

In this section, we presented an LP model for the optimization of premiums, introduced in Heras et al. (2004). Also, we introduced a MILP model for the optimization of transition rules. Finally, we presented two model modifications: unified transition rules (U) and non-unified ones (NU).

Some numerical experiments were presented for each model. In general, an optimal transition rule can be significantly better for distinguishing the policyholders. With more classes, the BMS performs better with optimized transition rules. Moreover, the NU type of transition rules could significantly improve the sorting ability when considering larger BMSs.

5 Joint optimization of transition rules and premiums

5.1 Stationary model

In this section, we present a modification of the model introduced in section 4.2. In this modification, we can jointly optimize the transition rules and premiums. In this case, if we use π_k ($\forall k$) as non-negative variables, we will get a quadratic constraint problem (MIQCP). Because solving a MILP usually takes less computational time than the corresponding MIQCP, we linearize the quadratic constraints. First, we consider the model without a profit constraint. Due to Theorem 8, it is sufficient to allow only finitely many possibilities for the premiums.

To this end, we start with default premiums for each class that can be increased if needed. We set each default premium to the expected claims of types with the lowest risk $\pi_k = \lambda^1$, $\forall k$. We then introduce ε as a value for changing the default premium and also consider various layers of these modifications. ε^ℓ denotes how much the premium changes in layer ℓ compared to the default premium.

By Theorem 8, setting the values of the changes to $\varepsilon^\ell = \lambda^\ell - \lambda^1$, $\ell = 2, \ldots, L$, and $L = I - 1$ is sufficient. Binary variable O_k^ℓ indicates whether we increase the premium in class k by ε^ℓ, i.e., if $O_k^\ell = 1$, then the final premium of class k is $\lambda^1 + \varepsilon^\ell = \lambda^\ell$. The final premiums should be monotonously decreasing:

$$\pi_k + \sum_{\ell=1}^{L} \varepsilon^\ell O_k^\ell \geq \pi_{k+1} + \sum_{\ell=1}^{L} \varepsilon^\ell O_{k+1}^\ell \qquad k = 0, \ldots, K \tag{MILP3.12}$$

Only one change should be active in each class:

$$\sum_{\ell=1}^{L} O_k^\ell \leq 1 \qquad \forall k \tag{MILP3.13}$$

In addition, the premium changes should be considered in the constraints (MILP1.10) and (MILP1.11). This would, however, change these linear constraints into quadratic ones. We can linearize these constraints with continuous non-negative variables $o_k^{\ell,i}$. With the following constraints, we can prescribe that if $O_k^\ell = 1$, then $o_k^{\ell,i}$ should be equal to $c_k^i \varepsilon^\ell$, otherwise 0.

$$o_k^{\ell,i} \geq \varepsilon^\ell \left(c_k^i - (1 - O_k^\ell)\right) \quad \forall i, k, \ell = 1, \ldots, I-1 \tag{MILP3.14}$$

$$o_k^{\ell,i} \leq \varepsilon^\ell c_k^i \quad \forall i, k, \ell = 1, \ldots, I-1 \tag{MILP3.15}$$

and

$$o_k^{\ell,i} \leq \varepsilon^\ell O_k^\ell \quad \forall i, k, \ell = 1, \ldots, I-1 \tag{MILP3.16}$$

We modify the constraints (MILP1.10) and (MILP1.11) as well:

$$\pi_k c_k^i + \sum_{\ell=1}^{I-1} o_k^{\ell,i} + g_k^i \geq \lambda^i c_k^i \quad \forall i, k \tag{MILP3.10}$$

$$\pi_k c_k^i + \sum_{\ell=1}^{I-1} o_k^{\ell,i} - g_k^i \leq \lambda^i c_k^i \quad \forall i, k \tag{MILP3.11}$$

For the joint optimization of transition rules and premiums, in the MILP, we would minimize (MILP1.obj), subject to (MILP1.1)-(MILP1.8) and (MILP3.12)-(MILP3.11).

If we consider the profit constraint (LP1.4), we may not get the global optimum with finitely many premium changes. Additionally, according to Theorem 9, there can be one additional premium in the optimal solution. We can include another layer for this extra premium with this unique premium's level. However, we do not know the exact value of this layer's ε beforehand. By adding multiple additional layers of premium changes, we can approximate the optimal solution with arbitrary precision.

We increase the number of layers (L) and then separate them into two sets $\mathcal{L}_1$ and $\mathcal{L}_2$; thus, $L = |\mathcal{L}_1| + |\mathcal{L}_2|$. The first set of layers denotes the modifications used previously to achieve the expected claims of the risk groups $\varepsilon^\ell = \lambda^\ell - \lambda^1$ if $\ell \in \mathcal{L}_1$; hence, $|\mathcal{L}_1| = I - 1$. The other type of layers is for the unique premium only. For this, we arbitrarily determine every ε^ℓ, if $\ell \in \mathcal{L}_2$. By Theorem 9, there can only be one type of unique premium, i.e., there can be at least one active layer in $\mathcal{L}_2$. For this, we introduce a binary variable S_ℓ for all $\ell \in \mathcal{L}_2$. This variable is equal to 1 if the classes' layer ℓ is active:

$$\sum_{k=0}^{K} O_k^\ell \leq (K+1) S_\ell \qquad \forall \ell \in \mathcal{L}_2 \tag{MILP4.17}$$

There can be at least one active layer in $\mathcal{L}_2$:

$$\sum_{\ell \in \mathcal{L}_2} S_\ell \leq 1 \tag{MILP4.18}$$

In this case, we also have to include the profit constraint in the model:

$$\sum_{i=1}^{I} \phi^i \sum_{k=0}^{K} \left(\pi_k c_k^i + \sum_{\ell=1}^{L} o_k^{\ell,i} \right) \geq \sum_{i=1}^{I} \phi^i \lambda^i \tag{MILP4.9}$$

The values ε^ℓ if $\ell \in \mathcal{L}_2$ are arbitrary, as well as $|\mathcal{L}_2|$. In theory, if we include a large number of layers of type $\mathcal{L}_2$, the model may give a good solution close to the global optimum. If $|\mathcal{L}_2|$ is large, then the computational time increases dramatically because of the “big-M” constraints, such as (MILP3.14)-(MILP3.16). In the numerical experiments, we used only one second-type layer ($|\mathcal{L}_2| = 1$). In this case, we iteratively reran the model to find the best ε of this layer. This reduced computational time, especially for larger cases, without significantly affecting the optimal solution.

5.2 Multi-period model

In section 4.2, we presented a MILP model based on the stationary distribution. However, in some cases, for the probabilities to reach the stationary level, more time periods are needed than the duration for which policyholders may remain in the system. In such cases, instead of the stationary distribution, using the probabilities in each period of the insurance contract would be more appropriate for the optimization. In this section, we introduce a modification to the model in section 5.1, where we do not use stationary probabilities.

Because this model does not require stationary probabilities, the classification process of the policyholders does not have to be a regular Markov chain. Hence, it is possible to optimize realistic situations where the claim probabilities of the policyholders or the ratios of the risk groups depend on time. However, for simplicity, we only present a model that considers the same assumptions in the stationary case. Although the model can be formulated to consider time-dependent transition probabilities, we did not investigate this prospect because the time required for finding the optimal solution was extremely long, even in the simplest case.

Take the first Θ periods of the insurance contract. The index of time is denoted by t ($t = 0, \ldots, \Theta$) where $t = 0$ indicates the beginning of the contract, and Θ is the end of it. The variables c_k^i, g_k^i and $d_{k,j,m}^i$ now depend on time, so we use the notations $c_{k,t}^i, g_{k,t}^i$ and $d_{k,j,m,t}^i$ accordingly. In the starting period (indexed with 0), each policyholder is assigned to the same initial class. We introduce a binary variables B_k for all classes to determine the initial class. When the variable B_k takes the value 1, class k is the initial class. Let’s assume that there is only one initial class:

$$\sum_{k=0}^{K} B_k = 1 \tag{MILP5.19}$$

$$c^i_{k,0} = B_k \qquad \forall i, k \tag{MILP5.20}$$

Transition rules are determined in the same way as in the previous case. This means that the constraints (MILP1.1)-(MILP1.4) remain unchanged in the multi-period model. Constraints (MILP1.5)-(MILP1.6) now become

$$\sum_{k=0}^{K} c^i_{k,t} = \phi^i \qquad\qquad \forall i, t = 1, \dots, \Theta \tag{MILP5.5}$$

$$c^i_{k,t} = \sum_{j=-(K-k)}^{k} \sum_{m=0}^{M} d^i_{k-j,j,m,t-1} \qquad \forall i, k, t \tag{MILP5.6}$$

$$d^i_{k,j,m,t} \geq \lambda^i_m c^i_{k,t} - (1 - T_{j,m,k})\phi^i \qquad \forall i, j, k, m, t = 0, \dots, \Theta - 1 \tag{MILP5.7}$$

Constraint (MILP1.8) can only be used for irreducibility if Θ is large enough. In the multi-period model, another approach for irreducibility would be more appropriate. Theorem 9 is still valid in the multi-period case, so for the joint optimization of the premiums and transition rules, we can use the same modifications as in section 5.1. Variables $o^{\ell,i}_k$, however, should be time-dependent and replaced by variables $o^{\ell,i}_{k,t}$.

There are two ways of including the profit-constraint in the multi-period model: either we prescribe the profitability over all the $\Theta + 1$ periods, or we do this for each period. If we consider the overall profit, the model should include the following constraint:

$$\sum_{t=0}^{\Theta} \sum_{i=1}^{I} \sum_{k=0}^{K} \left(\pi_k c^i_{k,t} + \sum_{\ell=1}^{L} o^{\ell,i}_{k,t} \right) \geq \sum_{i=1}^{I} (\Theta + 1)\phi^i \lambda^i \tag{MILP6.9.1}$$

On the other hand, if we consider the profitability in each period, then

$$\sum_{i=1}^{I} \sum_{k=0}^{K} \left(\pi_k c^i_{k,t} + \sum_{\ell=1}^{L} o^{\ell,i}_{k,t} \right) \geq \sum_{i=1}^{I} \phi^i \lambda^i \quad \forall t \tag{MILP6.9.2}$$

Furthermore, in the objective function, we should consider the absolute deviation of every period:

$$\min \sum_{t=0}^{\Theta} \sum_{i=1}^{I} \sum_{k=0}^{K} \phi^i g^i_{k,t} \tag{MILP5.obj}$$

5.3 Changing the number of classes

When there are many classes, some of them may not be reachable during the time of the contract. Hence, the probability to be classified into a class can be zero in every period if we consider multi-period optimization. Presumably, in the stationary case, more classes improve the sorting capability of the BMS. In the stationary models, the probability of these classes would be positive. However, a BMS where each class can be reached from the initial class may be more appealing to the designer of the system. Also, if we use the constraints (MILP1.8), we may get a relatively extreme transition rule. Furthermore, having a lot of classes with small probabilities may result in some numerical issues in the optimization.

Due to these reasons, we may allow for a change in the number of classes in the optimization. To this end, we introduce binary variables V_k for each class k. Whenever $V_k = 1$, we close the corresponding class k. Therefore, the policyholders cannot be assigned to this class, leading to the system having only K classes.

In other words, the probability of being in this class should be zero ($c^i_{k,t} = 0$) for each period. Therefore, the probability of getting into this class should be zero as well ($d^i_{k,j,m,t} = 0$, which holds because of constraint (19)).

To close a class, the probability of being in that class should be zero. Therefore, we need to add the next constraint to the model:

$$c^i_{k,t} \leq 1 - V_k \qquad \forall i, k, t \tag{MILP7.21}$$

The reason behind the closing could be to not enable that many nearly unreachable classes in the BMS. Hence, we may introduce a parameter that denotes the minimal probability of being in a class. We shall consider it in the last period, which is almost the same as the stationary level because the probabilities' variance is higher at the beginning of the process. For example, in a BMS where the claimless unified transition rule is 1 and the initial class is class 0, the probability of being in class 5 is zero for the first four periods.

If Θ is large enough, the probabilities of being in the classes in period Θ is close to the stationary probabilities. We may introduce a constraint to ensure that each class has a considerable amount of policyholders in the final period considered. Therefore, we consider a parameter that denotes the necessary probability for each class in the period Θ.

$$c^i_{k,\Theta} \geq \tau - V_k \qquad \forall i, k \tag{MILP7.8}$$

This approach is similar to the constraint (MILP1.8) in the non-unified transition rule's case. Hence, this constraint also leads to an irreducible Markov chain. The binary variables V_k enable the closing. Therefore, the probabilities of being in class ($c^i_{k,t}$) in the last period should be greater than τ only if the class is not closed. Otherwise, $c^i_{k,t} = 0$ because of the constraint (MILP7.21).

To keep the irreducibility of the system, a class should only be closed on one of the sides of the BMS. Thus, either class K or class 0 can be closed. For the sake of simplicity, we only present the constraint for initiating the closing from the upper side. The other type of closing needs similar constraints. For joint optimization, the side of closing is not essential. However, if we only consider the transition rule's optimization, the closing side has more importance because of the fixed premiums.

Upper-closing means that we may close class $k+1$ if class k is open:

$$V_k \leq V_{k+1} \quad k = 1, \dots K \tag{MILP7.22}$$

The constraint (MILP5.6) has to be changed because if class k is closed, the left-hand side of the constraint is zero. However, not all of the $d^i_{k-j,j,m,t-1}$ variables are zero. Hence, we have to prescribe the constraint to be an equation when class k is open:

$$c^i_{k,t} - V_k \leq \sum_{j=-(K-k)}^{k} \sum_{m=0}^{M} d^i_{k-j,j,m,t-1} \qquad \forall i, k, t \tag{MILP7.6.1}$$

$$c^i_{k,t} + V_k \geq \sum_{j=-(K-k)}^{k} \sum_{m=0}^{M} d^i_{k-j,j,m,t-1} \qquad \forall i, k, t \tag{MILP7.6.2}$$

Therefore, the equation only holds when $V_k = 0$; otherwise, the constraint is irrelevant.

However, if we consider the unified transition rules, the multi-period equivalent of the constraint (MILP2.6) differs more. In this case, we had to separate the constraints of the border and inner classes. Whenever a border class is closed, there will be a new class at the side of the system. Thus, closing should be possible for this class as well. For example, if class K closes, class $K-1$ will become the new upper class. Therefore, the last equation of the constraint (MILP2.6) should be applied to this class with the multi-period modification. In other words, if $V_K = 1$, then the following constraint should hold; otherwise, constraint (MILP2.6) should be valid.

$$c^i_{k,t} = \sum_{j=0}^{\overline{J}} \sum_{\ell=0}^{\min(j,k)} \sum_{m=0}^{M} d^i_{k-\ell,j,m,t-1} \qquad k = K-1, \forall i, t \tag{24}$$

Those $d^i_{k,j,m,t}$ variables, which would lead into a closed class constraint (MILP2.6) should be modified:

$$c^i_{k,t} + 1 - V_{k+1} + V_k \geq \sum_{j=0}^{\overline{J}} \sum_{\ell=0}^{\min(j,k)} \sum_{m=0}^{M} d^i_{k-\ell,j,m,t-1} \qquad k = 1, \ldots, K, \forall i, t \quad \text{(MILP8.6.1)}$$

$$c^i_{k,t} - 1 + V_{k+1} \leq \sum_{j=0}^{\overline{J}} \sum_{\ell=0}^{\min(j,k)} \sum_{m=0}^{M} d^i_{k-\ell,j,m,t} \qquad k = 1, \ldots, K, \forall i, t \qquad \text{(MILP8.6.2)}$$

These constraints will be met as an equation when $V_{k+1} = 1$ and $V_k = 0$.

The constraints (MILP2.6) should be applied for the inner classes, while the set of side classes depends on the binary variables V. Therefore, these constraints have to be modified as well:

$$c^i_{k,t} + V_{k+1} \geq \sum_{j=\max(\underline{J},-(K-k))}^{\min(\overline{J},k)} \sum_{m=0}^{M} d^i_{k-j,j,m,t-1} \qquad k = 1, \ldots, K, \forall i, t \qquad \text{(MILP8.6.3)}$$

$$c^i_{k,t} - V_{k+1} \leq \sum_{j=\max(\underline{J},-(K-k))}^{\min(\overline{J},k)} \sum_{m=0}^{M} d^i_{k-j,j,m,t-1} \qquad k = 1, \ldots, K, \forall i, t\ , \qquad \text{(MILP8.6.4)}$$

So, in this case, the constraint (MILP2.6) is valid if $V_{k+1} = 0$. This means that the class above class k is open.

5.4 Numerical experiments

For the numerical calculations, we used the same computer and same setup mentioned in section 4.4.

5.4.1 Joint optimization model without profitability constraint

We considered a 10-class BMS for each scenario and employed the model introduced in section 5.1. We then jointly optimized the transition rules with the premiums without taking into account the profitability constraint. We investigated cases where transition rules are allowed to be different in each class (introduced in section 4.2) and when they are are unified (mentioned in subsection 4.2.2).

Table 2 presents the transition rules of a 10-class BMS for each scenario. If there is no claim in a period, then in every situation, the policyholder moves one class upward. In contrast, if there is a claim during the period, then only unrealistic, "high" probability cases result in non-unified transition rules. In "small" probability situations, the results of the non-unified models gave the same solution as the unified

Current class	0 claim	1 claim							
		1%-2%		1%-5%		10%-20%		10%-50%	
		non-unified	unified	non-unified	unified	non-unified	unified	non-unified	unified
0	+1	0	0	0	0	0	0	0	0
1	+1	-1	-1	-1	-1	-1	-1	-1	-1
2	+1	-2	-2	-2	-2	-2	-2	-2	-2
3	+1	-3	-3	-3	-3	-3	-3	-3	-2
4	+1	-4	-4	-4	-4	-4	-4	-4	-2
5	+1	-5	-5	-5	-5	-5	-4	-5	-2
6	+1	-6	-6	-6	-6	-6	-4	-6	-2
7	+1	-7	-7	-7	-7	-7	-4	-4	-2
8	+1	-8	-8	-8	-8	-7	-4	-1	-2
9	0	-9	-9	-9	-9	-2	-4	-1	-2

Table 2: The transition rules of the 10-class BM systems for each situation

ones. In such cases, the policyholders will be classified into class 0 if they have a claim. In "high" risk situations, the higher classes however have less strict transition rules if the rules are non-unified. Overall, if the claim risks are higher, the transition rules are less strict. We investigated every scenario's BMS with more classes as well. The computational time of the models was limited to 3 hours. When the limit had been reached, we stopped, and the best solution found was recorded.

class	1%-2%			1%-5%			10%-20%			10%-50%		
	Obj. ch.	time NU	time U	Obj. ch.	time NU	time U	Obj. ch.	time NU	time U	Obj. ch.	time NU	time U
10	0%	22	0.2	0%	6	0.3	-0.08%	539	0.2	-0.52%	5	0.2
11	0%	589	0.2	0%	41	0.2	-0.10%	9311	0.3	-0.60%	16	0.2
12	0%	10606	0.3	0%	289	0.3	-0.13%	10800	0.3	-0.67%	423	0.3
13	0%	10800	0.4	0%	2992	0.3	-0.15%	10800	0.4	-0.68%	475	0.3
14	0%	10800	0.3	0%	10800	0.3	-0.16%	10800	0.4	-0.72%	1229	0.4
15	0%	10800	0.4	0%	10800	0.4	-0.17%	10800	0.5	-0.78%	10800	0.5

Table 3: Differences in the running time and objectives of 10–15-class BM systems. The time U indicates the running time of the unified model, and the time NU is the non-unified model's computational time (both in seconds). The columns titled 'Obj. ch.' represent the improvement of the objective value of the model with the non-unified transition rules.

Table 3 presents the differences in the objective function as well as the running time between the unified (U) and non-unified (NU) models. As we increased the number of classes, in "high" probability scenarios, the NU models gave an even better solution. For "small" probabilities, the change of U to NU was however of no significance. Besides, the NU models are computationally much harder than the U models because of the large number of binary variables. While the U models produced optimal solutions in every case within a second, we could not compute the optimal solution of the NU models within three hours in many cases. We further investigated the effect of the number of classes on unified transition rules. Every optimal solution was determined from a 3-class to a 120-class BMS.

In Figure 8, the unified transition rule for the case of a claim in each BMS can be seen.

As the number of classes increases, the transition rule generally gets stricter in

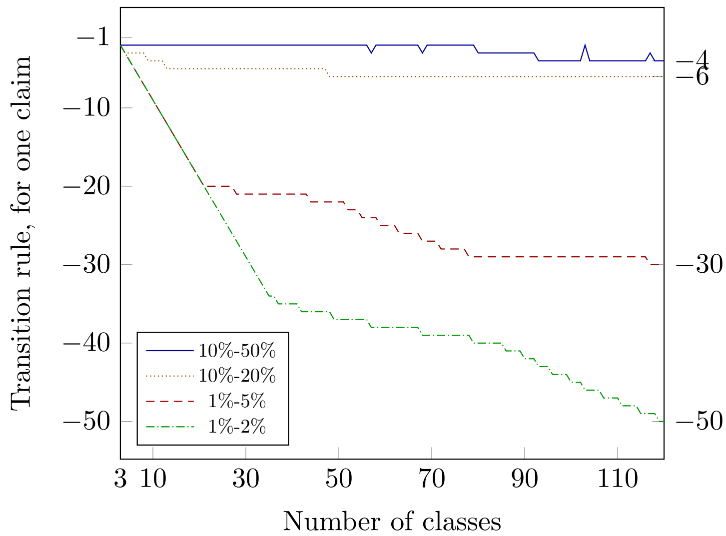

Figure 8: Changes to the transition rule with one claim as a function of the number of classes

every scenario. However, after reaching a certain number of classes, the penalty for a claim tends to decrease. In addition, if the probabilities are higher, the point where the rate of decline changes occurs at a much lower number of classes. The case where $m = 0$ resulted in one positive step in each situation.

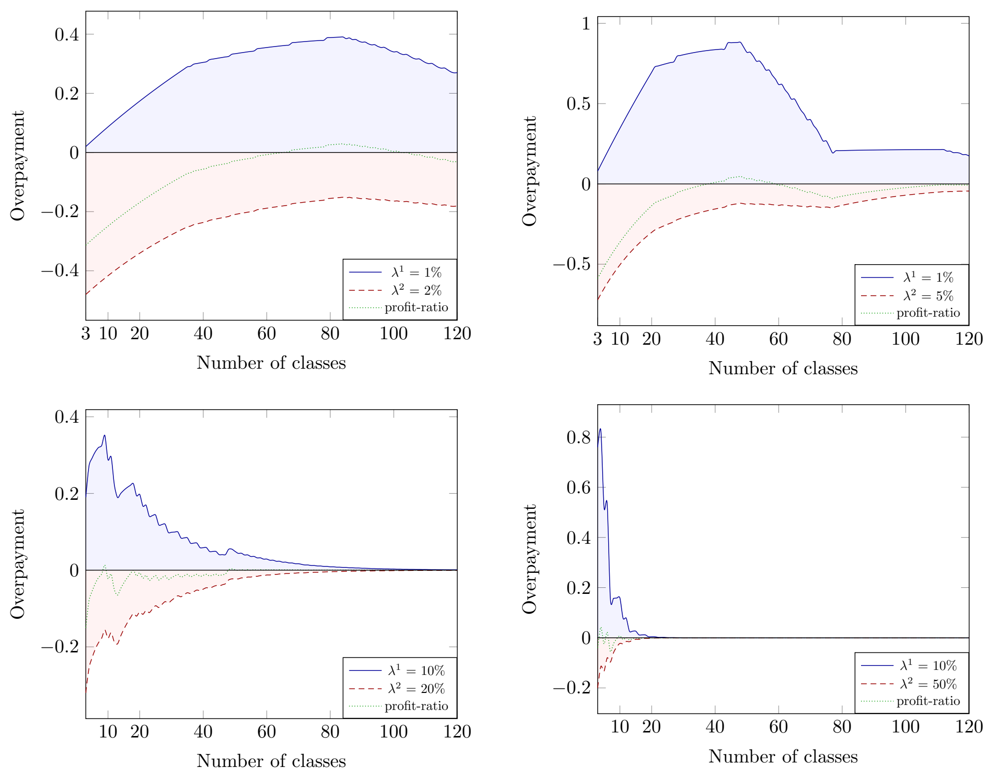

Figure 9: Changes in the OP^i and the profit ratio as a function of the number of classes

In every situation, Ω decreases with an increase in the number of classes; this means that a BMS with a larger number of classes would be a better sorting system. If the risk groups' parameters are higher, then Ω tends to level zero, meaning the BMS sorts the types almost perfectly. For the $10-50\%$ case, the BMSs with over 20 classes sort the types almost perfectly, whereas the $10-20\%$ situation needs about 100 classes. In lower-risk situations, even 120 classes are not enough to form a viable sorting system, but some decrease of Ω can be seen as the number of classes increases. It can also be seen that, in the lower-risk situations, a smaller number of classes brings the payment of "good" policyholders closer to their ideal level. As the number of classes increases, the payment for "bad" types gets closer to their ideal level too.

The dotted line represents the profit ratio of the insurer, which is the expected overall payment of the policyholders divided by the expected total claims minus one. If the profit ratio is positive, the BMS is profitable. Certainly, if every policyholder's expected payment equals to his/her ideal level, then the profit ratio is zero. With a lower number of classes, the policyholders pay less, but as the number of classes increases, the payment also increases.

5.4.2 Consideration of more than one claim per period.

In the previous models, the maximal number of claims that could happen in a period was one. We also considered a model when $M = 2$ where the probabilities of the claims are calculated according to the Poisson distribution. Figure 10 depicts the transition rules of the models. On the left-hand side, the first claim's transition rule is shown. The graph on the right side presents the additional reduction that the second claim would cause.

