Doctoral School of General and
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THESIS ON

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Inconsistency in multi-attribute decision problems

Ph.D. dissertation

Supervisor:
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Department of Operations Research and Actuarial Sciences

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I. Introduction

There are several well-known techniques for solving a Multi-Attribute decision making problem, the Analytic Hierarchy Process (AHP), based on the method of pairwise comparisons, is one of the most widely used methodologies. There has been an intense argument in the academic world about the inconsistency indices derived from pairwise comparisons.

Most papers are based on randomly generated matrices in order to analyse inconsistency. Gass and Standard [5] presented one of the first studies on empirical pairwise comparison matrices and pointed out some important differences between randomly generated and experimental matrices. Therefore the aim of Thesis is to characterize the origin of inconsistency and to define several techniques to decrease the level of inconsistency of pairwise comparison matrices. In order to achieve the goal we need to combine the experimental techniques with mathematical tools.

- There are significant differences between randomly generated and empirical pairwise comparison matrices
- Inconsistency is systematically higher in case of subjective problems
- Some elements may be missing from the matrix without an essential change in the results (the relevance of incomplete pairwise comparison matrices)
- The inconsistency is increasing with the size of the matrix
- Both misprints and other errors of the decision maker can be detected in real time.
- Inconsistency indices defined in several ways are able to identify the same matrix element if it is the only main source of inconsistency

1 Work methods

Graph representation is a convenient tool to analyse the whole structure of 3x3 submatrices (so called triads). By the help of the defined direct graph corresponding the pairwise comparison matrix we can prove the minimal number of inconsistent 3x3 submatrices and also can describe the matrices, which can be made consistent by the modification at most 3 matrix elements.

We introduce a mixed 0-1 linear programming technique to determine the minimal number of matrix elements whose modification makes the inconsistent matrix consistent. This approach can be extended with any inconsistency index (e.g. CR [9], CM [6], GCI [1], [7]) and corresponding threshold to answer the question: what is the minimal number of matrix elements by the modification of which the inconsistency of matrix will be lower than the threshold. The extended mixed 0-1 problem will be nonlinear, however strictly convex, therefore the local optimum is also global optimum.

2 Relevance and applications

A sample of near 600 matrices yields the empirical background of the thesis which guarantees higher accuracy and relevancy. The sample consists of 137 matrices originated from real decisions [8] which were published in academic journals and of another 450 matrices from own experiments with university students [3] where the fill-in order of elements was also documented. The real data will be processed and evaluated using MATLAB MINLP Solver [4] and SPSS.

It can happen in the practice that the decision maker (DM) who performs the pairwise comparisons works basically in a precise and (almost) consistent way, and makes errors only in a few cases. By using the thesis' results we can help the DM to identify the mistakes in the pairwise comparison matrix.

II. Multi-attribute decision problems

In solving a multi-attribute decision problem, one needs to express the importances/weights of the attributes by numbers as well as the evaluations of the alternatives with respect to the attributes. The method of pairwise comparison matrices [35] is one of the most often used techniques. Consider n items (weights of criteria, evaluations of the alternatives with respect to a criterion, or voting powers of individuals in group decision making) to be compared. The decision maker compares each pairs of the items and answers the question like 'How many times one is larger/better than the other one?' An $n \times n$ matrix

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & 1 & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & 1 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & 1 \end{pmatrix}$$

is called pairwise comparison matrix if it is positive and reciprocal, i.e.,

$$\begin{aligned} a_{ij} &> 0, \\ a_{ij} &= \frac{1}{a_{ji}}, \end{aligned}$$

for $i, j = 1, \dots, n$.

A pairwise comparison matrix A is consistent if it satisfies the transitivity property

$$a_{ij} a_{jk} = a_{ik}$$

for any indices $i, j, k,\ (i, j, k = 1, 2, \dots, n)$. Otherwise, A is inconsistent.

There is a number of methods for determining the