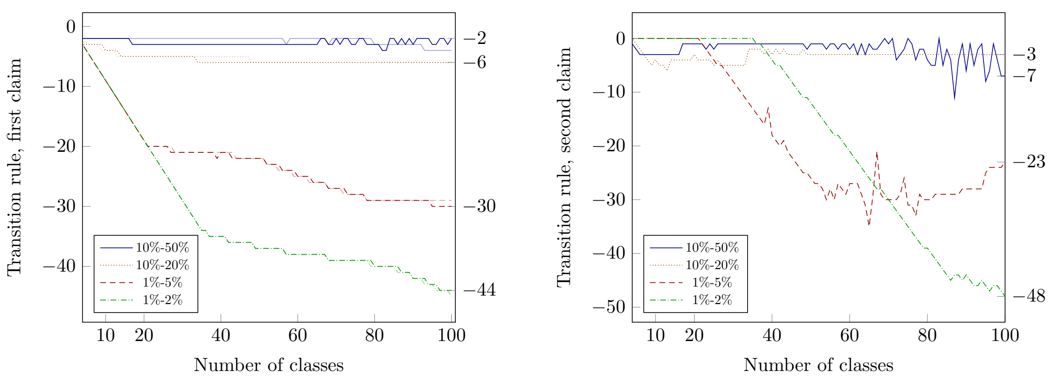

Figure 10: The 1st and 2nd claim's transition rules

In the graph showing the first claim's transition rule, we also display in paler colors the models' results with $M = 1$. These lines are almost the same; hence, the first claim is treated similarly across all scenarios.

As for the second claim, it is not treated too strictly when there are only a few classes in every scenario. Interestingly, in the scenario of $1-2\%$ claim probabilities, the second claim is not considered at all if there are less than 40 classes. However, as the number of classes increases, the smaller probability scenarios' transition rules decrease. Notably, the higher probability scenarios' transition rules do not decrease significantly.

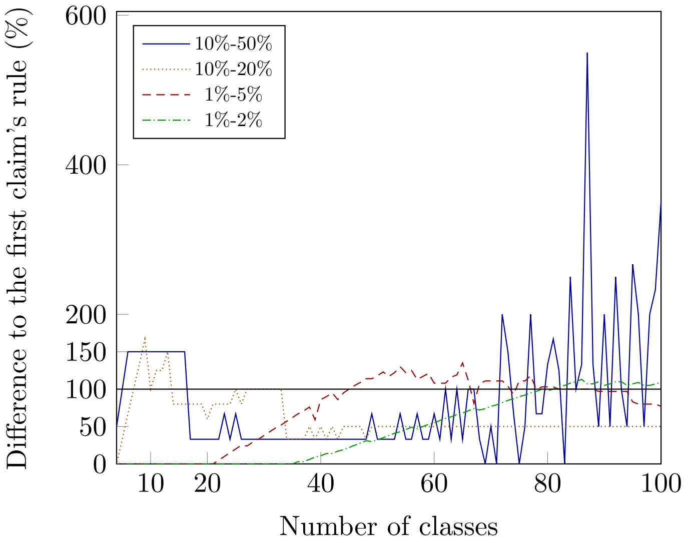

Figure 11: The ratio between the reduction associated with the first and second claim

Figure 11 illustrates the ratio between the first and second claims' reduction in the classification. If the relevant value is 100%, then the second claim is treated the same way as the first one is. When it is less than 100%, the second claim results in less class-reduction. Mostly, the transition rule for the second claim is not as strict as the first one. In the $10-50\%$ scenario, the rule for the second claim gets slightly unstable as the number of classes rises. This is because the Ω in these BM systems are nearly zero. Hence, the second claim does not influence the solution greatly.

In practice, the second claim is regularly treated similar to the first one. The probability of two claims occurring in the same period is in reality very small. It was also observed that the consideration of multiple claims does not change the Ω values of the models. Besides, the first claims' transition rule was very similar to the case where $M = 1$. Therefore, in the numerical examples, we considered only one possible claim for the sake of computational simplicity.

5.4.3 Model with profitability constraint

With the inclusion of the profitability constraint (MILP4.9), there is one unique premium. In section 5.1, we put forth a modification of the MILP by which we can determine the unique premium with additional premium changing layers. Multiple layers require more binary variables, producing a notable increase in computational time. Moreover, even with a fixed number of layers, we cannot be certain that we have found the global optimum. Therefore, for finding a (nearly) globally optimal solution within a reasonable time, we employed an Iterated Local Search (ILS) algorithm (see for instance Lourenço et al. (2010)).

Sol_0 = Initial solution
Sol^* = LocalSearch(Sol_0)
repeat
  Sol^p= Perturbation(Sol^*, history)
  Sol^{ls} = LocalSearch(Sol^p)
  if $\min(Sol^p; Sol^{ls}) < Sol^*$ **then**
    $Sol^* = \min(Sol^p; Sol^{ls})$
  end
until *termination condition met*;

Figure 12: Iterated Local Search (source: Lourenço et al. (2010))

We used this algorithm as a heuristic of the model. Hence, we reran the model in each iteration but with a different parameter for the unique premium. Thus, in this case, the *Sol* in Figure 12 determines the objective value of a model with a specified unique premium.

We started the initial solution (Sol_0) with a zero parameter, which means that the first model did not have unique premium. After the initial solution was calculated, we started to find a local optimum, close to the 0 unique premium. Hence, we searched for a better solution by iteratively increasing the value of the unique premium. After identifying the local optimum, we iteratively searched with a *Perturbation* function that increases the unique premium by a random value. We used a randomized *Perturbation* function, which increases over the iteration if we do not find a better solution. However, if we find a better solution, then the following perturbation will be a smaller increase again. To improve the running time, we only conduct *LocalSearch* on the perturbed solution if it is close to the best solution. We ran three random restarts of this algorithm to find a better solution.

Figure 13 shows the results of the transition rule of the case where $m = 1$ as the number of classes increases with the inclusion of the profitability constraint (MILP4.9). Similarly to the case that does not consider profitability, the parameter j decreases as the number of classes increases, but after getting to certain number of classes, the reduction trend changes. Interestingly, the initial decrease turns to increase only to start decreasing once again.

Figure 14 shows the OP^i values under different situations. The dashed lines indi-

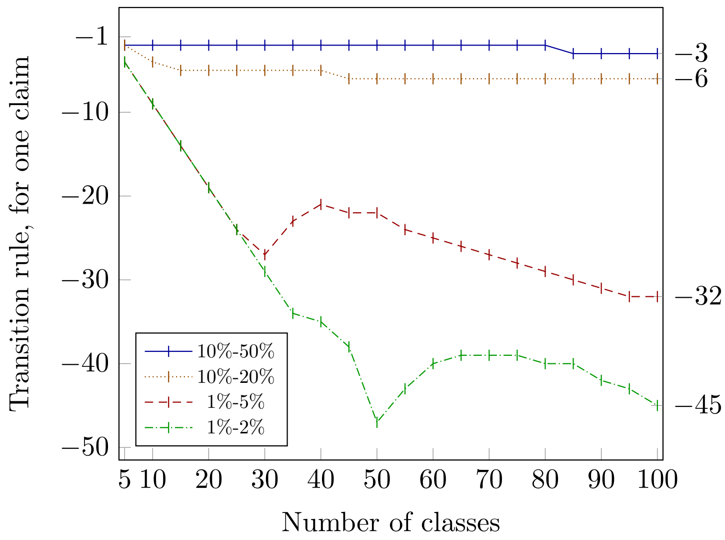

Figure 13: Transition rule changes for one claim as a function of the number of classes

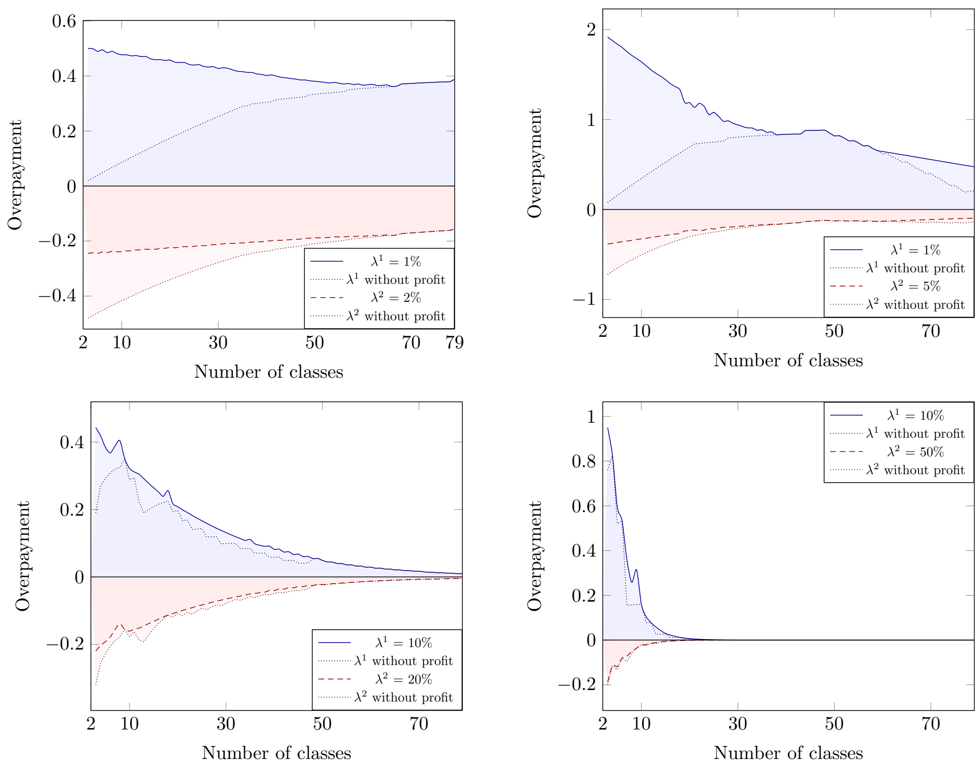

Figure 14: Changes of the OP^i as a function of the number of classes

cate the case where profitability is not considered. Profitability constraint typically affects BMSs having a small number of classes. Overall, profitability increases the payment by the policyholder, i.e., the "good" policyholders will pay even more than their ideal expected premium, but the "bad" policyholders pay closer to their ideal level.

5.4.4 Ratios of types

In the previous examples, we considered the two risk groups with the same ratio. We further investigated how the optimized BMS can sort risk groups when their ratio is not equal. For this, we calculated the optimal solution for 15-class BMSs for the two risk groups but with different proportions.

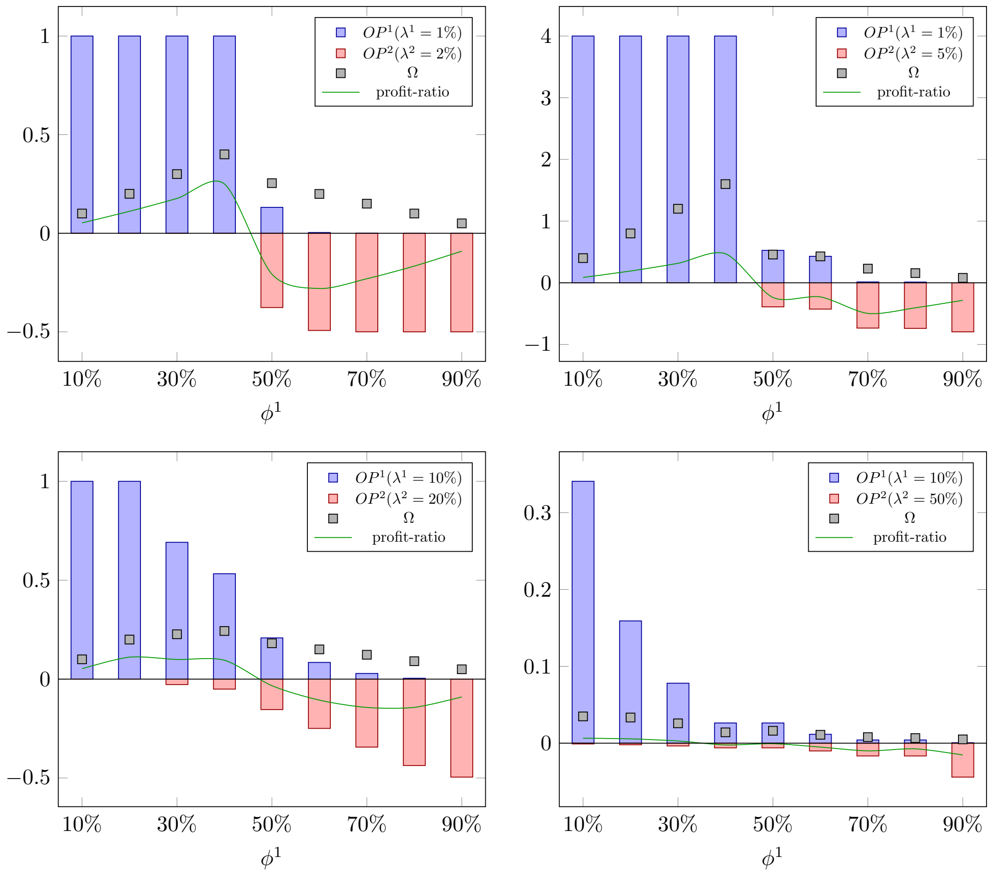

Figure 15: Changes of the ϕ^1.

Figure 15 presents the results of 15-class BMSs, with different ratios of the types (ϕ^i). The x-axis represents the proportion of the 'good' policyholders among all of the insurance contracts. If one of the types has a much higher proportion, then the system attributes this risk group more weightage. If the riskier ones constitute the majority, they pay their ideal expected premium, and the 'good' policyholders pay more than their risks. However, if the less risky policyholders form the majority, the 'good' policyholders pay their ideal expected premium, and the 'bad' policyholders pay less than their ideal level. Therefore, if there are more 'good' policyholders than the 'bad' ones, the BM system is not profitable.

The grey squares display the Ω of the systems. It has a lower value when the ratio between the two types further differs. Therefore, if the risk groups are equally sized, the BM system's sorting capability is not as efficient as it is otherwise.

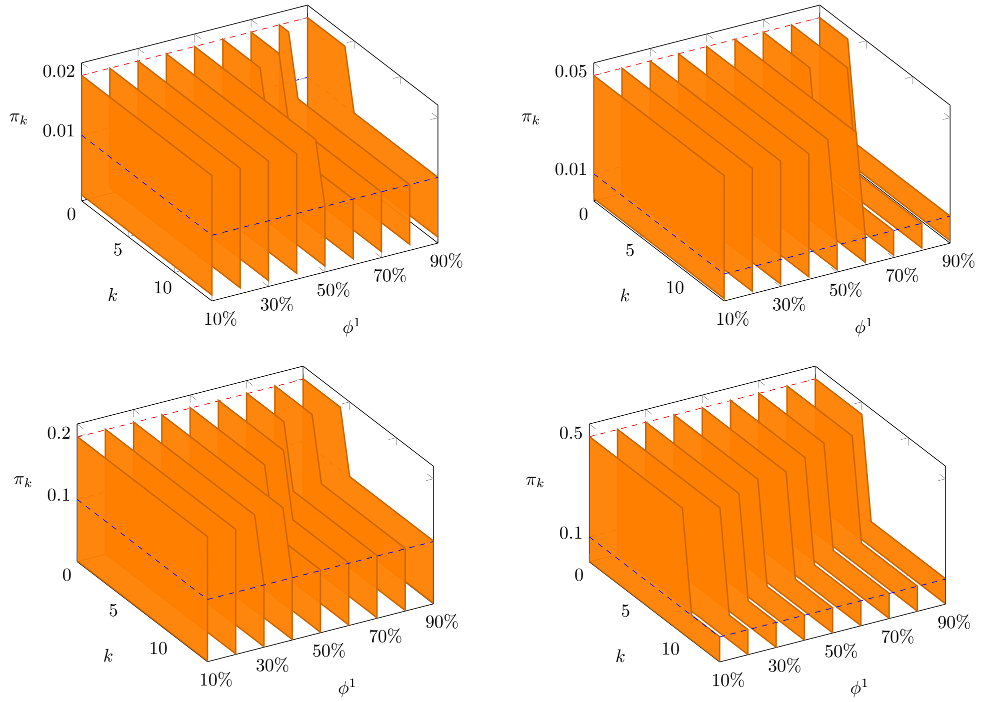

Figure 16: Premium-scales, when the ϕ^1 changes.

Figure 16 displays the premiums of each BMS. Each 'sheet' represents one system's premium vector. Therefore, there is a 'sheet' for each setup for the different risk group ratios. Hence, every 'sheet' has two dimensions: the class (k) and the premium (π_k).

If the proportion of the policyholders with lower risk is considerably smaller, then there are no ideal premiums for the 'good' policyholders in most cases. Consequently, the BMS only considers the 'bad' type. The 'good' policyholders surely pay more than their expected claim. The only exception is the 10- 50% case.

As the proportion of the 'good' type increases, the number of classes where the premiums are equal to their ideal premium increases as well. However, there is at least one class in every situation in which the premium is equal the 'bad' policyholders' ideal premium.

If the ratios of the risk groups are not equal, in a 15-class BMS, only the situations wherein the 'bad' policyholders are not the majority are affected by the profit constraint. Figure 17 presents the profit constraint's effect on the value of Ω in each situation. The unique premiums in the cases where there are very few 'bad' policyholders are becoming considerably higher than the other premiums. This is because, otherwise, the profitability would not be met. This premium scale can be optimal for the model; however, it will not be a useful solution in practice. In this case, causing a claim may increase the payment of a policyholder by a considerable amount. Therefore, this BM system may not reduce the risk of the policyholders.

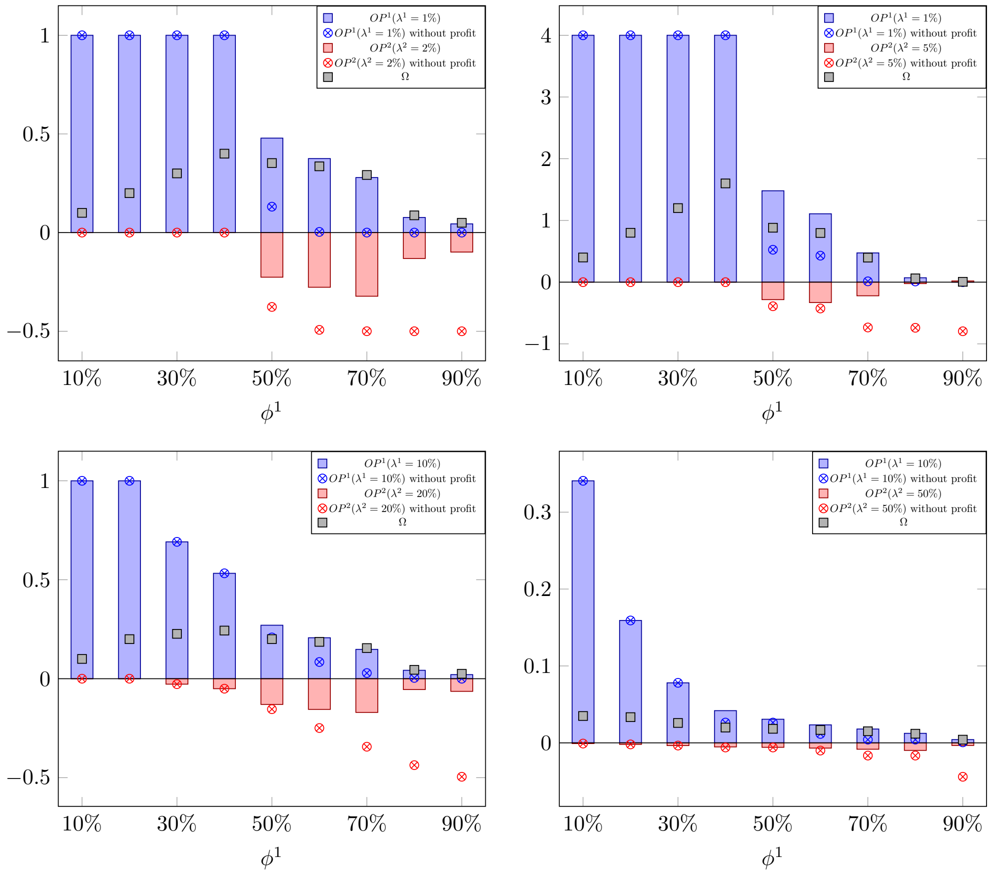

Figure 17: Changes of the ϕ^1 with the profit constraint.

5.4.5 Multi-period optimization without considering profitability

For the multi-period model, we studied the $\Theta = 20$ and $\Theta = 40$ cases. For treating the irreducibility of the system, constraints (MILP1.8) may not be suitable. Therefore, we fixed the $m = 0$ transition rule to 1. Because of the excessive amount of computations the multi-period model requires, we only examined BMSs where the number of classes was divisible by 5, up to 30 classes. Figure 18 shows the differences of the $m = 1$ transition rule in each case.

Figure 18 illustrates the transition rule of the multi-period models, compared with the results of the stationary models. In the "high" probability situations and in the $1-5\%$ case, the transition rules are much stricter in the multi-period model. In the $1-2\%$ risk situation, the transition rules however become less strict when there are 25 or 30 classes. In these cases, the value represents the average overpayment (or underpayment) of the types during the insurance contract period. Figure 19 shows the differences in the Ω values between the multi-period model and the stationary model.

As the number of the periods increases, the Ω of the multi-period model gets closer to the result of the stationary model. As Figure 19 depicts, if Θ is smaller, then

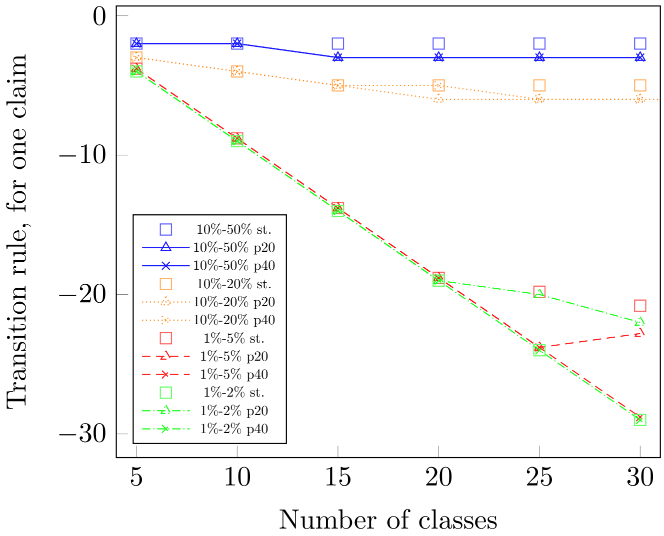

Figure 18: Changes in the transition rule with one claim as a function of the number of classes

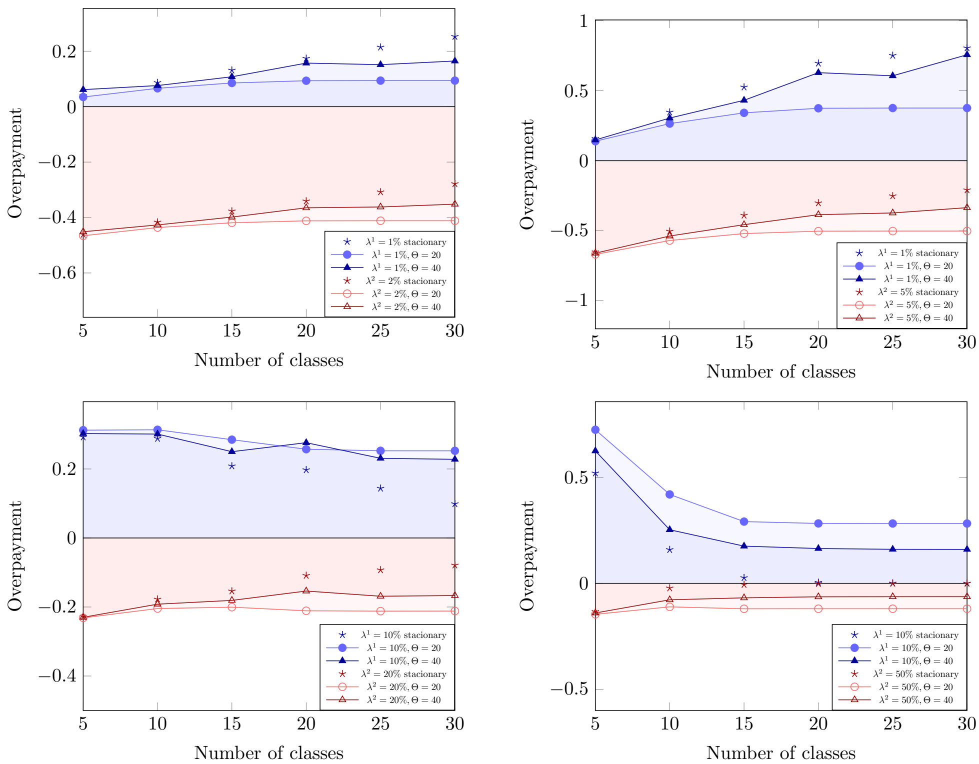

Figure 19: Changes of the OP^i as a function of the number of classes

the overpayment is less for the "good" policyholders and the underpayment is more for the "bad" ones. In addition, due to the limitation of the periods, if we increase the number of classes, then the sorting capability of the system does not necessarily improve. In the $10-50\%$ case, in the stationary model, each type of policyholder pays their ideal expected premium in class 25, whereas in the multi-period models we obtained considerably worse results.

5.4.6 Multi-period optimization considering profitability

Figure 20 depicts the Ω values of the multi-period models if we consider the profitability in each period (constraints (MILP6.9.2)).

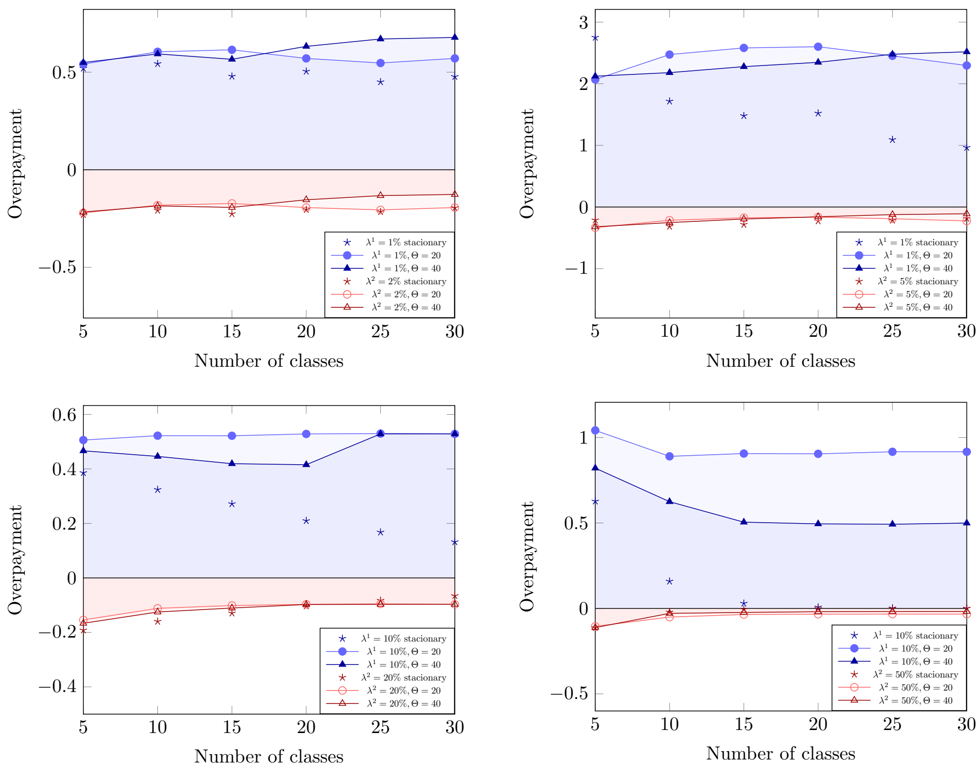

Figure 20: Changes of the OP^i as a function of the number of classes

We can observe results similar to those without profitability constraints. The improvement of the Ω is considerably smaller in the multi-period case if we increase the number of classes compared with the results of the stationary models.

5.4.7 Computational considerations

In the introduced MILP model, as we increase the number of classes, the number of types, or the maximal number of potential claims, finding the optimal solution may take more time. For example, the running times of the stationary model that did not factor in the profitability constraint changed as Figure 21 shows.

When $K = 3$, the running time, was generally less than a half-second, and when the number of classes was 120, it increased to above 30 minutes. When we investigated the multi-period model, even the instances with few classes needed slightly more time. The case where $\Theta = 40$ required more than one hour in all cases. When the profitability was considered, the running time also increased considerably because of the ILS's multiple runs. The realistic case also resulted in a long running time. The case where the profitability was not considered where $\Theta = 40$ needed

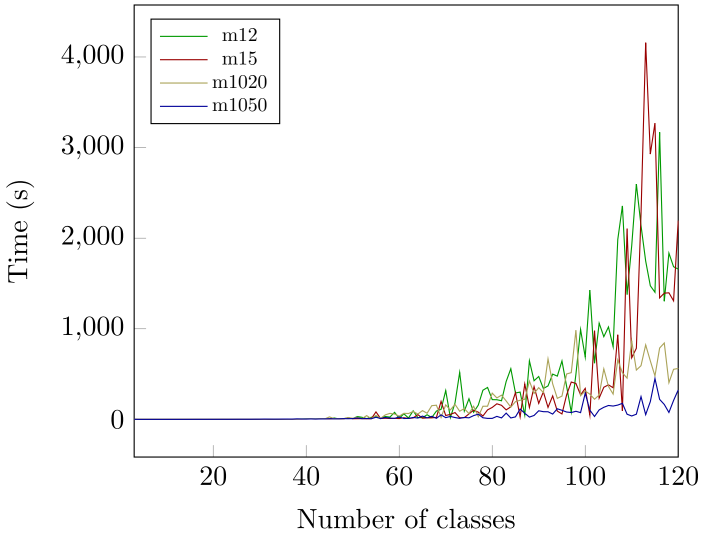

Figure 21: Running times of the stationary models without profit.

more than one day to find the optimal solution.

It is important to note that despite the running time being long in some cases, the aim to optimize such a BMS is not a time urgent problem. An optimized system should be valid for several years in the third-party liability insurance application. Also, the running time can be reduced using better hardware and a more significant tolerance level.

Another solution would be to approximate the result with a heuristic. The optimization models for the premiums and the transition rules can be calculated much faster separately than the joint optimization model. Hence, we may use an iterative method to approximate the optimal solution. First, we calculate the optimal transition rules with a fixed premium. Then, we find the optimal premiums to these transition rules, which we consider as parameters. Next, we use the optimal premiums of this model as parameters and re-optimize the transition rules. We continue this until we cannot improve the objective function further. The solution of this heuristic greatly depends on the initial model. In the initial model of transition rules' optimization, the premiums are outer parameters. We may also start with the premiums' optimization and then proceed with the optimization of the transition rules. In this case, the transition rules are the outer parameters in the first model. We present a comparison of this heuristic with the exact method in the next section.

5.4.8 Comparison of the iterative heuristic and joint optimization

In Ágoston and Gyetvai (2021), we made a comparison of the joint optimization of premiums and transition rules with the iterative method. In this section, we present the results of this study with an extension. We show that the iterative method is much faster than joint optimization, while the solution is close to the optimal solution of the joint MILP model. The performance of the iterative heuristic largely depends on the initial model. In the comparison, we considered four types of initial premiums:

- Proportional (prop): We introduce own classes for each risk group, which means the premium of these classes is equal to the risk groups' expected claim. The risk groups have that many own classes that are proportional to their percentage among all policyholders.
- Linear (lin): We take the lowest and highest risk groups' expected claim for the lowest and highest premium and set the classes' premium linearly.
- Minimal (min): The premium is equal to the highest expected claim in the worst class. In all other classes, it is equal to the lowest expected claim.
- Maximal (max): In this case, only one class premium is equal to the minimal expected claim, and the highest expected claim applies to all other classes.

We also considered two types of initial transition rules:

- TRK: In case of any claim, the policyholders move into the worst class. Without a claim, the policyholders move one class upward.
- TR1: In case of any claim, the policyholders move one class downward. Without a claim, the policyholders move one class higher.

Numerical experiments on the initial solutions

We considered BMSs with 15 classes in the comparison of the joint optimization model with the iterative heuristic. We then randomly chose their risk groups' claim probability to test the heuristic in as many setups as possible. Hence, we considered 100 randomized setups. In each setup, we considered five equally sized risk groups.

Two cases were examined: a realistic one, in which the risk parameters were generated from a 0.01 to 0.1 interval. And a non-realistic higher risk setup, in which the risks were chosen from the [0.1 : 0.3] interval. In each model, the maximal number of claims per period could be up to 2. For the claim probabilities, we considered the Poisson distribution. We compared the iterative heuristic with the joint optimization model with six different initial solutions. We highlighted the difference between the optimal solution and the running time. Figure 22 illustrates the relative deviation from the MILP model in both cases.

On the left side of the figure, the objective increase compared with the joint model is presented. The top row shows the low-risk case and the bottom one depicts the high-risk case. When the risks are low, the results are similar to the objective of the

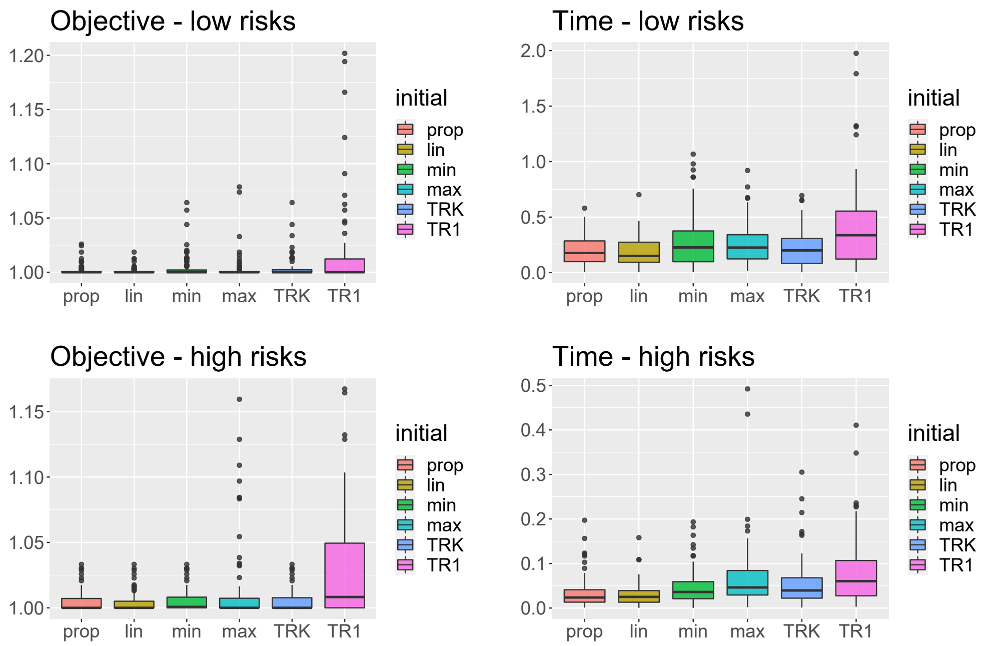

Figure 22: Objective and time changes compared with the joint optimization model

joint model. The *TR1* resulted in the highest difference in average. However, even in this case, the average increase was only 1.4%. When the risks are higher, the difference between the joint model and the heuristic is higher as well. The *TR1* led to the highest increase in average in this case too. The average increase was 2.9% for the *TR1* followed by the *max* with 1.1%

The running time of the iterative heuristic, was generally much faster than that of the joint optimization model with each initial solution. Again, the *TR1* seemed to be the worst. However, even so, the average running time was less than half of the joint model in both setups (40% in the low and 8% in the high setup).

Performance of the iterative method with the increase in BMS size

We investigated how the iterative method performs in comparison with joint optimization as the number of classes increases. We computed the solutions of models starting from a 2-class BMS up to an 80-class BMS. As the initial premium scales resulted in very similar results, we only considered the *prop* type initial premiums. We also considered the initial solution of *TRK* in the comparison. For each BMS, we considered only two risk groups, with expected claims generated from the [0.1 : 0.3] interval. We generated ten instances for each BMS.

Figure 23 presents the results of the models. On the top-left side, the computational time is represented. The increase in the computational time of the joint optimization model is significantly larger than that of the iterative method. Furthermore, the range of the running time fluctuates more (the background color). The

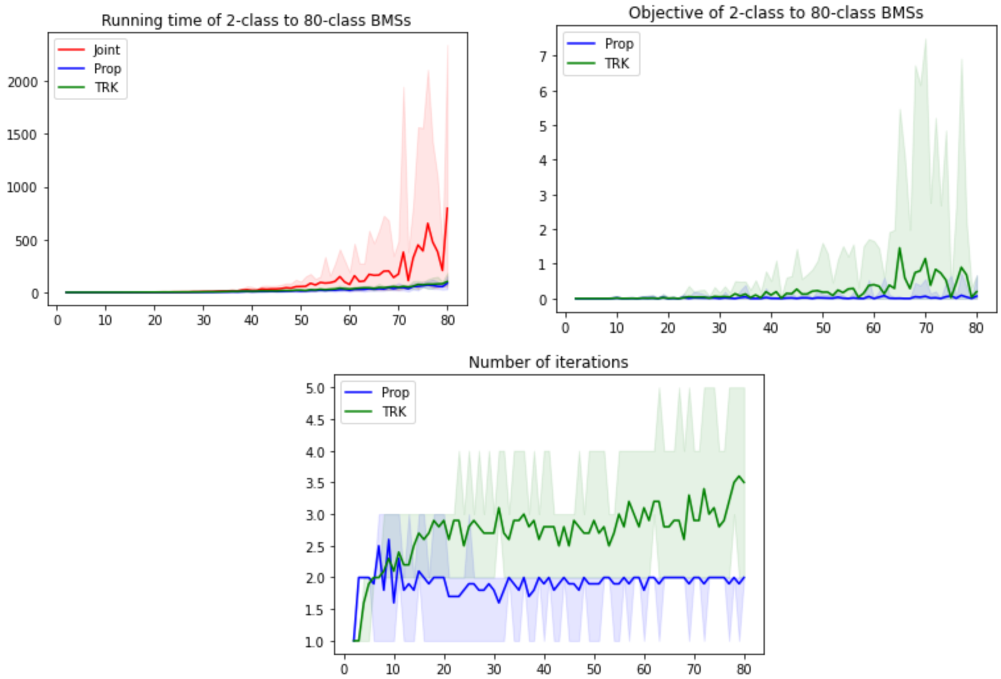

Figure 23: Results of models from a 2-class BMS to a 80-class BMS

top-right graph shows the difference between the objectives of the joint optimization model and the iterative methods. It is noticeable that as the size of the BMS increases, the *prop* increase becomes negligible, but the *TRK* gets very unstable. And even though the computational time required by the two iterative methods seems similar, the number of iterations (illustrated in the graph at the bottom) is more when we consider the *TRK* initial solution. In this case, it grew as the size of the BMS increased, while in the case of *prop*, the average number of iterations remained around two.

5.5 Case study: Optimizing the Hungarian BMS

Using data obtained from a Hungarian insurance company, we could work with realistic claim probabilities. We distinguished five different risk groups whose ratios and expected claims are provided in Table 4.

Type	1	2	3	4	5
Expected claim	1.8%	2.7%	3.2%	4.1%	5.0%
Ratio	11%	44%	26%	11%	7%

Table 4: Parameters of the risk groups

We solved the stationary models and the multi-period models with $\Theta = 20$ and $\Theta = 40$ respectively. In both cases, we carried out computations both with and without the consideration of the profitability constraint. Because of the excessive time requirement, we only considered the unified cases of a 20-class BMS. Table 5 lists the transition rules and the OP^i and Ω values of the models.

	Without profitability			With profitability		
	Stationary	$\Theta = 20$	$\Theta = 40$	Stationary	$\Theta = 20$	$\Theta = 40$
T_0	1	1	1	1	1	1
T_1	-6	-5	-6	-8	-2	-16
Objective	100%	100.01%	100.01%	155.53%	183.08%	163.91%
Ω	1.46	1.46	1.46	1.49	1.54	1.48
OP^1	0.50	0.50	0.50	0.68	0.78	0.69
OP^2	0.00	0.00	0.00	0.13	0.19	0.13
OP^3	-0.16	-0.16	-0.16	-0.05	0.00	-0.04
OP^4	-0.34	-0.34	-0.34	-0.25	-0.22	-0.24
OP^5	-0.46	-0.46	-0.46	-0.38	-0.36	-0.38

Table 5: The results of the models with realistic parameters. T_0, and T_1 denote the transition rules. *Objective* is the value of the objective function, compared with the stationary model that does not factor in profitability. Regarding multi-period models, the value of the objective function has been divided by the number of periods for the sake of comparison. The Ω shows the absolute overpayments, and the OP^i is the overpayment by the type i policyholders.

If we had not considered the profitability, the result of the stationary model and both multi-period models would not have been significantly different. The transition rule is less strict in the case where $\Theta = 20$, but the values of overpayments have not differed considerably. Considering profitability however results in a much different outcome. The value of the objective function gets much worse, and of course, the absolute overpayment increases as well compared with the financially not balanced models. The difference between the stationary and the multi-period models is more notable. The time of the multi-period model is more crucial here since the 20-period model's result differs more from the outcome of the stationary model than the result of the 40-period model. Even the case wherein $\Theta = 40$ results in a much worse optimal solution than that in the stationary case.

	Without profitability					With profitability					
Premium:	0.018	0.027	0.032	0.041	0.05	0.018	0.027	***0.03***	0.032	0.041	0.05
Stationary	-	5-19	0-4	-	-	-	-	19	0-18	-	-
$\Theta = 20$	-	5-19	0-4	-	-	-	-	-	8-19	1-7	0
$\Theta = 40$	-	4-19	0-3	-	-	-	-	7-19	2-6	0-1	-

Table 6: Premium scales of the realistic models

Table 6 shows the optimal premiums. Each cell shows the classes where the column's premium is present. Therefore, the $5-19$ in the column of 0.027 means that class $5, 6, \ldots, 19$ has the premium 0.027.

When we did not consider the profitability constraint, the optimal premium scale only contained two premiums: 0.027 and 0.032. It means that three risk groups cannot pay their fair price. Therefore, 11% of the policyholders surely overpay and 18% pay undoubtedly less. When the profit constraint was considered for the types with 0.018 and 0.027 expected claim, there was no class that provided a fair payment for them. The premiums were adjusted upward, but there were still up to three different premiums. However, because in this case we considered a metaheuristic, a better solution may exist. In these cases, we got 0.03 as unique premium, which only appeared in the stationary and the $\Theta = 40$ models. In the $\Theta = 20$ model, we did not get any unique premium.

Interestingly, we got fewer premiums than the number of risk groups in every case. The other objective of the BMS is to reduce the moral hazard. In other words, a system that motivates the policyholders to reduce their risks is needed. For this, the designer of the BMS may want more variability on the premiums. One possibility would be to specify a minimal difference between the premiums of each class. However, this would be rather difficult in the joint optimization approach. In this model, we can specify that each type's fair premium should appear in the premium scale. To this end, we have to add the following constraints:

$$\sum_{k=0}^{K+1} \sum_{\ell=1}^{L} O_k^\ell \leq K \tag{25}$$

$$\sum_{k=0}^{K+1} O_k^\ell \geq 1 \quad \forall \ell \in L \tag{26}$$

Premium:	0.018	0.027	0.032	0.041	0.05	T_1	ℓ^1 change	ℓ^2 change
Stationary						-17	1.709	1.151
$\Theta = 20$	19	3-18	2	1	0	-3	1.662	1.131
$\Theta = 40$						-16	1.719	1.172

Table 7: The results when each type has a premium

Table 7 provides the results of the extended models. We only considered the models without profitability constraint because in these cases, we obtained the exact solution. In these models, we got the same premium scales and only the transition rules differed. Again, the case wherein $\Theta = 20$ had a much different transition rule. When the considered time was longer, the model had a similar solution to the stationary model. The column related to ℓ^1 change presents the increase in the objective value if we add constraints (25) and (26). The ℓ^2 column shows the increase if we consider the ℓ^2 norm, i.e., the squared deviations instead of the absolute value of the deviations. In this case, the solution was worse, which means more variation on the premiums was not that good in this case as well.

5.6 Summary

This section introduced a MILP model for the joint optimization of the premiums and transition rules in a BMS. First, we put forth the model with stationary probabilities. Then, we presented a multi-period modification. From the model with multi-period probabilities, it is possible to omit some of the unrealistic assumptions required for the stationary model. Furthermore, with multi-period probabilities, we may determine the initial class. Moreover, in this section, we proposed a modification to optimize the number of classes. We also presented numerical experiment results on the models and a case study using real data.

6 Optimization of a Bonus-Malus System in practice

6.1 Introduction

In the practice of MTPL insurances, insurance companies often use other techniques besides the BMS. Usually, the policyholders' risk groups are calculated via some statistical methods and then part of the premium determined based on this classification. In modeling risks for MTPL insurance, the Generalized linear models (GLM) and Generalized additive models (GAM) are used most frequently (Giancaterino (2016); Kafková and Křivánková (2014); Burka et al. (2021)).

A better estimation is essential for the insurance company, but errors are unavoidable in practice. Therefore, it is also a necessary objective of the estimation method to reduce the classification errors. Hence, the practical application of a BMS is to minimize classification errors caused by unobservable variables.

As we discussed in section 1, with some observable parameters (such as the driver's age, location of the vehicle owner, age of the driving license, type of vehicle, etc.), we may estimate risk groups. However, there may be other unobservable parameters that influence the risk pertaining to each policyholder. Hence, the existence of unobservable parameters may result in some deviations from the estimated risk groups. For decreasing the error related to the estimated and "fair" premium, some tools such as the BMS can be used.

In section 5.5, we assumed that we might not know anything about the observable risk groups' deviation. Hence, we used these groups as parameters for the optimization model. However, with sufficiently large data, it is possible to calculate some kind of deviation from the estimated classifications. For an illustration of the problem, we introduce example 6.1.

Example 6.1. *For the sake of simplicity, let us assume that there is only one significant observable parameter with merely two categories. For example, this variable relates to the type of the location where the policyholder lives. A policyholder may live in a town or the countryside. Statistically, cities are a much riskier place for driving; hence, if the policyholder lives in a town, his/her premium should be higher.*

Let us imagine that the insurance company's statistical model suggests that the people who live in a city have a Poisson claim distribution with a 10% *mean. Those from the countryside have a risk with a* 5% *Poisson distribution. Hence, without a BMS, the city's policyholders' premium would be* 10%*, which would otherwise be* *simply* 5%.

However, presumably, not all policyholders from the city have a 10% *claim probability. With the statistical estimation, we could only evaluate the average values of the risk groups. Hence, the "fair" premium of many policyholders may differ from the estimation of the statistical model. This difference may be caused by some other variables that were not considered in the statistical model.*

Even though it is hard to find the deviation's source, using a BMS can reduce this error. Suppose that a policyholder is below the risk level corresponding to his/her estimated "fair" premium. Thus, in a BMS, the policyholder will have fewer claims; offering them the option to be in a better class than the initial classification of the insurance company. However, some error is still inevitable, e.g., if the policyholder has a smaller expected risk than the lowest premium of the BMS.

If the risk can be explained with an additional observable variable, the statistical model can be improved. It may also happen that the observable parameters of the policyholders are not fully describing the risk. Hence, there are some unobservable parameters such as the level of carefulness that are not considered. Some policyholders are careful and others not that much, regardless off their physical attributes.

Let us assume an unobservable categorical variable – the level of carefulness. We may separate policyholders into two types according to this variable: careful and careless policyholders.

Presumably, the insurance company has some data from the previous years. Hence, the effects of this unobservable variable can be estimated.

Let us assume that a careful policyholder's expected risk is smaller at a 3 *percentage point than the statistical model's estimated value. If the policyholder is careless, then the risk is higher than the observable risk with the same amount.*

Therefore, if the insurance company knows all the information, it classifies four risk groups among the policyholders: the countryside-careful with 2% *expected claim; the city-careful with* 7%*; the countryside-careless with* 8%*, and the city-careless with* 13%*.*

The level of carefulness is unobservable based on the policyholders' physical attributes. Therefore, the insurance company cannot precisely determine the policyholders' real "fair" premium values. Hence, using a tool for estimating it, such as the BMS, is necessary in this case.

We will refer to the two risk groups (city–countryside) as the types of the policyholders. Further, the four groups separated according to the level of carefulness variable as the unobservable risk groups.

In this section, we investigate the importance of using a BMS. In section 4, we assumed that the policyholders only pay the premium of the BMS. However, in practice, other statistical methods are also used. Therefore, in this section, we examine how the BMS can be optimized if we consider the statistical model too.

6.2 Preliminaries

This section explores a setup that is closer to practice. Hence, we once again consider only the number of claims in the BMS. Resultantly, even in the statistical model, we only consider the distribution of having claims and do not focus on its amount.

Let Λ^p denote the personal risk related to the policyholder p. The insurance company wants to set the premium as close to Λ^p as possible. Let π^p represent the premium of the policyholder p. The premium corresponds to the insurance company's estimation of the policyholder's risk ($\hat{\Lambda}^p$).

Furthermore, let $\eta^p \in \mathbb{R}^n$ be the n observable variables of the policyholder p. If the insurance company assumes that there are no unobservable variables or noticeable noise, then it would determine the risk according to equation (27).

$$\hat{\Lambda}^p = f(\eta^p), \tag{27}$$

where $f()$ is a function, usually in practice a GLM or GAM is used. We assume that there may be r unobservable parameters as well ($\rho_p \in \mathbb{R}^r$). A full model refers to a model wherein all of the information can be used. Therefore, the unobservable parameters can also be used as variables in the estimation:

$$\hat{\Lambda}^p = g(\eta^p, \rho^p), \tag{28}$$

where $g()$ is a function, similar to $f()$ but not the same. Of course, this is just a theoretical model because the unobservable variables cannot be used in this way in practice. However, for the sake of comparison, we also consider this model because it is the theoretically best estimation.

In practice, for estimating the unobservable parameters, some statistical methods can be employed. We consider the BMS for handling this problem. In a BMS, the policyholders are classified into classes according to their claim history. Therefore, for optimization, we consider the expected premium of the policyholders. Hence, the premium of policyholder p is $\mathbb{E}(\pi^p)$ when the BMS is applied.

There are several approaches for determining the premium:

- **Statistical model (SM) only:** In this case, a BMS is not utilized. Therefore, the unobservable variables are not considered. We use the model described in (27). We consider this case for the sake of comparison. If the effects of the unobservable parameters are small, considering a BMS besides a SM may not improve the fairness of the premiums that much.

- **BMS only:** The purpose of a BMS is to estimate the risk of the policyholders. Presumably, the risk has observable and unobservable aspects. We cannot consider only the unobservable effects in the BMS as it estimates only the overall risk. Hence, the BMS may distinguish policyholders efficiently; therefore, using an SM is not necessary. This is the model that we introduced in section 4.

- **Both SM and BMS:** In this case, the insurance company uses a SM and BMS jointly to estimate the policyholders' premium. Burka et al. (2021) compared GLM, GAM and machine learning models in the calculation of optimal premiums for MTPL insurance. In their models, the classification of the BMS was a variable. Hence, it means the premiums of the BMS are also determined inside the statistical model. However, as we can jointly optimize the premiums and the transition rules of the BMS, we only consider the premiums of the BMS in our analysis.

  We consider three methods:

  - ***Scaling***: The insurance company optimizes the BMS and SM separately. Then it finds an appropriate scale parameter ($0 \leq \alpha \leq 1$) between them to determine the final premium.

    $$\mathbb{E}(\pi^p) = \alpha SM^p + (1 - \alpha) BM^p, \tag{29}$$

    where BM^p denotes the expected premium of the BMS and SM^p represents the premium of the statistical model. For determining the α value, we presume that the insurance company has made assumptions about the unobservable parameters. Therefore, there are observable types and unobservable risk groups. The separation of the types is the result of the SM. The BMS represents the unobservable risks. Hence, we use an α to weight the difference between the $\mathbb{E}(\pi^p)$ and the assumed unobservable risk group.

  - ***Merging***: In this method, the expected premium comes from the BMS alone. Therefore, we consider the types, and consequently the SM, in the premiums of the classes. Therefore, the payment by two policyholders belonging to the same class may differ if their observable parameters are not the same.

Let us assume that we estimated H types of policyholders with the SM with each having λ^h $(h = 1, \ldots, H)$ expected claims. Therefore, the premiums for a policyholder of type h, who is classified into class k of the BMS is determined using the equation:

$$\pi_k^h = \lambda^h + BM_k, \tag{30}$$

where BM_k is considered a deviation from the expected value. Thus, the premium of class k can be negative as well.

Other approaches can also be adopted for merging the SM and BMS. However, we only investigated this additive approach because the MILP model we introduced in section 5.1 can be easily adjusted to consider the SM result.

In this case, we modify the default premium to be the result of the statistical model. Let SM^i denote the statistical model's result, calculated using only the observable parameters for the unobservable risk group i. Therefore, the default premium differs for each type of policyholder, and the constraints (MILP3.10) and (MILP3.11) changes into the following:

$$SM^i c_k^i + \sum_{\ell=1}^{I-1} o_k^{\ell,i} + g_k^i \geq \lambda^i c_k^i \quad \forall i, k \tag{MILP1.18”}$$

$$SM^i c_k^i + \sum_{\ell=1}^{I-1} o_k^{\ell,i} - g_k^i \leq \lambda^i c_k^i \quad \forall ik \tag{MILP1.19”}$$

However, with this modification, the premiums can be only higher than the default premiums. This change is not sufficient because the policyholders less riskier than the result of the SM would not have the option to pay their fair premium.

Therefore, we have to enable negative premiums in the BMS. Consequently, the variables $o_k^{\ell,i}$ should be able to take negative values as well. Therefore, in the optimization of the premiums, we use both positive and negative layers. Let set $\mathcal{L}$ denote the possible premium changes and let ε^ℓ represent the element ℓ in $\mathcal{L}$. We may construct $\mathcal{L}$ with negative and positive elements. Hence, by using the constraints of section 5.1, we can have negative and positive premiums of the BMS in the optimization.

For example, if the unobservable parameters cause a deviation of 3% from the SM value, then we set $\mathcal{L} = [-0.03; 0.03]$. With all these modifications, we can alter the BM premiums according to each fair value. If the effects of the unobservable risks are not that simplistic, other elements can be added to $\mathcal{L}$. For example, there are two types, with 0.02 and 0.03 expected claim. Suppose we can separate the 0.02 type of policyholders into 0.01 and 0.04 unobservable risk groups and the other type into 0.01 and 0.06. We can then get all of the fair premiums with $\mathcal{L} = [-0.02; -0.01; 0.02; 0.06]$.

We may consider the merging method in another way: the result of the BMS is considered in the SM. In this case, the BM premium, or the class, can be regarded as an observable variable in the estimation of the SM premium. We did not adopt this method because it requires a data-driven approach instead of a theoretical comparison.

– ***Independent***: In this case, we optimize a BMS independently for each observable risk group. Therefore, in example 6.1, we optimize two models. Hence, policyholders from the city may have different transition rules as opposed to the other policyholders. Also, the premiums may differ because the assumed unobservable risks determine the possible changes in the premiums.

This method would be hard to implement in a practical situation. Yet, we added it to the analysis because according to the results of Cooper and Hayes (1987), this method would give the best result in theory.

6.3 Computational analysis

6.3.1 Example

To compare the different methods, we considered the example 6.1. For the comparison of the models, we calculated the sum of the absolute differences between the fair and the expected premium (AD):

$$AD = \sum_{i=1}^{I} \phi^i |\mathbb{E}\pi^i - \Lambda^i|, \tag{31}$$

where I denotes the number of risk groups, considering all variables. Therefore in example 6.1 $I = 4$. The ϕ^i represents the ratio of risk group i. In all the experiments, we considered equally sized risk groups. The $\mathbb{E}\pi^i$ is the calculated expected premium, and the Λ^i is the fair premium, i.e., the expected claims.

- **Only SM:** In this case, the unobservable parameters are not considered. Thus, those who live in a city would pay 10%, and the rest would pay 5%. Hence, $AD = 3\%$.

- **Only the BMS:** In this case, we use all of the calculated risk groups to determine the optimal BMS. We assume that the insurance company knows all the information but cannot distinguish the policyholders. Therefore, the company knows all of the unobservable risk groups' mean and claim distributions. We considered a 20-class and 30-class BMS. We then optimized both systems according to the stationary model, introduced in section 5.1. In both cases, the claimless transition rules are one positive step. In the 20-class BMS, the downgrade is [6,10,19] classes, respectively for 1, 2, and 3 or more claims. $AD = 2.37\%$, which is better than the only SM's AD. In the 30-classes BMS, the results are even better, with $AD = 1.26\%$. The downgrades in the event of claims are [12,18,23] respectively for 1, 2, and 3 or more claims.

- **Both SM and BMS:**

  - ***scaling***: In the BMS, we cannot consider only the unobservable parameters without the observable ones. Therefore if we consider the BMS and the SM with equal weights, the observable parameters may be weighted higher than its actual effect. Therefore, we calculated the solution with multiple α values and chose the best result. Table 8 presents the AD values for both 30- and 20-class BM systems, with different α values.

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
AD_{20}	**2.37%**	2.36%	2.36%	2.35%	2.36%	2.37%	2.37%	2.38%	2.39%	2.39%	2.40%
AD_{30}	**1.26%**	1.26%	1.26%	1.27%	1.28%	1.30%	1.32%	1.34%	1.35%	1.37%	1.39%

Table 8: The AD values of the scaling BMS, with different α values.

  According to Table 8, BMS only has such an accurate solution so that it is not worth mixing the BMS with the SM. Therefore, the best α is 0 in both BMSs. When we determined the α more precisely, we got $\alpha = 0.02493$ for the 20-class BMS. However, we could not find a better α than 0 for the 30-class BMS.

  - ***merging***: When we considered the 20-class BMS, with the merging method, the AD value was 1.87%. That means, in a 20-class BMS, the merging method resulted in a better system than the scaling method. The transition rules were [8,19,19] class reduction in the event of claims.

    We designed the joint optimization so that the BMS premiums can be either 3% or −3%. Interestingly, only the highest class got a negative BMS premium. In the only BMS case, the premiums can only take the values of the fair premium. In that case, the lowest premium was 7%. Therefore, the risk group with a 2% expected claim probability cannot get its fair premium at all. In the merging method, it however can when it is classified into the highest class. Thus, the better result of the merging method.

When the number of classes increased to 30, the results changed as well. The merging method led to a 1.69% *AD* value, which is slightly higher than that of the scaling method (1.26%). The change in order is because of a 2% premium that appears in the scaling method's premium scale.

The improvement from 20 classes to 30 is not that great in the merging method, despite the transition rule now takes three different values in the event of claims ([11,23,29]).

– ***independent***: In the independent BMS case, the total AD value was 1.71% for the 20-class system and 1.41% when we considered 30 classes. Therefore, in the case with 20 classes, the independent BMS was the most effective. However, with 30 classes, it was not the best one. In that case, BMS only has such an effective outcome so that using any combination with the SM would decrease the sorting effectiveness.

6.3.2 Decreasing the effect of the unobservable variable

In the example 6.1, the deviation caused by the unobservable parameter was considerably high. Thus, based on the results, the BMS worked extremely well. In a real-world situation, the unobservable variables' effect may be much smaller than what we considered in the example. Therefore, we analyzed different levels of deviations. Table 9 presents the *AD* values of the cases with 3%, 2%, 1%, 0.5%, and 0.1% deviation.

Deviation	3%	2%	1%	0.5%	0.1%
only SM	3%	2%	1%	0.5%	0.1%
only 20-BMS	2.37%	2.06%	1.91%	1.85%	1.71%
scaling 20-BMS (α)	2.35% (0.03)	1.74% (0.26)	0.90% (0.58)	0.47% (0.76)	0.1% (0.95)
merging 20-BMS	1.87%	1.49%	0.87%	0.47%	0.1%
independent 20-BMS	1.72%	1.28%	0.86%	0.46%	0.1%
only 30-BMS	**1.26%**	1.36%	1.70%	1.60%	1.45%
scaling 30-BMS (α)	**1.26% (0)**	1.28% (0.24)	0.86% (0.52)	0.46% (0.76)	0.1% (0.95)
merging 30-BMS	1.69%	1.40%	0.85%	0.46%	0.1%
independent 30-BMS	1.41%	**1.16%**	**0.81%**	**0.45%**	**0.1%**

Table 9: The *AD* values with different deviations

We denoted the best result in boldface in Table 9. As the deviation gets smaller, the BMS only case gets worse, and the SM only case gets better. In every case, one of the 30-class BMS resulted in the smallest *AD*. Hence, an increased number of classes improve the sorting efficiency of the system.

In the cells of the scaling method, we also denoted the α value. As the deviation gets smaller, the α value increases. It means that the necessity for a BMS decreases with the deviation. However, even with a 0.1% deviation, the α is less than one. Thus, a BMS may improve the estimation in that case as well.

When we considered a 3% deviation, the SM only case was the worst, and using a BMS is highly recommended. Therefore, we can conclude that the BMS is more useful if the unobservable parameters play an essential role in the characteristic of the risk.

When we considered both the SM and BMS, the independent method led to the best outcome except for the 3% case. As the deviation gets smaller, the difference between the methods' results also gets smaller.

6.3.3 Unfairness between policyholders with an equal expected claim but with different observable parameters

As we saw in the previous example, the independent method seems to be the most effective one. Although having different BMSs for each observable risk group contribute to the best sorting capability, it may lead to an unfair outcome. Let us consider two types with 2% and 4% expected claims. Moreover, let the deviation be 1%. Therefore, by including the unobservable risks, we have three different unobservable risk groups. There are risk groups with 1% and 5% expected claims with a quarter of the policyholders from the two types. Furthermore, we have a risk group with a 3% claim probability. Half of the policyholders from this group are classified into the types with 2% claim probability, and the other half is put into the 4% one.

We examined 30-class BMSs using every method. Table 10 provides the *AD* values.

	SM	BMS	scaling	merging	independent
AD	1.00%	0.97%	0.91%	0.74%	0.72%
RG(2,1)	2.00%	3.02%	2.66%	1.40%	1.48%
RG(2,3)	2.00%	3.15%	2.78%	2.04%	2.14%
RG(4,3)	4.00%	3.15%	2.82%	4.04%	3.82%
RG(4,5)	4.00%	3.43%	3.07%	4.46%	4.27%
RG(4,3)/RG(2,3)	2	1	1.01	1.98	1.79

Table 10: *AD* values of the methods and the expected payment of each risk group (RG(2,1) refers to the type with 2% observable risk but with 1% risk considering the unobservable variables.

In this case, once again, the independent method provides the best *AD* value. The worst one is noted in the SM only case. Besides the *AD* values, Table 10 presents the expected payment of each risk groups in every solution. We denoted

$RG(i,j)$ as the payment of a risk group, with observable risk i and unobservable risk j (in percentage).

The risk groups $RG(2,3)$ and $RG(4,3)$ have the same fair premium. Hence, these policyholders should pay a similar premium. However, because they are different types of policyholders, they pay the same amount only in the BMS. The RG(4,3)/RG(2,3) row shows how much the policyholders from the RG(4,3) pay more than those from the RG(2,3). According to the AD value, the independent and the merging methods seem to be the best ones. However, the difference between the payment of these groups is much higher compared with the scaling method.

Optimal premium-scales

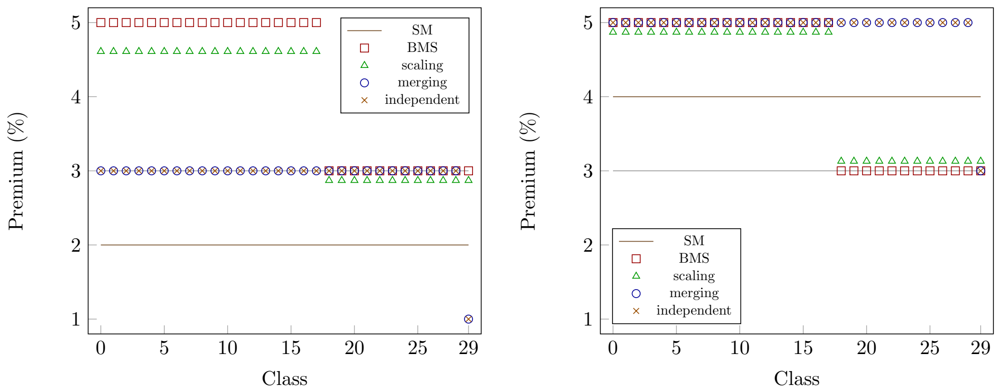

Figure 24: Payment of the $RG(2,3)$ and $RG(4,3)$ in each class with every method

Figure 24 shows the premiums of the risk group $RG(2,3)$ and $RG(4,3)$, of each case. The SM payment is also displayed in the graph with a horizontal line. In the BMS only case, the payment is the same for both groups. Two different premiums exist in this system: the 5% for class 0–17 and 3% for every other class. The scaling method brings the payments a bit closer to the observable level.

The merging and the independent methods have the same premium scale. The only difference between their results is the transition rules. In the merging case, every policyholder has the transition rule [21,29,29] respective to the number of claims. In the independent case, the $RG(2,3)$ group has [27,29,29] transition rules. For the other groups, it is a bit less strict, [14,29,29] respectively to the claims.

Therefore, the $RG(4,3)$ risk group pays more or at least an equal premium compared to their fair premium in all results. Furthermore, the $RG(2,3)$ may pay less than 3%.

6.4 Case study on Hungarian data

6.4.1 By making assumption on the unobservable parameters

Estimating the observable risks

We used the 2008–2009 vehicle insurance data of an insurance company that operates in Hungary for using realistic parameters in the optimization. For the observable risk groups, we employed the statistical classification tree method. For the classification variables, we did not consider some variables. We excluded the sex of the policyholders because insurance companies in the European Union cannot use this variable in the estimation [1]. Also, we took out the previous years' insurance-related variables because they also may not be fully available for the insurance company. A policyholder may be a beginner driver or it can be a new contract for this insurance company. Hence, we used location-based variables, age, and vehicle's age variables to separate the policyholders. Presumably, any insurance company can ask the policyholders about these variables. Hence, we consider the leaves of the classification tree as the observable types of policyholders. With this method, we could divide the policyholders into four parts.

Estimating the unobservable risks

We could not observe any trends of deviation from the separated types because we only had two years of insurance data. Therefore, we assumed that the leftover variables also influence the policyholders' risk. Thus, with those variables that we did not consider in the estimation of the types, we further divided the policyholders. We again utilized classification trees for making the unobservable part, but this time separately on each type. In the estimation, we used the insurance-related variables (previous years' payment and BMS classification) and the sex of the policyholders.

We could divide each type into two unobservable parts. Therefore, we determined eight different unobservable risk groups. The mean of the observable types and unobservable risk groups and their ratio are given in Table 11.

Observable	1.8%		2.7%		2.9%		3.9%	
Unobservable	0.5%	3.1%	2.3%	6.1%	1.6%	5.4%	3.0%	8.5%
Ratio	6%	6%	4%	1%	37%	15%	26%	5%

Table 11: The means of the observable and unobservable risk groups, as well as their ratio

[1] https://ec.europa.eu/commission/presscorner/detail/en/IP_12_1430

Results of methods

We optimized four BMSs with 10, 15, 20, and 30 classes. Moreover, we considered the result of the observable classification tree as the output of the SM. Besides the SM and the BMS, we calculated the merging, scaling, and independent methods. Table 12 presents the *AD* values of all the considered models.

class	10	15	20	30
only BMS	1.34%	1.29%	1.25%	1.16%
scaling BMS (α)	1.17% (0.54)	1.16% (0.52)	1.14% (0.49)	1.1% (0.48)
merging BMS	1.16%	1.15%	1.1%	1.01%
independent BMS	0.99%	0.95%	0.91%	0.81%

Table 12: The *AD* values of the models

Using only the SM (in this case, the observable classification tree) results in an *AD* of 1.53%. As Table 12 shows, using the BMS with any method increased the premiums' overall fairness level. Even when we only considered the BMS with ten classes, the AD was 1.34%. Also, it is worth noting that as we increased the BMS size, the expected payments got closer to the fair premiums. The consideration of both the SM and BMS produced the most beneficial outcome. The result of the independent method shows the real limits of the BMS. Theoretically, this method gives the best feasible solution. Hence, presumably, we cannot decrease the difference between the "fair" and the real expected premium under 0.8% in a 30-class BMS.

The merging and scaling methods had similar results in smaller BMSs. In a 30-class BMS, the merging method had a slightly better result. The absolute deviation only decreased below 1% with the independent method. The independent method's separation was so superior that a 10-class BMS of this method was better than any other method's 30-class BMS.

Difference between the expected payment and claims of each policyholder

Figure 25 depicts the distance from the model's premium and each risk group's fair premium. The x-axis shows the risk groups' parameter, and we marked their payment of each model on this graph. When the payment is on the diagonal red line, the risk group's expected payment is equal to its fair premium.

In general, the policyholders whose risk is more than 3% pay less than their "fair" premium in every model. The scheme is quite similar to the results we presented in section 5.4 because all of these risk groups have a higher risk than what we can observe. Even though it seems the payments of these risk groups are far away from the diagonal, they only constitute 27% of the policyholders. The blue area represents the ratio of the risk groups. Hence the merging method did not give the worst solution, despite being farthest from the diagonal.

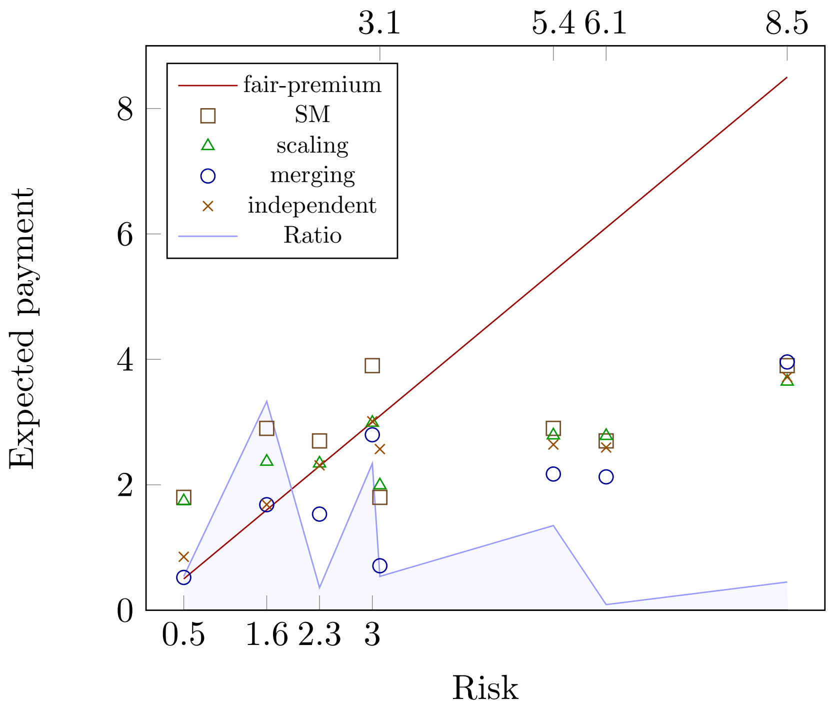

Figure 25: Payments of the risk groups

Conclusion

Consequently, the risk groups with smaller probabilities define the result because of their sizes. Moreover, in every smaller-probability risk groups, the independent method's premium is very close to the diagonal. The SM's premium is the farthest from the diagonal in these risk groups. Thus, using a BMS mostly reduces the unfairness to the less risky policyholders. Interestingly, none of the policyholders pays more than their fair premium in the merging method.

Overall, using the BMS with any method could decrease the difference between the expected payment and the fair premium. However, the effect was not large. The independent method provided the best results, followed by the merging method.

6.4.2 Without any assumption on the unobservable parameters

In the previous examples and in this case study, we used relatively few unobservable risk groups. We created two unobservable risk groups from each type to calculate the optimal solution within a reasonable time. More risk groups may significantly increase the running time, thus calculating the optimal BMS, even with eight risk groups took 155 hours on a desktop computer.

However, in the previous example, the optimal solution of the BMS was built around the two largest risk group. As Figure 25 illustrates, these risk groups' payment is almost "fair" in the merging and the independent method. We also wanted to calculate the solution with more unobservable risk groups. However, with the classification tree method, we could not categorize the policyholders into nearly equal-sized groups.

Estimating the unobservable risks

To create equally sized unobservable risk groups, we estimated the policyholders' claim probabilities with a logit model. We used all available variables for the estimation (age, sex, location of the policyholder, the previous year's insurance premiums, and the vehicles' type). We will refer to these probabilities as the "real" probabilities of the policyholders because all of their parameters were utilized in the estimation. The distribution of the logit model's probabilities is presented in Figure 26.

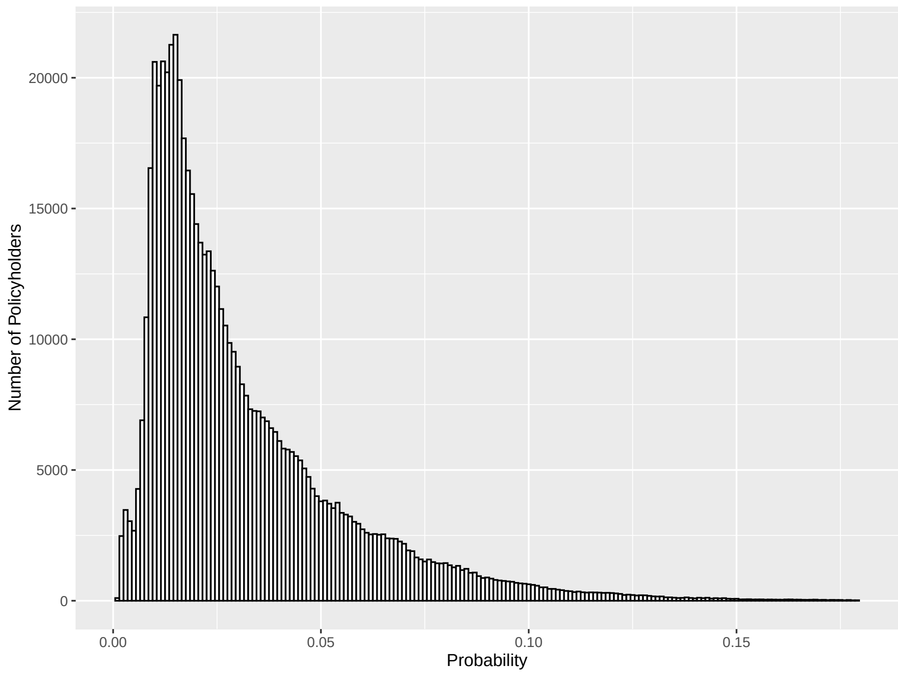

Figure 26: Distribution of the policyholders by their "real" claim probabilities.

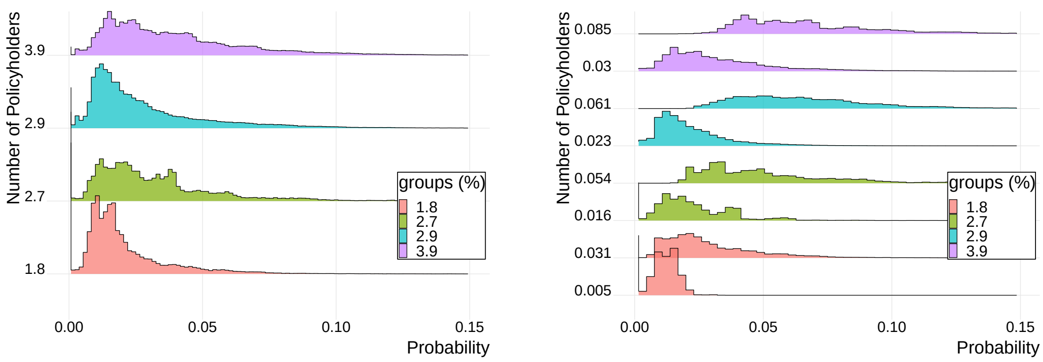

Figure 27: Distribution of the "real' claim probabilities by the observable types (left) and the unobservable risk groups (right)

Figure 27 illustrates the "real" claim probabilities classified by the types and the unobservable risk groups that we used previously. On the left, the observable types are presented. The 2.7% and 3.9% types of policyholders' distribution seems flatter compared with the other two types' distribution. It can be noted that the unobservable risk groups can be separated into smaller segments (on the right). Some risk groups' distribution has a larger deviation. Hence, considering more unobservable risk groups in the optimization may lead to a better estimation of the fair premiums.

Therefore, we created 100 equally sized unobservable risk groups, according to the "real" probabilities, with the consideration of the observable risk groups. Hence, we separated the types into 1%-sized parts. For example, from those types of policyholders, whose observable risk is 1.8%, we created 12 equally sized risk groups.

Calculation of optimal solutions

As there are many risk groups in this case, the computation of the optimal solution would be a very long process. Therefore, we used the iterative method, introduced in section 5.4.8 to approximate the optimal solution. First, we calculate the optimal transition rules with a fixed premium. Then, we find the optimal premiums to these transition rules, which we consider as parameters. Next, we use the optimal premiums of this model as parameters and re-optimize the transition rules. We continue this until we cannot improve the objective function further. In the first fixed premiums, we use a proportional premium scale.

	Deviation	3%	2%	1%	0.5%	0.1%
20 class	Time	15.46%	47.58%	38.72%	94.94%	87.65%
	Objective	102.58%	100.00%	100.05%	100.00%	100.00%
30 class	Time	11.08%	1.62%	6.57%	7.80%	9.42%
	Objective	100.00%	100.18%	100.65%	100.38%	100.45%

Table 13: Differences between the objective value and the running time of the exact solution and the result of the iterative algorithm

Table 13 presents the time and the objective value of the iterative method compared with the exact solution of the examples, put forth in section 6.3.1. We considered the 20- and 30-class BMSs. The table shows that the time of the iteration was decreased significantly with the iterative optimization. The reduction was more noticeable in the 30-class systems. Nevertheless, even in the case of the 20-class BMSs, it was still noticeable. The objective value, however, did not increase that much. The highest increase was observed in the 3% 20-class case, but even in this case, it was less than 3%. Therefore, we assume that the iterative optimization's optimal solution may not be too far away from the global optimum in this model.

The problem with this iterative optimization is that we can only determine a BMS independently from the SM. Hence, the merging method cannot be utlized with this type of optimization. Since we found that the merging method's solution was usually between that of the independent and scaling method, we focused only on the latter two.

Results of methods

With the 100 unobservable risk groups, we found that the SM model's AP was only 0.56%. The value is much smaller than in the previous study. The reason behind the decrease is that when we consider more risk groups, the policyholders with extreme probabilities have their own group. Hence, the deviation from the mean of the observable risk group is much smaller.

With the 30-class BMS only case, we got a 0.7% AP value. It was worse than the SM's AP, but with the scaling-method, we could get a slightly better AP of 0.55%. In the scaling method, the BMS was considered with a weight $\alpha = 20\%$. The independent-method resulted in the best AP value. Nevertheless, even in this case, it was only 0.52%.

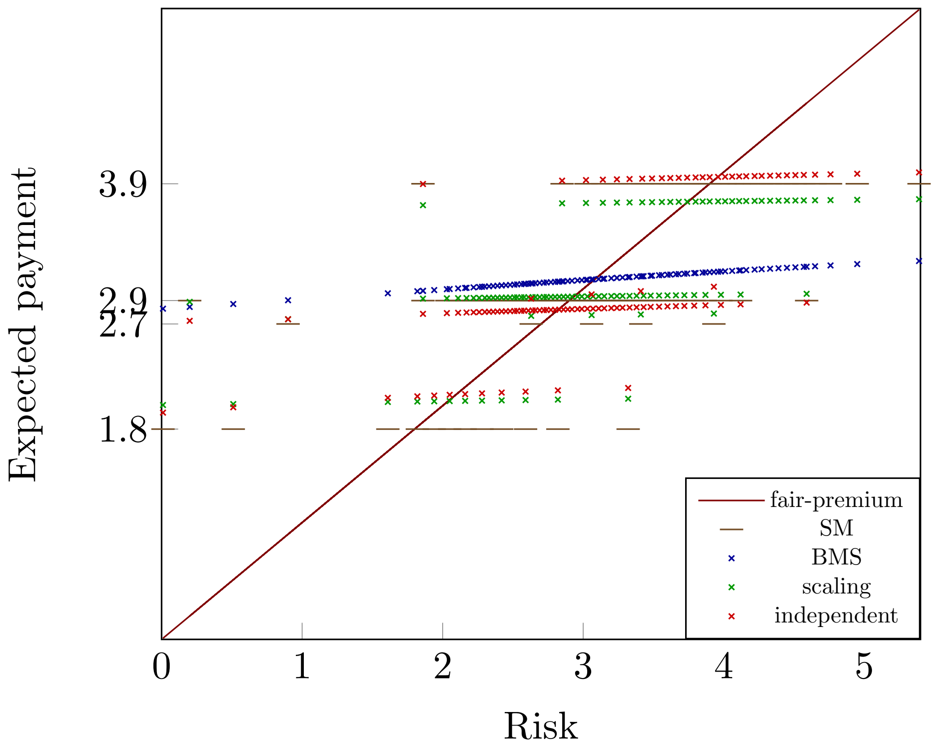

Figure 28: Payments of the risk groups

Difference between the expected payment and the claims of each policy-holder

Figure 28 shows how far the unobservable risk groups payment was from their "fair" premium. The "fair" premium is the diagonal red line in this figure. The SM premiums are represented as flat lines at the observable risks that we marked on the y-axis. The blue line represents the payments of the BMS. It is noticeable that in the BMS, the payments almost linearly increase. Compared with the SM model, the BMS has one line, but it tries to 'rotate' into the diagonal. The SM payments have a stepwise increase; hence, the risk groups farthest from the diagonal differ from the two methods. In the BMS, the risk groups whose payment differs the most from their "fair" premium are those far from the mean of all policyholders' risk. In the SM, the policyholders who have the least fair payment are far away from the mean of their observable parameter.

Therefore, there are multiple lines in the methods where we consider both SM and BMS, but none is flat. The steepness differs for each observable risk group, but neither the scaling method nor the independent method can rotate the lines much.

Conclusion
Overall, as we separate more unobservable risk groups, the effect of the BMS keeps decreasing. In the first study, the improvement was noticeable, but when we considered 100 risk groups, even the independent method could not improve the classification significantly. However, there was a slight improvement, and even in the scaling method, the BMS was considered with 20% weight. Furthermore, there may exist a better solution than what we found with the iterative optimization.

To make an insurance contract more "fair" for the policyholders, considering the adverse selection problem is essential. The BMS, even in extreme cases, seems to be able to decrease the welfare loss, even though its effect is not enormous.

6.5 Summary

In this section, we investigated how the optimization model can be used in practical situations. In section 3.1, we assumed that the payment of the policyholders only depends on the BMS' premium. However, the insurance company may estimate part of the premium using statistical methods (SM) in practice. This section further compared the methods on how the BMS and SM can be applied together. We considered the *scaling* method, in which the insurance company optimizes the BMS and SM separately. The final premium is determined by the weighted sum of the two premiums. We also examined the *Merging* method. In this case, the transition rules differ for each observable risk groups. Moreover, we investigated the *independent* method, where the whole BMS differs for each observable risk group.

Overall, the *independent* method was the most efficient. However, in general, using the BMS and statistical method jointly almost always resulted in a better solution than considering only one.

We also presented a case study based on realistic data. We proposed two approaches for determining the unobservable parameters. We found that the optimized BMS's effectiveness depends on the risk groups' parameters. Hence, if the applied statistical method is accurate, the BMS cannot improve the solution greatly. However, in both realistic models, the BMS could improve the results.

7 The transition rules based on the size of claims

In practice – most often – the transition rules are based on the number of claims. Therefore, the size of the claims does not affect the transition rules.

This is reasonable since empirical studies suggest that 'good' and 'bad' policyholders differ more in terms of the probability of the number of claims than in amount (assuming there is at least one claim). Despite the fact that differences in the number of claims are more significant than differences in the claim amounts, we can observe deviations in the (conditional) amount of claims as well.

Because, in practice, the classification in the BMS depends only on the number of claims, the literature on claim amount–dependent BMS is limited. Pinquet (1997) and Frangos and Vrontos (2001) use a Bayesian model to analyze a rating system where the cost of claims (considered as the size of the claim) is also considered. The BMS premiums are used to calculate the final premiums in these articles; thus, the claim amount is not factored in the classification.

Bonsdorff (2005) investigates the Markov chain properties of a BMS, where both the claim number and amount are considered in the transition rules. In Bonsdorff (2008), the analysis is extended to the optimization of the premiums in such a system. In the article, an algorithm to find the optimal solution of a BMS is also proposed. The algorithm focuses on the optimization of premiums and calculates the stationary probabilities with simulation.

In Ágoston et al. (2019), we investigated a case in which we considered a BMS where the transition rules depend only on the size of claims. We introduced a method to jointly optimize the premiums and transition rules of such a system. This section summarizes the results of this article and extends it with a more realistic model.

7.1 Preliminaries

Some parameters and assumptions are the same for this model as in the model mentioned in section 4. Among the policyholders, there are I disjoint risk groups. We denote ϕ^i as the ratio of the group i among all of the policyholders. The insurance company knows the size and distributions of each group but cannot distinguish them. Hence, the insurer does not know which group a specific policyholder belongs to. The (aggregate) claim amount is described with a random variable L^i for group i, which differs in each risk group.

We consider a BM system with $K+1$ classes, but the transition rules depend on the claim amount instead of the number. Class 0 is the worst one (i.e., it has the highest premium) and class K is the best one (with the lowest premium). We denote the premium of class k with π_k. We then assume that the payment of the policyholders only depends on the BMS premium.

To jointly optimize the premiums and re-classifications based on the claim amount, the non-linear model would look as follows:

$$\min \sum_{i=1}^{I} \sum_{k=0}^{K} \phi^i g_k^i \tag{NLP1.obj}$$

Subject to

$$\pi_k c_k^i + g_k^i \geq L^i c_k^i \qquad \forall i, k \tag{NLP1.1}$$

$$\pi_k c_k^i - g_k^i \leq L^i c_k^i \qquad \forall i, k \tag{NLP1.2}$$

$$\pi_{k-1} \geq \pi_k \qquad k = 1, \ldots, K \tag{NLP1.3}$$

$$\sum_{k=0}^{K} c_k^i = \phi^i \qquad \forall i \tag{NLP1.4}$$

$$c_k^i = \sum_{j=0}^{K} c_j^i p_{j,k}^i \qquad \forall i, k \tag{NLP1.5}$$

$$\ell_h^k \geq \ell_{h+1}^k + \zeta \qquad h = 1, \ldots K-1 \tag{NLP1.6}$$

$$\begin{aligned}
p_{k,h}^i &= 1 - F_i(\ell_0^k) && \forall i, k; h = 0 \\
p_{k,h}^i &= F_i(\ell_{h-1}^k) - F_i(\ell_h^k) && \forall i, k; h = 1, \ldots, K-1 \\
p_{k,h}^i &= F_i(\ell_h^k) && \forall i, k; h = K
\end{aligned} \tag{NLP1.7}$$

$$\begin{aligned}
\pi_k &\geq 0 && \forall k \\
g_k^i, c_k^i &\geq 0 && \forall k, i \\
\ell_h^k &\geq 0 && \forall k, h = 1, \ldots, K \\
1 \geq p_{k,h}^i &\geq 0 && \forall k, i, h = 0, \ldots, K
\end{aligned}$$

Here, $p_{k,h}^i$ represents the transition probabilities of the type i policyholders from class k to class h. This value depends on the transition rules, which are, in this case, determined by the claim amount.

To consider the transition rules based on the claim amount, we introduce K breakpoint variables for every class k: $\ell_1^k > \ell_2^k > \cdots > \ell_K^k$. We have to presume a strict inequality between the breakpoints, otherwise, the Markov chain of the classification may not be regular. Hence, we introduce a positive ζ auxiliary parameter to ensure the strict monotony of the breakpoints with constraint (NLP1.6).

The transition rules are based on these breakpoints: if a policyholder is assigned to class k and its claim amount is between ℓ_h^k and ℓ_{h+1}^k, the policyholder will be transitioned to class h in the next period. If the policyholder's claim amount is higher than ℓ_1^k, he/she gets into class 0; if it is less than ℓ_K^k, then the policyholder gets into class K.

Constraint NLP1.7 connects the transition probabilities to the claim amount distribution. $F_i(\ell_h^k)$ denotes the cumulative distribution function (CDF) of the claim amounts for the type i policyholders; hence, $F_i(\ell_h^k) = \mathbb{P}(x < \ell_h^k)$, where x represents a claim amount.

Note that (NLP1.1), (NLP1.2) and (NLP1.5) are non-linear constraints. Moreover, we assume that the CDF is nonlinear.

In this case, for the transition rules, we have to find K^2+K optimal breakpoints. As section 2.1 mentions, usually there are more than 15 classes in practice. Hence, the number of optimal breakpoints can be quite large.

We can reduce the number of breakpoints if they do not differ in the classes. Accordingly, we define $2K+1$ breakpoints $\ell_{-K} > \ell_{-(K-1)} > \cdots > \ell^{-1} > \ell_0 > \ell_1 > \cdots > \ell_K$.

In this case, if the claim amount is between ℓ_h and ℓ_{h+1}, the policyholder moves h classes upward (if $h < 0$, it will be a downward move). Surely, the policyholders cannot move higher than class K or lower than class 0; thus, in such cases, the policyholder will get to (or remain in) class K or class 0 respectively. For instance, suppose that a policyholder is currently in class 2, and they should move three classes downward, meaning the policyholder should go to (the non-existing) class -1. In this case, the policyholder will be classified into class 0, so the total number of downward steps will only be two.
This approach is equivalent to the unified transition rules of the number of claims, introduced in 4.3. Therefore, each class has the same transition rule. However, instead of the number of claims, claim amount intervals determine the classification of the next period.

In the second approach, we can further reduce the number of breakpoints. We can consider breakpoints $\ell_{-D} > \ell_{-(D-1)} > \cdots > \ell^{-1} > \ell_0 > \ell_1 > \cdots > \ell_U$ with $U, D < K$. Thus, the policyholder can move downward at most D classes and upward at most U classes.

In both of the previously shown approaches, the policyholder can move upward and downward one class, with positive probability. Hence, if we consider the BMS classification process as a Markov chain, it will have the irreducibility property.

Besides, because of the finite number of classes associated with these transition rules, the Markov chain is aperiodic as well. Thus, the policyholders' classification will be a regular Markov chain, as in the previously researched models. Therefore, there exist unique $c_0^i, ..., c_K^i$ stationary probabilities.

7.2 Optimization process

The goal of the optimization is to determine the optimal breakpoints and premiums. The only purpose of the BMS in this model is to decrease the welfare loss caused by adverse selection.

The perfect outcome would entail each policy holder paying their expected claim amount value ($\mathbb{E}(L^i)$). However, in the investigated BM system, this target is unreachable. Therefore, our goal is to get as close to the perfect situation as possible.

The optimization problem is non-linear. When transition rules depend on the number of claims, we could linearize the model by introducing binary variables for each possibility of the transition rule. If the claim amount determines the transition rules, this approach is inadequate since there are too many possibilities. In this case, we would need to introduce binary variables for each possible interval. Hence, we approximate the optimal solution.

With fixed breakpoints and an assumption about the policyholders' claim amount distribution, we can calculate the stationary probabilities c_k^i.

After determining the stationary probabilities, we can calculate the optimal premiums with the LP model presented in section 4.1. We also consider the profitability constraint in the model.

To solve this linear programming model, we have to compute the stationary probabilities that depend on the breakpoints. We (randomly) generate these breakpoints and then optimize the LP with the calculated stationary probabilities.

Let us assume a BM system with two classes. We use the second approach for the transition rules; thus, we have to determine only two breakpoints. We assume that there are two equally sized groups among the policyholders.

Figure 29 shows the values of the objective function of the LP model discussed in section 4.1 with changing breakpoints. For the graph on the left, we assumed 6200- and 9300-exponential distributions and for the graph on the right side, these results come from a 3500- and 7400-exponential distributions. In both cases, the objective function's value gives a non-convex, smooth surface with local minima.

If we have more classes, then the dimension of the optimization problem increases quadratically. If we use a method like the Monte Carlo method, we will need a very large number of random points, which is not fortunate.

Using a classical optimization tool, such as the Newton–Raphson method, is also

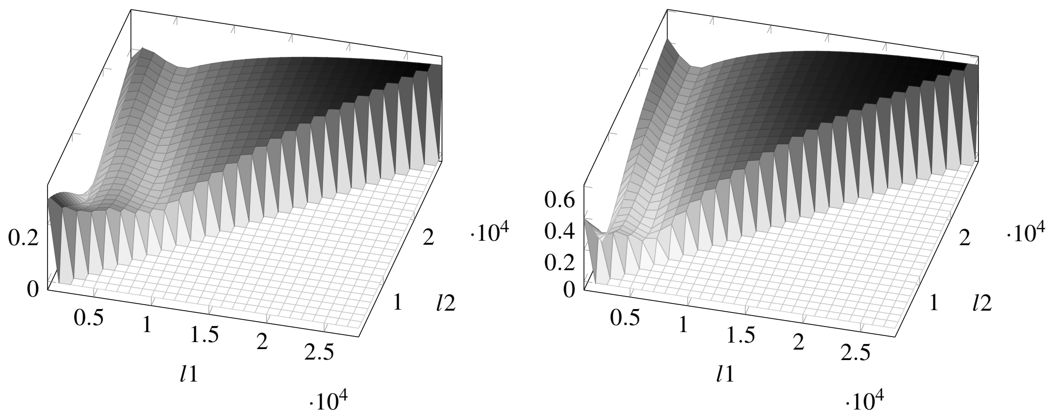

Figure 29: The expected value of absolute deviation in the case of two groups, depending on breakpoints

not a suitable solution because we expect multiple local optima, and the surface is not convex.

An initial solution should be determined to optimize the model with the Newton–Raphson method. However, in this case, it is not simple to give such an initial solution. Although it is possible to approximate a good initial solution and then improve it with the Newton–Raphson method, we cannot be sure that the Jacobian matrix of the conditions is invertible since it depends on the claim amount distribution.

Furthermore, as Figure 29 shows, the surface is non-convex, even in this very simple example. If the size of the problem is more extensive (with more classes and risk groups), the optimal solution may not be calculated within a reasonable time using classical optimization methods.

Due to these reasons, we decided to only approximate the optimal solution of this model. We assumed that by using a metaheuristic, a result close to the optimal solution could be found.

Although it may be possible to determine the exact solution with classical optimization tools, such as the Newton–Raphson method, we did not investigate it, even in smaller instances. This was because we think the results would not be much better, but a classical optimization tool may not be generally applicable.

Therefore, we use simulated annealing, which is more suitable for problems such as these.

7.2.1 Simulated annealing

Simulated annealing is a very simple heuristical search algorithm (see for instance Datta et al. (2019)). It first appeared in Kirkpatrick et al. (1983). It starts by considering a random solution as the actual solution: s_{act}. In each iteration, the algorithm examines a neighbor of s_{act}. If the chosen neighbor is more optimal regarding the objective function, we consider that the new s_{act}. In the following

s_{act} = Random solution
while *temperature > min_temperature* **do**
  for *t in Time* **do**
    s_n = neighbour(s_{act}) **if** $s_n > s_{act}$ **then**
      if $e^{(-\frac{\delta}{temperature})} > random(0,1)$ **then**
        $s_{act} = s_n$
      end
    else
      $s_{act} = s_n$
    end
  end
  decrease *temperature*
end

Figure 30: Simulated annealing (Kirkpatrick et al. (1983))

iterations, we examine the neighbors of the new actual solutions.

However, to avoid local extremes of the objective function, we can also change s_{act} if the Boltzmann condition is accepted: $e^{(-\frac{\delta}{T})} > random(0,1)$, where δ is the difference between s_{act} and the neighbor in the objective function, and T is a parameter that controls the maximum number of iterations of the algorithm. The Boltzmann condition basically means that if δ is not really high (i.e., the objective function of the neighbor is not much worse than s_{act}), then we can accept the neighbor as the new s_{act} with a high probability, even if it is less optimal than s_{act}. With this extra condition in the search process, we can escape local extremes present on our non-convex surfaces as seen in Figure 29.

The parameter T is the temperature parameter that controls the annealing schedule. The starting value for T is given as a parameter, and it continuously decreases with each iteration of the algorithm until it reaches a minimum temperature, given also as a parameter. When the minimum temperature is reached, the algorithm terminates. We can also specify a time parameter that gives the number of iterations to use the same T. This way the decrease of T is not continuous but monotonous. Together with the Boltzmann condition, the annealing schedule controlled by T can indicate that the probability for accepting a worse solution as s_{act} is high at the first few iterations and relatively low at the last iterations when we need the formation of a stable convergence.

This algorithm is very similar to annealing in metallurgy, a technique involving heating and controlled cooling of a material to increase the size of its crystals and reduce their defects. This is why the algorithm is named simulated annealing. In our applications, we chose the starting value of T as 0.001 and the minimum temperature as 0.0001. The time spent on each T is 10 iterations, and T decreases by 0.0001 with every 10 iterations. This way, if we have constant δ of 0.001, the probability of accepting a worse neighbor as s_{act} decreases from 0.3679 to 0.00005 during the iterations.

7.2.2 Numerical results

We used an Intel Core i5-7300HQ CPU 2,50 GHz computer with 8 GB DDR3 RAM for calculating the numerical results. We ran the program in Python 3.7.3. and used the Cbc 2.10. solver for the optimization of the model introduced in section 4.1. We considered the profitability constraint in the model.

In our numerical examination, we assumed two risk groups with different expected claim amounts with exponential distribution. We then calculated a BM system with four classes, and we used the second type approach. Therefore, in this case, we only need to determine three breakpoints.

$\mathbb{E}(L_1)$	$\mathbb{E}(L_2)$	Best Obj.	SD of Obj.	π_0	π_1	π_2	π_3	OP_1	OP_2
500	550	22.15	0.0091	550	550	500	500	4.88%	-3.62%
500	600	23.13	1.8348	600	500	500	500	9.29%	-2.97%
500	700	38.11	0.0363	700	500	500	500	5.61%	-3.99%
500	900	45.83	1.2438	900	500	500	500	10.91%	-4.12%
500	1200	39.05	3.1273	1200	1200	500	500	7.83%	-3.25%
500	1600	21.78	0.4441	1600	1600	500	500	4.36%	-1.36%
500	2100	10.83	0.0671	2100	2100	500	500	2.17%	-0.52%
500	3100	3.90	0.0005	3100	3100	500	500	0.78%	-0.13%
500	5100	1.14	0.0004	5100	5100	500	500	0.23%	-0.02%
500	9100	0.30	0.0024	9100	9100	500	500	0.06%	0.00%

Table 14: Results of exponential distributions
The first two columns show the parameters of the two groups, and the two adjacent the best objective value and the standard deviations of the objective values, followed by the premiums of each class, and lastly the overpayment (OP) of each group

The results of ten different models are shown in Table 14. In each case, the optimal premium scale is of a staircase type with only two steps. The lower premium is always equal to the expected claim amount of the less risky group, while the higher premium is consistently matched with the same parameter of the riskier one.

The simulated annealing always stopped after 470 seconds. The standard deviations of the objective values are relatively small, as seen in the fourth column.

The columns OP show the overpayment of the risk groups, compared with the expected claim amount. The less risky group is always paying more than what corresponds to their real risk, while the riskier group almost always pays less. However, the rate of the overpayment becomes smaller if the difference between the groups' parameters is higher. Thus, the BM system can handle adverse selection in a better way if the difference between the risks of the policyholders is high.

The breakpoints of the models can be seen in Figure 31. The lowest breakpoint

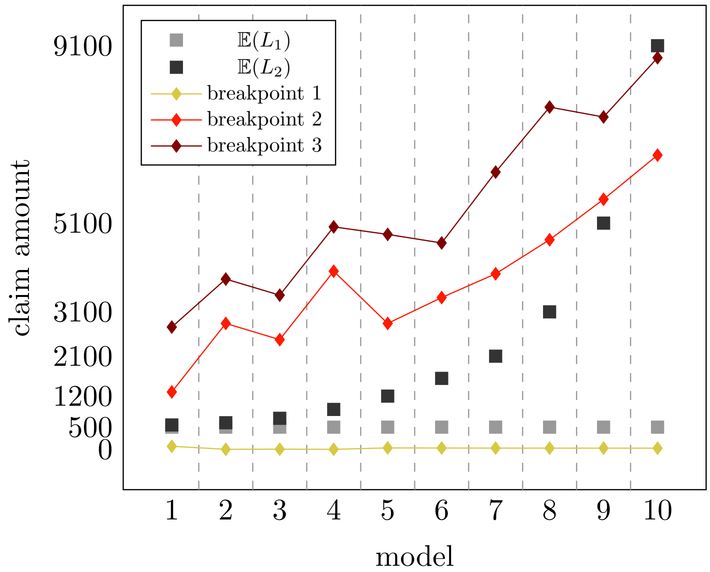

Figure 31: The changes of the breakpoints in the models.

always stays close to zero; however, the other two higher breakpoints grow together with the difference in parameters. The two higher breakpoints in most cases are over the higher-risk policyholder's parameter, except the last case where the difference between the policyholders is the highest.

7.3 Consideration of the claim occurrence with the claim amount

In the previous example, the probability of claim occurrence is relatively high because we only considered the exponential distribution for the claim amount. We also wanted to investigate a more realistic model wherein the claim numbers are also considered. Hence, in the model, we assume a distribution for the number of claims and another distribution for the claim amount. For the sake of simplicity, we suppose that the policyholders move one class upward in the BMS if they did not have any claim during a period. And in this model, only the claim amount determines the downward steps.

Hence, if at least a claim occurs in the period, the policyholders move downward. The number of classes the policyholder will move in this case is solely determined by the claim amount. Thus, we only have to determine breakpoints for moving downward in the system.

For the sake of simplicity, we considered a simple model with either no claim or at least one claim. If there is at least one claim, we only consider the cumulative claim amount. For the distribution of the claim amount, we consider exponential distribution.

We wanted to investigate BMSs with more classes and therefore with more breakpoints. To optimize this model, using a classical optimization tool is not suitable because of the same reasons mentioned in section 7.2. However, we wanted to compare the simulated annealing with at least one alternative approach. Hence, we also tested a genetic algorithm, called the biased random-key genetic algorithm (BRKGA) for this model.

7.3.1 Biased random-key genetic algorithm

```
k = 1
while termination conditions not met do
|   if k = 1 then
|   |   Generate P initial random-keys
|   |   for p in Population do
|   |   |   Solution[k][p] = LP(initial random-keys)
|   |   end
|   else
|   |   for p in Population do
|   |   |   for the size of the elite population:
|   |   |      Solution[k][p] = Elites[k-1][p]
|   |   |   for the size of the mutant population:
|   |   |      Solution[k][p] = LP(new random-keys)
|   |   |   for the rest:
|   |   |      Solution[k][p] = LP(mating(Elites[k-1],
|   |   |    Rest[k-1], biased-probability))
|   |   end
|   end
|   Separate Solution[k] into Elite[k]-set and Rest[k]-set.
|   k += 1
end
```

Figure 32: Biased random-key genetic algorithm (Gonçalves and Resende (2010))

This genetic algorithm was introduced in Gonçalves and Resende (2010). In the genetic algorithm, we consider a population of solutions that evolve during a specified time. In this type of algorithm, we consider random-keys to determine the solutions. The random-keys are vectors with $[0, 1)$ elements; often, the elements are called alleles. In our application, we considered the breakpoints as the random-keys. Hence, we multiply the elements of the vector to be suitable breakpoints. From a random-key vector, we calculate the corresponding solution with the model of section 4.1.

At the beginning of the algorithm, we generate a population with P random-keys. For each random-key (breakpoint), we calculate its optimal solution. Then, we separate the solutions into an elite set and a non-elite set. The better solutions belong to the elite set; the rest is classified into the non-elite set. A parameter determines the size of the elite set.

We produce new generations until we cannot improve the best solution for some periods. In each period, we copy the elite set into the subsequent period's population. We also generate some mutant solutions that are generated from new random-keys. For all of the population's remaining solutions, we mate a random elite with a non-elite from the previous period. In the mating, we use a parametrized uniform crossover. Let us call the new random-key the child of an elite and non-elite parent. Each element (allele) of the child's random-key vector comes from one of the parents. Therefore, for each allele, we flip a biased coin to choose which parent passes the allele to the child. There is a bias toward the elite parent in the mating; hence, the child is more likely to get alleles with a good solution. At the end of each period, we separate the period's solutions into an elite set and a non-elite set. At the end of the algorithm, we choose the best solution.

Parameter tuning

The parameters of the BRKGA algorithm are the ratio of elites, mutants, the mating bias toward the elites, and the size of a generation. Gonçalves and Resende (2010) recommends intervals for the parameters that we have provided in Table 15.

	recommended
ratio of elites	(0.1;0.25)
ratio of mutants	(0.1;0.3)
mating bias towards the elites	(0.5;0.8)

Table 15: Recommended BRKGA parameter intervals

To find parameters that result in stable solutions, we used a bootstrap-type search. We generated 100 small BMS instances with five classes and two random risk groups (claim numbers are the same, only the claim amount differs). The expected claims were generated from a [10% : 30%] interval, and the expected claim amounts were generated from a [100 : 2000] interval. For each instance, we generated ten different combinations of BRKGA parameters, and we optimized the model five times with each parameter. We then calculated the standard deviations of the optimal solutions (standardized to the best optimal solution). In the end, the parameters that had the smallest average standard deviation of optimal solutions were chosen. We considered 20 for the size of a population for the determination of the other parameters. We found that the results were the most stable when the ratio of the elites was 20%, the ratio of mutants was 25%, and the mating bias was 70%.

For the size of the population, Gonçalves and Resende (2010) recommend using the multiplication of the number of keys. We have as many keys as the breakpoints, which is equal to the number of classes minus one. To find the optimal population size, we generated 1000 random instances and optimized the BRKGA with different populations. We considered populations of multiplication of the breakpoints by $0.5, 1, 2, \ldots, 10$. To have at least one elite and mutant in each population, we increased the number of classes to eleven. Thus, there are 10 breakpoints in this model. Figure 33 presents the objective compared with the smallest size of population and the runtimes of the computations.

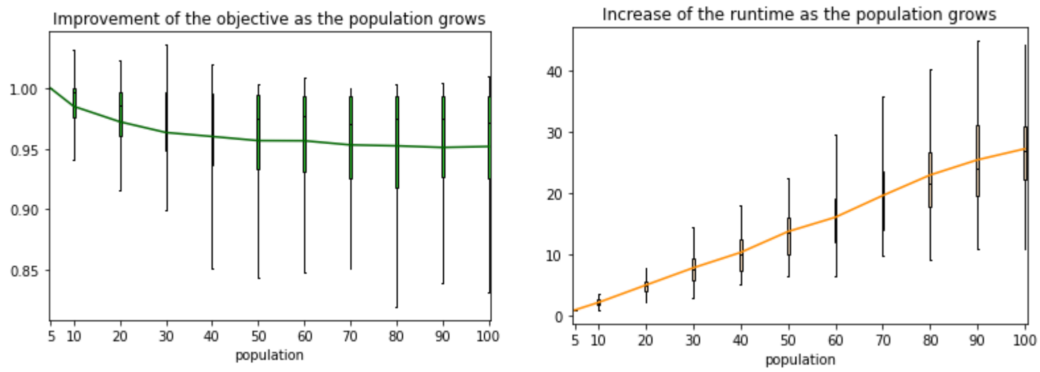

Figure 33: Objective and time change as the size of the population increases

On the left side, the objective improvement is shown, compared with the smallest population. As the population increases, a decrease in the objective is noticeable. However, the improvements of the populations that are larger than 15 are negligible. On the right side, the runtime is depicted. As the population increases, the running time, on average, almost linearly increases. For getting a good solution within a reasonable time, we chose a population that is three times greater than the number of breakpoints because it seems that the objective cannot improve much further, but the computational time increases significantly.

7.3.2 Numerical results

We used the same desktop computer for the numerical calculation as in section 4.4. In the numerical experiments, we considered two parameters for the risk groups: expected claim amount and the probability of a claim in a period.

We considered two types of policyholders regarding their claim amount distribution. Also, we assumed two types of policyholders in terms of claim occurrence. The policyholders with a higher probability of a claim are considered type A policyholders, and otherwise as type B. Policyholders with a less expected value of the claim amount are type L policyholders, and otherwise type H. Hence, in each case, we considered four risk groups: AL; AH; BL and BH.

Comparison of the genetic algorithm and simulated annealing

To choose between simulated annealing (SA) and the genetic algorithm, we generated 100 instances of four different setups. We considered setups with low (between 0.01 and 0.05) and high (between 0.1 and 0.3) expected claims. Furthermore, we utilized setups where the difference between the expected claim size is low and high. We searched for an optimal solution of 5-, 10-, and 15-class BMSs with SA and BRKGA.

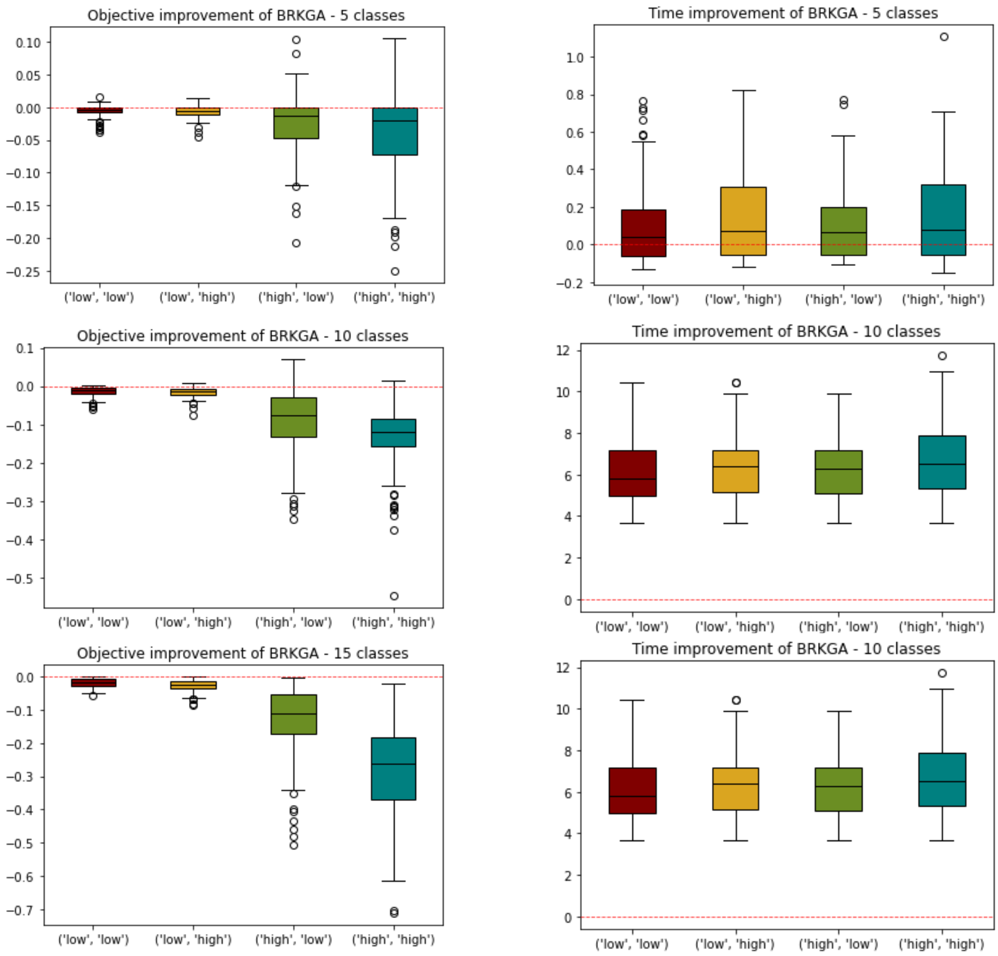

Figure 34: Objective and time increase of the BRKGA compared to the SA

Figure 34 presents the objective and the time difference distribution of the BRKGA and the SA of all instances in each setup. On the left, the objective change of the BRKGA compared with the SA is given. A value indicates by how many percentage points the BRKGA changed the objective of the SA. Hence, a negative value means that BRKGA had a better result. All of the setups' distributions are presented (('low', 'high') corresponds to the cases with low expected claim number and high difference between the claim size). The 5-class results are at the top; the 10-class is in the middle; and the 15-class case's results are at the bottom.

When the expected claims are lower, the BRKGA improvement is almost negligible. However, when the expected claims are higher, the improvement of the BRKGA in comparison with the SA is more noticeable. When the claim size differs more, the improvement is generally much greater. As the number of classes increases, the BRKGA gets even better than the SA.

In terms of the running time, the SA was much faster than the BRKGA. On the right side, the increase of the BRKGA's computational time compared with the SA is presented. In all of the cases, the SA is much faster than the BRKGA. And if the number of classes increases, the difference becomes much higher.

Even though the SA was significantly faster, the BRKGA could be calculated within a reasonable time. And because the BRKGA resulted in better objectives, we decided to use the BRKGA for the larger, more realistic instances.

Testing the impact of the policyholders' parameters

Table 16 presents the four scenarios we investigated.

	Risk group	AL	AH	BL	BH
case 1	claim probability	10%	10%	20%	20%
	expected claim amount	1000	2000	1000	2000
case 2	claim probability	10%	10%	20%	20%
	expected claim amount	1400	1600	1400	1600
case 3	claim probability	1%	1%	2%	2%
	expected claim amount	1000	2000	1000	2000
case 4	claim probability	1%	1%	2%	2%
	expected claim amount	1400	1600	1400	1600

Table 16: Parameters of the risk groups in each case

Hence in cases 1 and 2, the claims happen more frequently than in cases 3 and 4. Also, in the first two cases, the difference between the A and B types of policyholders is higher. According to the results of section 5.4, the BMS, which only considers the claim number, would be more efficient than the BMS of cases 3 and 4.

In cases 1 and 3, the difference between the claim amount is smaller for the risk groups than in cases two and four. In these experiments, we left out the profitability constraint.

Table 17 presents the objective values and the OP values of the risk groups. In each case, we considered a BMS with 10, 20, or 30 classes. The row pertaining to objective in the table shows the improvement of the objective value over the result of the 10-class BMS. In each case, it is decreasing as the size of the BMS increases. In cases 1 and 2, it is more noticeable than in cases 3 and 4. Hence, if the probability of claim occurrence is small, adding more classes to the BMS does not improve the solution. Thus, the results are similar to the outcome described in section 5.4. When the claim amount was not considered and the claim probabilities were small, even 100 classes could not decrease the risk groups' OP values significantly.

	Case 1			Case 2		
Class	10	20	30	10	20	30
Objective	100%	88%	76%	100%	76%	60%
OP^{AL}	101.89%	101.77%	100.77%	45.07%	30.31%	24.95%
OP^{AH}	5.69%	8.45%	7.25%	27.47%	14.99%	11.06%
OP^{BL}	5.51%	9.05%	9.68%	-12.60%	-10.50%	-4.62%
OP^{BH}	-39.22%	-27.64%	-19.97%	-23.22%	-21.20%	-15.75%

	Case 3			Case 4		
Class	10	20	30	10	20	30
Objective	100%	100%	100%	100%	98%	97%
OP^{AL}	100.28%	100.40%	100.77%	17.08%	18.69%	20.37%
OP^{AH}	0.93%	1.29%	2.12%	2.55%	4.12%	5.73%
OP^{BL}	0.31%	0.42%	0.85%	-40.03%	-38.40%	-36.71%
OP^{BH}	-49.04%	-48.67%	-47.75%	-47.42%	-45.84%	-44.21%

Table 17: Results of the 4 cases

Similarly, in this model, when the claim probabilities are smaller in general, the system cannot distinguish the risk groups that efficiently.

The OP values present the relative overpayments of the risk groups. In this consideration, case 1 and case 3 are similar to each other. In both cases, the overpayment is close to zero for the two middle groups (the AH and the BL. Their payment is relatively fair, compared with the other two risk groups. The very risky policyholders (BH) pay much less than their fair premium. The least risky policyholders, however, pay premiums more than twice their fair price. In these two cases, every risk group except BH pays more than the fair premium. Cases 2 and 4 are similar in a way that all of the risk groups' over- and underpayment is similar: AL pays much more, and AH pays a bit more than their fair premium. Furthermore, BL and BH pay less. Hence, if the difference between the expected claim amount is large, the system is not fair to the more extreme policyholders. Those who have a small claim amount probability pay much more, and those who have really high expected claims pay much less.

Breakpoints in solutions

Figure 35 presents the breakpoints of each case's 30-class BMS. For better visibility, in the figure, we depicted the breakpoint as the percentage to the highest breakpoint. In our computations, we set the highest breakpoint to the value, where the exponential distribution probability becomes exactly one if we round the probability to 4 digits.

None of the breakpoints increases linearly. Also, it is noted that in each case, there is one large step between the breakpoints. In case 1, the breakpoints start

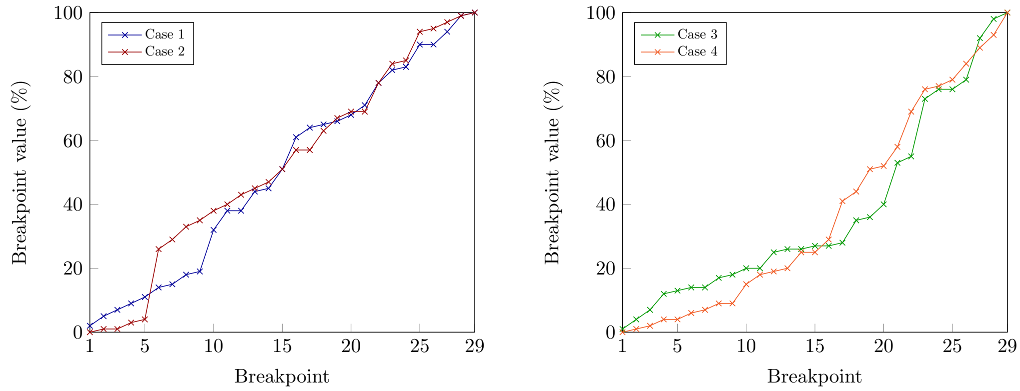

Figure 35: The breakpoints of the four cases. Represented in the percentage of the highest breakpoint(29).

almost linearly, and then at around the tenth breakpoint, there is a huge increase. This increase occurs around the fifth breakpoint for case 2 and around fifteenth breakpoint for case 4. For case 3, there is a big step between breakpoints 20 to 24.

For cases 2 and 4, the first few breakpoints are really close to zero. Hence, in these cases, in the event of any claim, the policyholders are likely to fall multiple classes.

However, even though the change in the breakpoints does not seem to increase linearly, the breakpoints with a higher number are not as relevant. For comparison, we also calculated the optimal premiums with linearly increasing breakpoints. It was thus found that the objective value of the models with linearly increasing breakpoints was worse in each case. In case 1, it was worse with 5.9%. In Case 2, it was much worse, with the decrease being 24.3%. Presumably, the result was much worse for Case 2 because there is a huge step after the 5th breakpoint. On the other hand, for case 1, the first ten breakpoints increased almost linearly. Therefore, the difference is much smaller.

Interestingly for Cases 3 and 4, the difference was not that huge: for case 3, the increase was only 0.4%, and for case 4, it was 3.0%. Therefore, even though the overall increase between the breakpoints does not seem to be linear, the first few breakpoints have more weights.

To further analyze the reasons behind it, we calculated the probabilities of reclassification between the periods if there is a claim.

Probability of reclassification

Figure 36 illustrates the reclassification distribution of a claim from the class 29. Therefore, for each class, the probability of the policyholders will be assigned to this class if they are currently in class 29 and have a claim.

In case 1, if the policyholders have any claim, they are very likely to fall less than

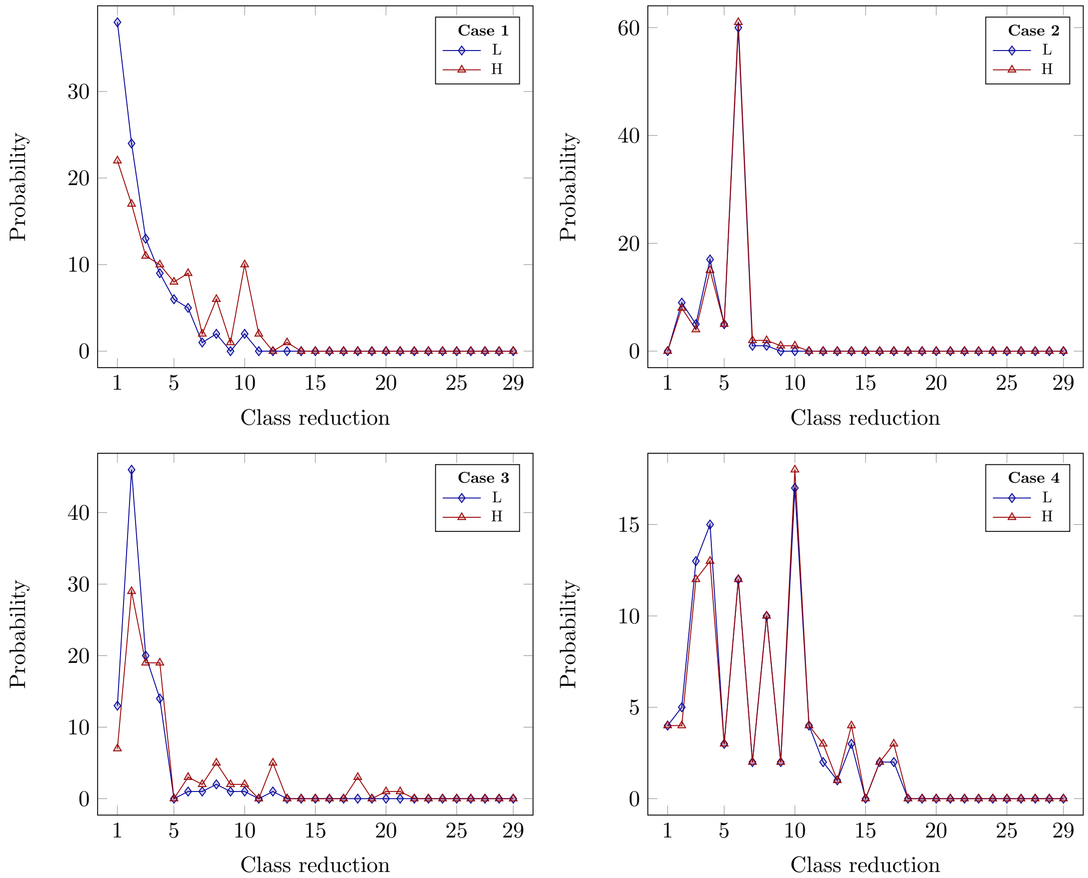

Figure 36: The probabilities to reclassifications

5 classes. The type L policyholders have more than 40% probability for one class reduction when there is a claim. The other type, however, has a higher probability of stepping down more classes in the system.

In case 2, the policyholders' class reduction is around 6 classes, with almost any claim amount. If we only consider the number of claims, the optimal transition rules will be exactly 5 classes reduction in case of a claim. Hence, if the expected claim amount is similar among the risk groups, the consideration of only the number of claims has almost the same outcome. However, if we consider the claim amount, it gives slightly more possibilities to distinguish the policyholders since in case 2, the five class reduction is not with 100% probability.

In case 3, it is around four classes. In case 4, the policyholders are likely reclassified around 3–17 class under. If we only consider the number of claims, the transition rules would be 29 classes reduction when a claim occurs. Hence the transition rules of the claim amount have very different results, unlike in cases 1 and 2.

Compared with the breakpoint values, in Figure 35, the large leap corresponds to the reclassification probability only in case 2. In this case, the huge leap results in a large possibility of reduction around five classes when at least one claim occurs. In the other cases, the huge increase in the breakpoint value does not seem noticeable in the probabilities. In case 1, the probability of stepping more than ten classes (where the increase in the breakpoint value is noticeable) is around zero for type L policyholders but higher for the other one. In cases 3 and 4, the large leap is over the 20th breakpoint. However, the probability of more than 20 class reductions is almost zero in both cases.

7.3.3 Case Study

Currently, in Hungary, only the number of claims determines the transition rules in the MTPL insurance. In the previously presented case studies, we analyzed vehicle insurance data of an insurance company that operates in Hungary. Unfortunately, the claim amounts that were paid are not listed in this data. However, two variables indicate the claim amounts. Each policyholder with at least a claim has a variable showing which decile the claim amount belonged to. Also, another variable shows the classifications of the claim amounts if the claims are separated into ten equal length intervals.

Because we analyzed the Hungarian BMS in the previous sections of the dissertation, we did not seek to analyze a different system for this model. Hence, we estimated claim amounts from another dataset. We used an open data library from R, the dataCar dataset from the insuranceData library.

By considering the deciles and range variables, we generated a claim amount for each policyholder with at least one claim in the Hungarian data. We used the insurance data's claim amount distribution as a reference for the generation.

Our intention was to compare two models: when the transition rule depends only on the number of claims (currently used system) and when there is a claim, the transition depends only on the claim amount (as in section 7.3.2). When only the number of claims matters, we used the model introduced in section 5.1. The profit constraint was not considered in any of these models. To make the comparison easier, we assumed that during any period, one claim can occur at most.

We used the same risk groups as in section 5.5. However, we separated each risk groups further by the claim amounts. We estimated a regression tree for each risk group to further separate the group. In each regression tree, the target variable was the claim amounts, and the features were the same variables that we used to create the original risk groups. Table 18 provides the estimated risk groups expected claims amounts and claim occurrence probabilities (first two columns).

Overall, we could separate 17 risk groups. The first column presents the probability of claim occurrence in a period, followed by the expected claim amount and the risk group ratio within the whole population. For simplicity, we assumed that all of the risk groups' claim amount distribution is exponentially distributed.

Claim probability	Expected claim size	Ratio	OP		
			claim size	claim number	number with size
1.8%	1156	2%	262.3%	50.0%	327.7%
	1858	1%	125.4%		166.1%
	2216	2%	89.0%		123.1%
	2897	6%	44.7%		70.7%
2.7%	1922	3%	45.3%	0.0%	71.6%
	2792	35%	0.1%		18.1%
	3244	6%	-13.7%		1.6%
3.2%	1739	3%	35.5%	-15.6%	60.1%
	2300	16%	2.5%		21.0%
	2718	4%	-13.2%		2.4%
	3244	3%	-27.1%		-14.2%
4.1%	3813	1%	-51.3%	-34.0%	-43.0%
	5136	7%	-63.1%		-57.7%
	5924	3%	-67.4%		-63.3%
5.0%	5144	5%	-69.4%	-45.8%	-65.3%
	7520	1%	-77.3%		-76.2%
	8895	1%	-79.8%		-79.9%

Table 18: Parameters of the risk groups and the results of the models.

With both methods, we optimized a 30-class BMS. The three OP columns present the OP values of each risk groups for each model. The first column presents the OP values of the model where the negative transition rules depend only on the claim amount (we estimated the results with the BRKGA). The next two columns present the OP results of the model that were introduced in section 5.1. The column on the left relates to the assumption that the claim amount is equal in each risk groups. The right column presents the OP values of the claim number model with the consideration of the estimated claim amounts.

Overall, the trend of the OP values are similar in each column: risk groups with lower risks generally pay more than what would be fair, and the riskier policyholders pay less. However, the amount of overpayment differs. In general, the more extreme risk groups pay less when the transition rules depend on the claim amount. The least risky groups pay less – they get closer to their fair premium. Nevertheless, the risky policyholders also pay less – they get farther from their fair premium.

Even though the OP values differ significantly for some risk groups in the two solutions, the claim amount model also faces similar limitations as the claim number model. Figure 37 presents the difference between expected payment and expected claim amounts for each risk groups in both models. In general, when the claim occurrence determines the classification solely, almost any type of policyholder pays more. And, when transition rules depend on the claim amount, the policyholders generally pay less.

The blue line in the background depicts the ratio of the risk groups. It is also similar in both models that the risk group with the majority gets the fairest expected

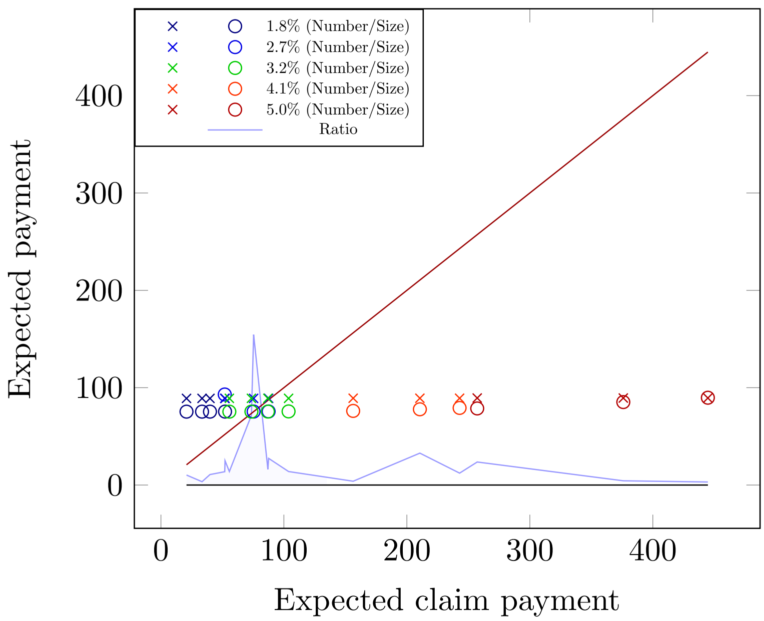

Figure 37: Difference between the expected payments and claim-payments

premium. It seems that the risk group with the majority has to pay more than their fair premium in the model, where the transition rules depend on the claim number. However, it only appears that way because the claim amount was not considered in the optimization of this model. With uniform claim amounts, the OP value of the risk group 2.7% is almost 0.

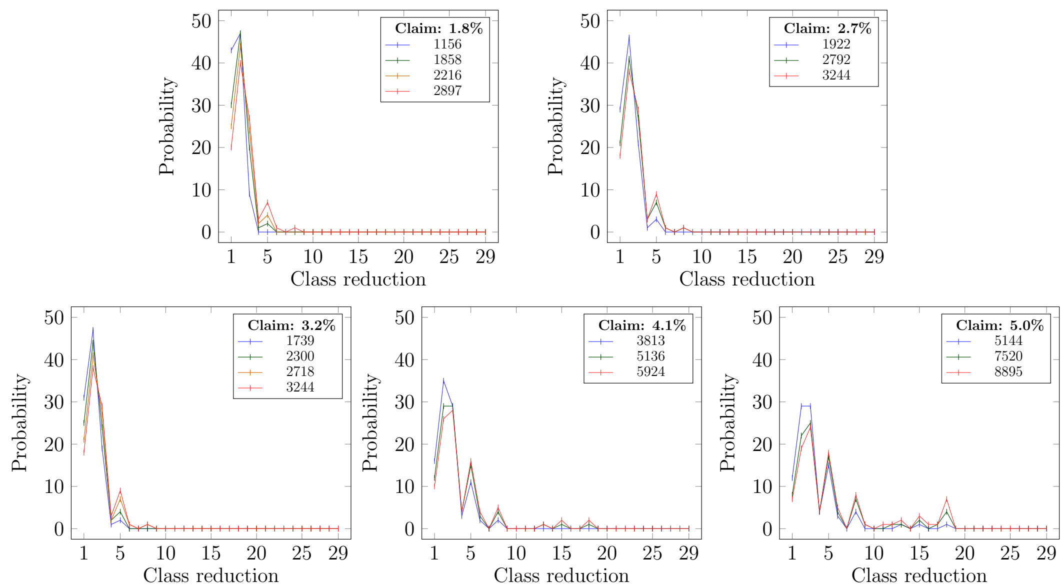

Figure 38: The probabilities to reclassifications

Figure 38 presents the probability of reclassification for each risk group from class 30 if there is a claim in a period. It is noticeable that if the size of the expected claim is small, there is a high probability that the policyholder will be reclassified

1–3 classes. When the expected claim size is larger, there is also a chance for more class reduction. Over 5000 expected claims, it is also possible that a policyholder falls 18 classes. When we considered only the number of claims in the transition rules, the rule was 9 class reduction if a claim occurs. Seemingly, the expected claim size model is more indulgent. Hence, there is a small probability that a policyholder will fall as many classes. Nevertheless, in this case, it is also possible to fall more classes. Therefore, transition rules that depend on the claim amount give more flexibility to the reclassifications.

7.3.4 Summary

In practice, usually, the transition rules only depend on the number of claims, and the sizes of the claims does not influence the classification. In this section, we considered BMS, where the classification only depends on the claim amount. Finding optimal classification rules and premiums require a non-linear optimization model. We compared simulated annealing (SA) and biased random-key genetic algorithm (BRKGA) to approximate the optimal solution. We found that the SA is much faster than the BRKGA at larger instances; however, the BRKGA generally found better results.

We compared the claim size model with the original models – where the transition rules only depend on the number of claims. A case study on realistic data is also presented. We found that even though claim amounts can result in more flexible classifications, the efficiency of BMS (in terms of separating the risk groups) was very similar to the results of the claim number model.

8 Summary

In the dissertation, we presented research on the Bonus-Malus System (BMS). This system is mostly used in Motor third-party liability insurances to separate the risky drivers from the less risky ones. Another aim is to incentivize risk aversion for the drivers.

A BMS consists of several classes, and each class has a premium. The policyholders (drivers) are classified into these classes over the periods of the insurance contract. Hence, in the base model, the policyholder's payment in a period only depends on the class where he/she is classified into.

In the literature on optimizing a BMS, the rules over the classification are usually considered as the parameters, and the premiums are the variables. In some countries, the rules of classification are determined by law. Hence, the insurance companies can only optimize the premiums. However, as we presented in section 2.1, in many European countries, the regulations became liberated in the past 25 years. Hence in these countries the insurance companies can optimize their own BMS.

In section 4.1, we presented an LP model where the premiums were the optimization variables. In section 4.2, we put forth a MILP model where we may optimize the transition rules instead of the premiums. In this model, we introduced binary variables for each possible transition rule.

This section also presented a model for the joint optimization of the premiums and transition rules. We found that there are finite possibilities of the premiums in the optimal solution with the considered objective function. Hence, we introduced binary variables for each possibility in each class.

We also proposed a modification of this model, where instead of the stationary probability, we may consider multi-period optimization. We also extended this MILP model to optimize the number of classes besides the premiums and the transition rules.

We conducted numerical tests of these models. We found that the BMS is the most effective if there are many BMS classes and the difference between the policyholders' parameters is large. However, using too many classes in a BMS may not be viable in practice due to the contract exists only for a finite number of periods. Hence, we investigated the multi-period model, where we got a slightly different result than the stationary model, which assumes the contract will never end. However, even in this case, the optimal result was to use more classes than usually used in practice.

We also conducted research on realistic parameters calculated from Hungarian data. We found that more classes and more rigid transition rules would separate the policyholders better than the currently used system.

In the models of section 4, we assumed that the policyholders only pay the BMS's premium. However, in practice, insurance companies often use other statistical methods besides the BMS to approximate the policyholders' real probabilities. In section 6, we compared techniques for matching the BMS and statistical method's premium in the optimization of the BMS. Using the BMS and statistical method jointly has almost always resulted in a better solution than when we considered only one of them. The most effective method was when we optimized BMSs for each type of policyholder, and we also considered the statistical method's result in the payments. However, the results can be a bit unstable because an optimized BMS's effectiveness depends on the risk groups' parameters.

In section 6.4, we compared the techniques using parameters of real data. We considered two different approaches for the determination of the parameters. We obtained slightly different results, but in both cases, considering both BMS and statistical methods resulted in the most effective outcome.

In practice, only the number of claims are considered in the transition rules of the BMS. Hence, the premiums are independent of the claim amount. In section 7, we presented a study where instead of the number of claims, the claim amount determines the transition rules. In this BMS, we consider breakpoints of the claim amount for the classification rule. The interval of two breakpoints determines the policyholder's class in the subsequent periods. In the optimization model, we have to optimize the premiums as well as the breakpoints' exact values. For the premiums, we used the LP model of section 4.1. For finding relatively good breakpoints, we used heuristics. Similarly to the study, where only the number of the claims mattered, the BMS of this model is more efficient when there are more classes. Also, it can separate the policyholders better if the difference between their parameters is large. When the claim occurrence probability was higher, the separation of the system was also more effective.

9 Appendix

9.1 List of Notations

List of mathematical notations, used in models.

Greek letter

ε^{ℓ} How much the premium changes in layer ℓ compared to the default premium.

ζ Auxiliary parameter to enforce the strict monotony of the breakpoints

η^p Observable variables of the policyholder p

Θ Final period in multi-period optimization

λ^i Expected claim amount for risk group i

λ^i_m Probability of the occurrence of m claims for the policyholders of type i

Λ^p Personal risk related to the policyholder p

$\hat{\Lambda}^p$ Insurance company's estimation of the p policyholder's risk

ρ_p Unobservable variables of the policyholder p

π_k Premium of class k

τ Very small value to exclude the non-irreducible Markov chains from the optimization

ϕ^i Proportion of the type i policyholders among all of the policyholders

Ω The overall cross-financing of the policyholders in a BMS

Lowercase letter

c^i_k Stationary probability that type i policyholders is classified into class k

$c_{k,t}$ Probability that the policyholder in period t is classified into class k

$c^i_{k,t}$ Probability that type i policyholders is classified into class k at period t

$d^i_{k,j,m,t}$ Probability of an individual from group i and from class k at period t moves to class $k+j$ in the next period

$d^i_{k,j,m}$ Probability of an individual from group i and from class k moves to class $k+j$ in the next period

g^i_k Absolute deviations of expected payment to expected claims of the group i in class k

$g^i_{k,t}$ Absolute deviations of expected payment to expected claims of the group i in class k at period t

n Number of observable parameters

$o^{\ell,i}_k$ Continuous nonnegative variable, if $O^\ell_k = 1$, then $o^{\ell,i}_k$ is equal to $c^i_k \varepsilon^\ell$, otherwise 0

$o^{\ell,i}_{k,t}$ Continuous nonnegative variable, if $O^\ell_k = 1$, then $o^{\ell,i}_{k,t}$ is equal to $c^i_{k,t} \varepsilon^\ell$, otherwise 0

$p^i_{j,k}$ Transition probability of the type i policyholders from the class j to k

$p_{k,l}(t)$ The t-th step transition probabilities from class l to k

r Number of unobservable parameters

Special character

$\mathcal{L}_1$ Layers for the premium increases to the expected claims

$\mathcal{L}_2$ Layers for the premium increases to the unique premium

OP^i Ratio of the paid and the ideal payment of type i policyholders

Uppercase letter

B_k Binary variable, if it is equal to one, then class k is the initial class

C_t Row vector form of the $c_{k,t}$

$F_i()$ Cumulative distribution function of the ith risk group

I Number of risk groups

J^* Set of transition rules that lead to a non-irreducible Markov-chain

J_k Domain of j. $J_k = [\underline{J_k} : \overline{J_k}]$ for class k

K The class with the lowest premium (there are $K+1$ classes indexed from 0 to K)

L Number of layers

M Highest number of possible claims in a period

O_k^ℓ Binary variable, if it is equal to one, then the premium of class k is increased by ε^ℓ

$P(t)$ Matrix with entries $p_{k,l}(t)$

S_ℓ Binary variable, if it is equal to one, if the classes' layer $\ell \in \mathcal{L}_2$ is active

T^m Transition matrix of claim m

$T_{j,m,k}$ Binary variable of non-unified transition rule. If it is equal to one, then policyholders with m claims are moved from class k, j classes in the following period

$T_{j,m}$ Binary variable of unified transition rule. If it is equal to one, then policyholders with m claims are moved j classes in the following period

V_k Binary variable, if it is equal to one, then class k is closed

X_t The class, where the policyholder is classified in period t

9.2 Summary of Models

In this section, we present the whole models that we introduced in the dissertation.

9.2.1 Premium scale optimization model with fixed transition rules

The model is presented in section 4.1.

$$\min \sum_{i=1}^{I} \sum_{k=0}^{K} \phi^i g_k^i \tag{LP1.obj}$$

Subject to

$$\pi_k c_k^i + g_k^i \geq \lambda^i c_k^i \qquad \forall i, k \tag{LP1.1}$$

$$\pi_k c_k^i - g_k^i \leq \lambda^i c_k^i \qquad \forall i, k \tag{LP1.2}$$

$$\pi_{k-1} \geq \pi_k \qquad k = 1, \ldots, K \tag{LP1.3}$$

$$\begin{aligned} \pi_k &\geq 0 \qquad \forall k \\ g_k^i &\geq 0 \qquad \forall k, i \end{aligned}$$

9.2.2 Premium scale optimization model with fixed transition rules with profit constraint

The model is presented in section 4.1.1.

$$\min \sum_{i=1}^{I} \sum_{k=0}^{K} \phi^i g_k^i \tag{LP1.obj}$$

Subject to

$$\pi_k c_k^i + g_k^i \geq \lambda^i c_k^i \qquad \forall i, k \tag{LP1.1}$$

$$\pi_k c_k^i - g_k^i \leq \lambda^i c_k^i \qquad \forall i, k \tag{LP1.2}$$

$$\pi_{k-1} \geq \pi_k \qquad k = 1, \ldots, K \tag{LP1.3}$$

$$\sum_{i=1}^{I} \sum_{k=0}^{K} \left(\pi_k c_k^i\right) \geq \sum_{i=1}^{I} \phi^i \lambda^i. \tag{LP1.4}$$

$$\begin{aligned} \pi_k &\geq 0 \qquad \forall k \\ g_k^i &\geq 0 \qquad \forall k, i \end{aligned}$$

9.2.3 Non-unified transition rules optimization model with fixed premiums

The model is presented in section 4.2.

$$\min \sum_{i=1}^{I} \sum_{k=0}^{K} \phi^i g_k^i \tag{MILP1.obj}$$

Subject to

$$\sum_{j=\underline{J_k}}^{\overline{J_k}} T_{j,m,k} = 1 \ , \qquad \forall m, k \tag{MILP1.1}$$

$$\sum_{j=\min(\overline{J_k},1)}^{\overline{J_k}} T_{j,0,k} = 1 \ , \qquad \forall k \tag{MILP1.2}$$

$$\sum_{j=\underline{J_k}}^{\max(\underline{J_k},-1)} T_{j,M,k} = 1 \ , \qquad \forall k \tag{MILP1.3}$$

$$\sum_{\ell=j}^{\overline{J_k}} T_{\ell,m,k} \geq T_{j,m+1,k} \qquad \forall j, k, \ m = 0, \ldots, M-1 \tag{MILP1.4}$$

$$\sum_{k=0}^{K} c_k^i = \phi^i \qquad \forall i \tag{MILP1.5}$$

$$c_k^i = \sum_{j=-(K-k)}^{k} \sum_{m=0}^{M} d_{k-j,j,m}^i \qquad \forall i, k \tag{MILP1.6}$$

$$d_{k,j,m}^i \geq \lambda_m^i c_k^i - (1 - T_{j,m,k})\phi^i \qquad \forall i, j, k, m \tag{MILP1.7}$$

$$\sum_{i=1}^{I} c_k^i \geq \tau \qquad \forall k \tag{MILP1.8}$$

$$\pi_k c_k^i + g_k^i \geq \lambda^i c_k^i \qquad \forall i, k \tag{MILP1.10}$$

$$\pi_k c_k^i - g_k^i \leq \lambda^i c_k^i \qquad \forall i, k \tag{MILP1.11}$$

$$T_{j,m,k} \in (0,1) \qquad \forall j,m,k$$
$$g_k^i \geq 0; c_k^i \geq 0 \qquad \forall k,i$$
$$d_{k,j,m}^i \geq 0 \qquad \forall k,j,m,i.$$

The profit-constraint can be added to this model as well:

$$\sum_{i=1}^{I}\sum_{k=0}^{K}\left(\pi_k c_k^i\right) \geq \sum_{i=1}^{I}\phi^i\lambda^i. \tag{MILP1.9}$$

9.2.4 Unified transition rules optimization model with fixed premiums

The model is presented in section 4.3.

$$\min \sum_{i=1}^{I}\sum_{k=0}^{K}\phi^i g_k^i \tag{(MILP1.obj)}$$

Subject to

$$\sum_{j=\underline{J}}^{\overline{J}} T_{j,m} = 1 \qquad \forall m \tag{MILP2.1}$$

$$\sum_{j=1}^{\overline{J}} T_{j,0} = 1 \tag{MILP2.2}$$

$$\sum_{j=\underline{J}}^{-1} T_{j,M} = 1 \tag{MILP2.3}$$

$$\sum_{\ell=j}^{\overline{J}} T_{\ell,m} \geq T_{j,m+1} \qquad \forall j,\ m=0,...,M-1 \tag{MILP2.4}$$

$$\sum_{k=0}^{K} c_k^i = \phi^i \qquad \forall i \tag{MILP1.5}$$

$$\begin{aligned}
c_k^i &= \sum_{j=\underline{J}}^{0}\sum_{\ell=j}^{0}\sum_{m=0}^{M} d_{k-\ell,j,m}^i && k=0,\forall i\\
c_k^i &= \sum_{j=\max(\underline{J},-(K-k))}^{\min(\overline{J},k)}\sum_{m=0}^{M} d_{k-j,j,m}^i && k=1,\ldots,K-1,\forall i\\
c_k^i &= \sum_{j=0}^{\overline{J}}\sum_{\ell=0}^{j}\sum_{m=0}^{M} d_{k-\ell,j,m}^i && k=K,\forall i
\end{aligned} \tag{MILP2.6}$$

$$d^i_{k,j,m} \geq \lambda^i_m c^i_k - (1 - T_{j,m})\phi^i \qquad \forall i, j, k, m \tag{MILP2.7}$$

$$\sum_{m=0}^{M} T_{j_m,m} \leq M \qquad \forall (j_0, j_1, \ldots, j_M) \in J^* \tag{MILP2.8}$$

$$\pi_k c^i_k + g^i_k \geq \lambda^i c^i_k \qquad \forall i, k \tag{MILP1.10}$$

$$\pi_k c^i_k - g^i_k \leq \lambda^i c^i_k \qquad \forall i, k \tag{MILP1.11}$$

$$\begin{aligned} T_{j,m} \in (0,1) &\qquad \forall j, m \\ g^i_k \geq 0; c^i_k \geq 0 &\qquad \forall k, i \\ d^i_{k,j,m} \geq 0 &\qquad \forall k, j, m, i. \end{aligned}$$

The profit-constraint can be added to this model as well:

$$\sum_{i=1}^{I} \sum_{k=0}^{K} \left(\pi_k c^i_k\right) \geq \sum_{i=1}^{I} \phi^i \lambda^i. \tag{MILP1.9}$$

9.2.5 Joint optimization of non-unified transition rules and premiums - stationary model

The model is presented in section 5. Only the non-unified transition rules' model was introduced in the dissertation. However, changing the constraints (MILP1.1)-(MILP1.7) to the unified transition rules optimization model (MILP2.1)-(MILP2.7) constraints would allow optimizing the unified transition rules with the premiums.

$$\min \sum_{i=1}^{I} \sum_{k=0}^{K} \phi^i g_k^i \tag{MILP1.obj}$$

Subject to

$$\sum_{j=\underline{J_k}}^{\overline{J_k}} T_{j,m,k} = 1 \ , \qquad \forall m, k \tag{MILP1.1}$$

$$\sum_{j=\min(\overline{J_k},1)}^{\overline{J_k}} T_{j,0,k} = 1 \ , \qquad \forall k \tag{MILP1.2}$$

$$\sum_{j=\underline{J_k}}^{\max(\underline{J_k},-1)} T_{j,M,k} = 1 \ , \qquad \forall k \tag{MILP1.3}$$

$$\sum_{\ell=j}^{\overline{J_k}} T_{\ell,m,k} \geq T_{j,m+1,k} \qquad \forall j, k, \ m = 0, \ldots, M-1 \tag{MILP1.4}$$

$$\sum_{k=0}^{K} c_k^i = \phi^i \qquad \forall i \tag{MILP1.5}$$

$$c_k^i = \sum_{j=-(K-k)}^{k} \sum_{m=0}^{M} d_{k-j,j,m}^i \qquad \forall i, k \tag{MILP1.6}$$

$$d_{k,j,m}^i \geq \lambda_m^i c_k^i - (1 - T_{j,m,k}) \phi^i \qquad \forall i, j, k, m \tag{MILP1.7}$$

$$\sum_{i=1}^{I} c_k^i \geq \tau \qquad \forall k \tag{MILP1.8}$$

$$\pi_k c_k^i + \sum_{\ell=1}^{I-1} o_k^{\ell,i} + g_k^i \geq \lambda^i c_k^i \quad \forall i, k \tag{MILP3.10}$$

$$\pi_k c_k^i + \sum_{\ell=1}^{I-1} o_k^{\ell,i} - g_k^i \leq \lambda^i c_k^i \quad \forall i, k \tag{MILP3.11}$$

$$\pi_k + \sum_{\ell=1}^{I-1} \varepsilon^\ell O_k^\ell \geq \pi_{k+1} + \sum_{\ell=1}^{I-1} \varepsilon^\ell O_{k+1}^\ell \qquad k = 0, \ldots, K \tag{MILP3.12}$$

$$\sum_{\ell=1}^{I-1} O_k^\ell \leq 1 \qquad \forall k \tag{MILP3.13}$$

$$o_k^{\ell,i} \geq \varepsilon^\ell \left(c_k^i - (1 - O_k^\ell) \right) \quad \forall i, k, \ell = 1, \ldots, I-1 \tag{MILP3.14}$$

$$o_k^{\ell,i} \leq \varepsilon^\ell c_k^i \quad \forall i, k, \ell = 1, \ldots, I-1 \tag{MILP3.15}$$

$$o_k^{\ell,i} \leq \varepsilon^\ell O_k^\ell \quad \forall i, k, \ell = 1, \ldots, I-1 \tag{MILP3.16}$$

$$\begin{aligned}
T_{j,m,k} \in (0,1) &\qquad \forall j, m, k \\
g_k^i \geq 0; c_k^i \geq 0 &\qquad \forall k, i \\
d_{k,j,m}^i \geq 0 &\qquad \forall k, j, m, i \\
O_k^\ell \in (0,1) &\qquad \forall k, \ell = 2, \ldots, I-1 \\
o_k^{\ell,i} \geq 0 &\qquad \forall k, i, \ell = 2, \ldots, I-1
\end{aligned}$$

9.2.6 Joint optimization of non-unified transition rules and premiums with the profit constraint - stationary model

The model is presented in section 5.

$$\min \sum_{i=1}^{I} \sum_{k=0}^{K} \phi^i g_k^i \tag{MILP1.obj}$$

Subject to

$$\sum_{j=\underline{J_k}}^{\overline{J_k}} T_{j,m,k} = 1 \ , \qquad \forall m, k \tag{MILP1.1}$$

$$\sum_{j=\min(\overline{J_k},1)}^{\overline{J_k}} T_{j,0,k} = 1 \ , \qquad \forall k \tag{MILP1.2}$$

$$\sum_{j=\underline{J_k}}^{\max(\underline{J_k},-1)} T_{j,M,k} = 1 \ , \qquad \forall k \tag{MILP1.3}$$

$$\sum_{\ell=j}^{\overline{J_k}} T_{\ell,m,k} \geq T_{j,m+1,k} \qquad \forall j,k,\ m=0,\ldots,M-1 \tag{MILP1.4}$$

$$\sum_{k=0}^{K} c_k^i = \phi^i \qquad \forall i \tag{MILP1.5}$$

$$c_k^i = \sum_{j=-(K-k)}^{k} \sum_{m=0}^{M} d_{k-j,j,m}^i \qquad \forall i,k \tag{MILP1.6}$$

$$d_{k,j,m}^i \geq \lambda_m^i c_k^i - (1 - T_{j,m,k})\phi^i \qquad \forall i,j,k,m \tag{MILP1.7}$$

$$\sum_{i=1}^{I} c_k^i \geq \tau \qquad \forall k \tag{MILP1.8}$$

$$\sum_{i=1}^{I} \phi^i \sum_{k=0}^{K} \left(\pi_k c_k^i + \sum_{\ell=1}^{L} o_k^{\ell,i} \right) \geq \sum_{i=1}^{I} \phi^i \lambda^i \tag{MILP4.9}$$

$$\pi_k c_k^i + \sum_{\ell=1}^{L} o_k^{\ell,i} + g_k^i \geq \lambda^i c_k^i \quad \forall i,k \tag{MILP3.10}$$

$$\pi_k c_k^i + \sum_{\ell=1}^{L} o_k^{\ell,i} - g_k^i \leq \lambda^i c_k^i \quad \forall i,k \tag{MILP3.11}$$

$$\pi_k + \sum_{\ell=1}^{L} \varepsilon^\ell O_k^\ell \geq \pi_{k+1} + \sum_{\ell=1}^{L} \varepsilon^\ell O_{k+1}^\ell \qquad k=0,\ldots,K \tag{MILP3.12}$$

$$\sum_{\ell=1}^{L} O_k^\ell \leq 1 \qquad \forall k \tag{MILP3.13}$$

$$o_k^{\ell,i} \geq \varepsilon^\ell \left(c_k^i - (1 - O_k^\ell) \right) \quad \forall i,k,\ell=1,\ldots,L \tag{MILP3.14}$$

$$o_k^{\ell,i} \leq \varepsilon^\ell c_k^i \quad \forall i,k,\ell=1,\ldots,L \tag{MILP3.15}$$

$$o_k^{\ell,i} \leq \varepsilon^\ell O_k^\ell \quad \forall i,k,\ell=1,\ldots,L \tag{MILP3.16}$$

$$\sum_{k=0}^{K} O_k^\ell \leq (K+1) S_\ell \qquad \forall \ell \in \mathcal{L}_2 \tag{MILP4.17}$$

$$\sum_{\ell \in \mathcal{L}_2} S_\ell \leq 1 \tag{MILP4.18}$$

$$\begin{aligned}
T_{j,m,k} \in (0,1) &\qquad \forall j, m, k \\
g_k^i \geq 0; c_k^i \geq 0 &\qquad \forall k, i \\
d_{k,j,m}^i \geq 0 &\qquad \forall k, j, m, i \\
O_k^\ell \in (0,1) &\qquad \forall k, \ell \\
o_k^{\ell,i} \geq 0 &\qquad \forall k, i, \ell \\
S_\ell \in (0,1) &\qquad \forall \ell \in \mathcal{L}_2
\end{aligned}$$

9.2.7 Joint optimization of non-unified transition rules and premiums - multi-period model without profit constraint

The model is presented in section 5.2.

$$\min \sum_{t=0}^{\Theta} \sum_{i=1}^{I} \sum_{k=0}^{K} \phi^i g_{k,t}^i \tag{MILP5.obj}$$

Subject to

$$\sum_{j=\underline{J_k}}^{\overline{J_k}} T_{j,m,k} = 1 \ , \qquad \forall m, k \tag{MILP1.1}$$

$$\sum_{j=\min(\overline{J_k},1)}^{\overline{J_k}} T_{j,0,k} = 1 \ , \qquad \forall k \tag{MILP1.2}$$

$$\sum_{j=\underline{J_k}}^{\max(\underline{J_k},-1)} T_{j,M,k} = 1 \ , \qquad \forall k \tag{MILP1.3}$$

$$\sum_{\ell=j}^{\overline{J_k}} T_{\ell,m,k} \geq T_{j,m+1,k} \qquad \forall j, k, \ m = 0, \ldots, M-1 \tag{MILP1.4}$$

$$\sum_{k=0}^{K} c_{k,t}^i = \phi^i \qquad \forall i, t = 1, \ldots, \Theta \tag{MILP5.5}$$

$$c_{k,t}^i = \sum_{j=-(K-k)}^{k} \sum_{m=0}^{M} d_{k-j,j,m,t-1}^i \qquad \forall i, k, t \tag{MILP5.6}$$

$$d_{k,j,m,t}^i \geq \lambda_m^i c_{k,t}^i - (1 - T_{j,m,k})\phi^i \qquad \forall i, j, k, m, t = 0, \ldots, \Theta - 1 \tag{MILP5.7}$$

$$\sum_{i=1}^{I} c_{k,\Theta}^i \geq \tau \qquad \forall k \tag{MILP5.8}$$

$$\pi_k c_{k,t}^i + \sum_{\ell=1}^{I-1} o_{k,t}^{\ell,i} + g_{k,t}^i \geq \lambda^i c_{k,t}^i \quad \forall i, k, t \tag{MILP5.10}$$

$$\pi_k c_{k,t}^i + \sum_{\ell=1}^{I-1} o_{k,t}^{\ell,i} - g_{k,t}^i \leq \lambda^i c_{k,t}^i \quad \forall i, k, t \tag{MILP5.11}$$

$$\pi_k + \sum_{\ell=1}^{I-1} \varepsilon^\ell O_k^\ell \geq \pi_{k+1} + \sum_{\ell=1}^{I-1} \varepsilon^\ell O_{k+1}^\ell \qquad k = 0, \ldots, K \tag{MILP3.12}$$

$$\sum_{\ell=1}^{L} O_k^\ell \leq 1 \qquad \forall k \tag{MILP3.13}$$

$$o_{k,t}^{\ell,i} \geq \varepsilon^\ell \left(c_{k,t}^i - (1 - O_k^\ell)\right) \qquad \forall i,k,t,\ell = 1,\dots,I-1 \tag{MILP5.14}$$

$$o_{k,t}^{\ell,i} \leq \varepsilon^\ell c_{k,t}^i \qquad \forall i,k,t,\ell = 1,\dots,I-1 \tag{MILP5.15}$$

$$o_{k,t}^{\ell,i} \leq \varepsilon^\ell O_k^\ell \qquad \forall i,k,t,\ell = 1,\dots,I-1 \tag{MILP5.16}$$

$$\sum_{k=0}^{K} B_k = 1 \tag{MILP5.19}$$

$$c_{k,0}^i = B_k \qquad \forall i,k \tag{MILP5.20}$$

$$\begin{aligned}
T_{j,m,k} &\in (0,1) && \forall j,m,k \\
g_{k,t}^i \geq 0; c_k^i &\geq 0 && \forall k,i \\
d_{k,j,m,t}^i &\geq 0 && \forall k,j,m,i \\
O_k^\ell &\in (0,1) && \forall k, \ell = 2,\dots,I-1 \\
o_{k,t}^{\ell,i} &\geq 0 && \forall k,i,\ell = 2,\dots,I-1 \\
B_k &\in (0,1) && \forall k
\end{aligned}$$

9.2.8 Joint optimization of non-unified transition rules and premiums - multi-period model with profit constraint

The model is presented in section 5.2.

$$\min \sum_{t=0}^{\Theta} \sum_{i=1}^{I} \sum_{k=0}^{K} \phi^i g_{k,t}^i \tag{MILP5.obj}$$

Subject to

$$\sum_{j=\underline{J_k}}^{\overline{J_k}} T_{j,m,k} = 1 \ , \qquad \forall m, k \tag{MILP1.1}$$

$$\sum_{j=\min(\overline{J_k},1)}^{\overline{J_k}} T_{j,0,k} = 1 \ , \qquad \forall k \tag{MILP1.2}$$

$$\sum_{j=\underline{J_k}}^{\max(\underline{J_k},-1)} T_{j,M,k} = 1 \ , \qquad \forall k \tag{MILP1.3}$$

$$\sum_{\ell=j}^{\overline{J_k}} T_{\ell,m,k} \geq T_{j,m+1,k} \qquad \forall j, k, \ m = 0, \ldots, M-1 \tag{MILP1.4}$$

$$\sum_{k=0}^{K} c_{k,t}^i = \phi^i \qquad \forall i, t = 1, \ldots, \Theta \tag{MILP5.5}$$

$$c_{k,t}^i = \sum_{j=-(K-k)}^{k} \sum_{m=0}^{M} d_{k-j,j,m,t-1}^i \qquad \forall i, k, t \tag{MILP5.6}$$

$$d_{k,j,m,t}^i \geq \lambda_m^i c_{k,t}^i - (1 - T_{j,m,k})\phi^i \qquad \forall i, j, k, m, t = 0, \ldots, \Theta - 1 \tag{MILP5.7}$$

$$\sum_{i=1}^{I} c_{k,\Theta}^i \geq \tau \qquad \forall k \tag{MILP5.8}$$

$$\sum_{t=0}^{\Theta} \sum_{i=1}^{I} \sum_{k=0}^{K} \left(\pi_k c_{k,t}^i + \sum_{\ell=1}^{L} o_{k,t}^{\ell,i} \right) \geq \sum_{i=1}^{I} (\Theta + 1) \phi^i \lambda^i \tag{MILP6.9.1}$$

$$\pi_k c_{k,t}^i + \sum_{\ell=1}^{L} o_{k,t}^{\ell,i} + g_{k,t}^i \geq \lambda^i c_{k,t}^i \quad \forall i, k, t \tag{MILP5.10}$$

$$\pi_k c_{k,t}^i + \sum_{\ell=1}^{L} o_{k,t}^{\ell,i} - g_{k,t}^i \leq \lambda^i c_{k,t}^i \quad \forall i, k, t \tag{MILP5.11}$$

$$\pi_k + \sum_{\ell=1}^{L} \varepsilon^\ell O_k^\ell \geq \pi_{k+1} + \sum_{\ell=1}^{L} \varepsilon^\ell O_{k+1}^\ell \qquad k = 0, \ldots, K \tag{MILP3.12}$$

$$\sum_{\ell=1}^{L} O_k^\ell \leq 1 \qquad \forall k \tag{MILP3.13}$$

$$o_{k,t}^{\ell,i} \geq \varepsilon^\ell \left(c_{k,t}^i - (1 - O_k^\ell)\right) \qquad \forall i, k, t, \ell = 1, \ldots, L \tag{MILP5.14}$$

$$o_{k,t}^{\ell,i} \leq \varepsilon^\ell c_{k,t}^i \qquad \forall i, k, t, \ell = 1, \ldots, L \tag{MILP5.15}$$

$$o_{k,t}^{\ell,i} \leq \varepsilon^\ell O_k^\ell \qquad \forall i, k, t, \ell = 1, \ldots, L \tag{MILP5.16}$$

$$\sum_{k=0}^{K} O_k^\ell \leq (K+1) S_\ell \qquad \forall \ell \in \mathcal{L}_2 \tag{MILP4.17}$$

$$\sum_{\ell \in \mathcal{L}_2} S_\ell \leq 1 \tag{MILP4.18}$$

$$\sum_{k=0}^{K} B_k = 1 \tag{MILP5.19}$$

$$c_{k,0}^i = B_k \qquad \forall i, k \tag{MILP5.20}$$

$$\begin{aligned}
T_{j,m,k} &\in (0,1) && \forall j, m, k \\
g_{k,t}^i \geq 0; c_k^i &\geq 0 && \forall k, i \\
d_{k,j,m,t}^i &\geq 0 && \forall k, j, m, i \\
O_k^\ell &\in (0,1) && \forall k, \ell = 1, \ldots, L \\
o_{k,t}^{\ell,i} &\geq 0 && \forall k, i, \ell = 1, \ldots, L \\
B_k &\in (0,1) && \forall k
\end{aligned}$$

The following constraint can be used instead of (MILP6.9.1):

$$\sum_{i=1}^{I} \sum_{k=0}^{K} \left(\pi_k c_{k,t}^i + \sum_{\ell=1}^{L} o_{k,t}^{\ell,i} \right) \geq \sum_{i=1}^{I} \phi^i \lambda^i \quad \forall t \tag{MILP6.9.2}$$

9.2.9 Joint optimization of non-unified transition rules, premiums and number of classes - multi-period model

The model is presented in section 5.3.

$$\min \sum_{t=0}^{\Theta} \sum_{i=1}^{I} \sum_{k=0}^{K} \phi^i g^i_{k,t} \qquad \text{(MILP5.obj)}$$

Subject to

$$\sum_{j=\underline{J_k}}^{\overline{J_k}} T_{j,m,k} = 1 \ , \qquad \forall m, k \qquad \text{(MILP1.1)}$$

$$\sum_{j=\min(\overline{J_k},1)}^{\overline{J_k}} T_{j,0,k} = 1 \ , \qquad \forall k \qquad \text{(MILP1.2)}$$

$$\sum_{j=\underline{J_k}}^{\max(\underline{J_k},-1)} T_{j,M,k} = 1 \ , \qquad \forall k \qquad \text{(MILP1.3)}$$

$$\sum_{\ell=j}^{\overline{J_k}} T_{\ell,m,k} \geq T_{j,m+1,k} \qquad \forall j, k,\ m = 0, \dots, M-1 \qquad \text{(MILP1.4)}$$

$$\sum_{k=0}^{K} c^i_{k,t} = \phi^i \qquad \forall i, t = 1, \dots, \Theta \qquad \text{(MILP5.5)}$$

$$c^i_{k,t} + 1 - V_{k+1} + V_k \geq \sum_{j=0}^{\overline{J}} \sum_{\ell=0}^{\min(j,k)} \sum_{m=0}^{M} d^i_{k-\ell,j,m,t-1} \qquad k = 1, \dots, K, \forall i, t \quad \text{(MILP8.6.1)}$$

$$c^i_{k,t} - 1 + V_{k+1} \leq \sum_{j=0}^{\overline{J}} \sum_{\ell=0}^{\min(j,k)} \sum_{m=0}^{M} d^i_{k-\ell,j,m,t} \qquad k = 1, \dots, K, \forall i, t \qquad \text{(MILP8.6.2)}$$

$$c^i_{k,t} + V_{k+1} \geq \sum_{j=\max(\underline{J},-(K-k))}^{\min(\overline{J},k)} \sum_{m=0}^{M} d^i_{k-j,j,m,t-1} \qquad k = 1, \dots, K, \forall i, t \qquad \text{(MILP8.6.3)}$$

$$c^i_{k,t} - V_{k+1} \leq \sum_{j=\max(\underline{J},-(K-k))}^{\min(\overline{J},k)} \sum_{m=0}^{M} d^i_{k-j,j,m,t-1} \qquad k = 1, \dots, K, \forall i, t \ , \quad \text{(MILP8.6.4)}$$

$$d^i_{k,j,m,t} \geq \lambda^i_m c^i_{k,t} - (1 - T_{j,m,k})\phi^i \qquad \forall i, j, k, m, t = 0, \ldots, \Theta - 1 \tag{MILP5.7}$$

$$\sum_{i=1}^{I} c^i_{k,\Theta} \geq \tau \qquad \forall k \tag{MILP5.8}$$

$$\pi_k c^i_{k,t} + \sum_{\ell=1}^{I-1} o^{\ell,i}_{k,t} + g^i_{k,t} \geq \lambda^i c^i_{k,t} \quad \forall i, k, t \tag{MILP5.10}$$

$$\pi_k c^i_{k,t} + \sum_{\ell=1}^{I-1} o^{\ell,i}_{k,t} - g^i_{k,t} \leq \lambda^i c^i_{k,t} \quad \forall i, k, t \tag{MILP5.11}$$

$$\pi_k + \sum_{\ell=1}^{I-1} \varepsilon^\ell O^\ell_k \geq \pi_{k+1} + \sum_{\ell=1}^{I-1} \varepsilon^\ell O^\ell_{k+1} \qquad k = 0, \ldots, K \tag{MILP3.12}$$

$$\sum_{\ell=1}^{L} O^\ell_k \leq 1 \qquad \forall k \tag{MILP3.13}$$

$$o^{\ell,i}_{k,t} \geq \varepsilon^\ell \left(c^i_{k,t} - (1 - O^\ell_k) \right) \qquad \forall i, k, t, \ell = 1, \ldots, I - 1 \tag{MILP5.14}$$

$$o^{\ell,i}_{k,t} \leq \varepsilon^\ell c^i_{k,t} \qquad \forall i, k, t, \ell = 1, \ldots, I - 1 \tag{MILP5.15}$$

$$o^{\ell,i}_{k,t} \leq \varepsilon^\ell O^\ell_k \qquad \forall i, k, t, \ell = 1, \ldots, I - 1 \tag{MILP5.16}$$

$$\sum_{k=0}^{K} B_k = 1 \tag{MILP5.19}$$

$$c^i_{k,0} = B_k \qquad \forall i, k \tag{MILP5.20}$$

$$c^i_{k,t} \leq 1 - V_k \qquad \forall i, k, t \tag{MILP7.21}$$

$$V_k \leq V_{k+1} \quad k = 1, \ldots K \tag{MILP7.22}$$

$$
\begin{aligned}
T_{j,m,k} \in (0,1) &\qquad \forall j, m, k \\
g^i_{k,t} \geq 0; c^i_k \geq 0 &\qquad \forall k, i \\
d^i_{k,j,m,t} \geq 0 &\qquad \forall k, j, m, i \\
O^\ell_k \in (0,1) &\qquad \forall k, \ell = 2, \ldots, I-1 \\
o^{\ell,i}_{k,t} \geq 0 &\qquad \forall k, i, \ell = 2, \ldots, I-1 \\
B_k \in (0,1) &\qquad \forall k \\
B_k \in (0,1) &\qquad k = 1, \ldots, K
\end{aligned}
$$

9.2.10 Joint optimization of unified transition rules, premiums and number of classes - multi-period model

The model is presented in section 5.3.

$$\min \sum_{t=0}^{\Theta} \sum_{i=1}^{I} \sum_{k=0}^{K} \phi^i g^i_{k,t} \tag{MILP5.obj}$$

Subject to

$$\sum_{j=\underline{J}}^{\overline{J}} T_{j,m} = 1 \qquad \forall m \tag{MILP2.1}$$

$$\sum_{j=1}^{\overline{J}} T_{j,0} = 1 \tag{MILP2.2}$$

$$\sum_{j=\underline{J}}^{-1} T_{j,M} = 1 \tag{MILP2.3}$$

$$\sum_{\ell=j}^{\overline{J}} T_{\ell,m} \geq T_{j,m+1} \qquad \forall j,\ m = 0, ..., M-1 \tag{MILP2.4}$$

$$\sum_{k=0}^{K} c^i_{k,t} = \phi^i \qquad \forall i, t = 1, \ldots, \Theta \tag{MILP5.5}$$

$$c^i_{k,t} - V_k \leq \sum_{j=-(K-k)}^{k} \sum_{m=0}^{M} d^i_{k-j,j,m,t-1} \qquad \forall i, k, t \tag{MILP7.6.1}$$

$$c^i_{k,t} + V_k \geq \sum_{j=-(K-k)}^{k} \sum_{m=0}^{M} d^i_{k-j,j,m,t-1} \qquad \forall i, k, t \tag{MILP7.6.2}$$

$$d^i_{k,j,m,t} \geq \lambda^i_m c^i_{k,t} - (1 - T_{j,m,k})\phi^i \qquad \forall i, j, k, m, t = 0, \ldots, \Theta - 1 \tag{MILP5.7}$$

$$\sum_{i=1}^{I} c^i_{k,\Theta} \geq \tau \qquad \forall k \tag{MILP5.8}$$

$$\pi_k c^i_{k,t} + \sum_{\ell=1}^{I-1} o^{\ell,i}_{k,t} + g^i_{k,t} \geq \lambda^i c^i_{k,t} \quad \forall i, k, t \tag{MILP5.10}$$

$$\pi_k c^i_{k,t} + \sum_{\ell=1}^{I-1} o^{\ell,i}_{k,t} - g^i_{k,t} \leq \lambda^i c^i_{k,t} \quad \forall i, k, t \tag{MILP5.11}$$

$$\pi_k + \sum_{\ell=1}^{I-1} \varepsilon^\ell O^\ell_k \geq \pi_{k+1} + \sum_{\ell=1}^{I-1} \varepsilon^\ell O^\ell_{k+1} \qquad k = 0, \ldots, K \tag{MILP3.12}$$

$$\sum_{\ell=1}^{L} O^\ell_k \leq 1 \qquad \forall k \tag{MILP3.13}$$

$$o^{\ell,i}_{k,t} \geq \varepsilon^\ell \left(c^i_{k,t} - (1 - O^\ell_k)\right) \qquad \forall i, k, t, \ell = 1, \ldots, I - 1 \tag{MILP5.14}$$

$$o^{\ell,i}_{k,t} \leq \varepsilon^\ell c^i_{k,t} \qquad \forall i, k, t, \ell = 1, \ldots, I - 1 \tag{MILP5.15}$$

$$o^{\ell,i}_{k,t} \leq \varepsilon^\ell O^\ell_k \qquad \forall i, k, t, \ell = 1, \ldots, I - 1 \tag{MILP5.16}$$

$$\sum_{k=0}^{K} B_k = 1 \tag{MILP5.19}$$

$$c^i_{k,0} = B_k \qquad \forall i, k \tag{MILP5.20}$$

$$c^i_{k,t} \leq 1 - V_k \qquad \forall i, k, t \tag{MILP7.21}$$

$$V_k \leq V_{k+1} \quad k = 1, \ldots K \tag{MILP7.22}$$

$$
\begin{aligned}
T_{j,m,k} &\in (0,1) && \forall j, m, k \\
g_{k,t}^{i} \geq 0; c_k^i &\geq 0 && \forall k, i \\
d_{k,j,m,t}^{i} &\geq 0 && \forall k, j, m, i \\
O_k^{\ell} &\in (0,1) && \forall k, \ell = 2, \ldots, I-1 \\
o_{k,t}^{\ell,i} &\geq 0 && \forall k, i, \ell = 2, \ldots, I-1 \\
B_k &\in (0,1) && \forall k \\
B_k &\in (0,1) && k = 1, \ldots, K
\end{aligned}
$$

9.3 Program for optimizing a BMS

During the research, we implemented the previously presented models in python. While I wrote the dissertation, I had to do additional researches with some of these models. For the sake of simplicity, I connected all of the implemented models in a framework program. This program can be downloaded from: `https://github.com/gyetmarton/BonMal`

9.4 BMS in practice websites

The internet search was concluded in January 2021.

- Austria
  - https://www.oesterreich.gv.at/en/themen/gesundheit_und_notfaelle/unfall/4/Seite.2892002.html
  - https://europa.eu/youreurope/citizens/vehicles/insurance/validity/austria/index_en.htm
- Belgium
  - https://economie.fgov.be/en/themes/consumer-protection/insurance/car/civil-liability/bonus-malus-and-certificate
  - https://www.mon-assurance-auto.be/accident/bonus-malus-auto.html
  - https://europa.eu/youreurope/citizens/vehicles/insurance/validity/belgium/index_en.htm
  - https://www.aginsurance.be/Retail/nl/mobiliteit/auto/Paginas/bonus-malus-schadevrij.aspx
  - https://www.ethias.be/content/campaigns/ethias-campaigns/nl/lp-auto-new.html
  - https://www.mon-assurance-auto.be/accident/joker-assurance-auto.html
- Czech Republic
  - https://europa.eu/youreurope/citizens/vehicles/insurance/validity/czechia/index_en.htm
- Denmark

- https://europa.eu/youreurope/citizens/vehicles/insurance/validity/denmark/index_en.htm

- Finland
  - https://europa.eu/youreurope/citizens/vehicles/insurance/validity/finland/index_en.htm
  - https://www.lvk.fi/en/obligation-to-insure/obligation-to-insure/
  - https://www.if.fi/en/private-customers/insurances/car-insurance
  - https://www.internationallawoffice.com/Newsletters/Insurance/Finland/HPP-Attorneys-Ltd/New-Motor-Liability-Insurance-Act-enters-into-force
  - https://www.nordea.fi/en/personal/our-services/insurance/if-insurance/vehicle-insurance.html
- France
  - https://europa.eu/youreurope/citizens/vehicles/insurance/validity/france/index_en.htm
  - https://www.service-public.fr/particuliers/vosdroits/F2655
- Germany
  - https://europa.eu/youreurope/citizens/vehicles/insurance/validity/germany/index_en.htm
  - https://www.allianz.de/auto/kfz-versicherung/schadenfreiheitsklasse/
  - https://www.axa.de/kfz-versicherung
- Hungary
  - https://www.mnb.hu/fogyasztovedelem/biztositasok/gepjarmu-biztositas/a-bonus-malus-besorolasi-rendszer
- Italy
  - https://europa.eu/youreurope/citizens/vehicles/insurance/validity/italy/index_en.htm

  - \protect\tolerance9999\emergencystretch3em\hfuzz.5\p@\vfuzz\hfuzz{https://www.chiarezza.it/guide/bonus-malus-che-cos-e-e-come-funziona}

- Luxembourg
  - https://www.axa.lu/en/blog-car-no-claims-bonus-scale
  - https://www.lalux.lu/en/info-tools/faq

- Netherlands
  - https://www.centraalbeheer.nl/verzekeringen/autoverzekering/bonus-maluskorting
  - https://www.fbto.nl/autoverzekering/premie-berekenen/stel-samen?utm_campaign=Auto&utm_content=tekstlink&awc=8354_1610448652_c0d2084e6d148bf61db0f3d402b83385&utm_medium=Affiliate&utm_source=Zanox&zanuid=666995&zanpub=httpwwwexpaticacom
  - https://www.aegon.nl/particulier/verzekeren/autoverzekering/schadevrije-jaren
  - https://www.abnamro.nl/en/personal/insurance/car-insurance/claim-free-years.html

- Norway
  - https://eika.no/forsikre/bilforsikring
  - https://www.codanforsikring.no/privat/forsikringer/bilforsikring/bonus
  - https://www.dnb.no/bedrift/forsikring/skadeforsikring/bilforsikring
  - https://www.frende.no/forsikringer/bilforsikring/velg-frende/bonusmesteren/

- Portugal
  - https://europa.eu/youreurope/citizens/vehicles/insurance/validity/portugal/index_en.htm
  - https://www.mapfre.pt/seguros-pt/particulares/automovel/net-auto/
  - https://www.libertyseguros.pt/Produto/Sobre-Rodas/prod-7

- Romania
  - https://asfromania.ro/en/legislation/sectorial-legislation/insurance-reinsurance-market/secondary-legislation-csa/rules-csa/6014-rule-no-20-2017-on-motor-vehicle-insurance-in-romania
  - https://europa.eu/youreurope/citizens/vehicles/insurance/validity/romania/index_en.htm
- Switzerland
  - https://www.helvetia.com/ch/web/en/private-customers/vehicles-and-travel/vehicles/car-insurance/bonus-malus.html

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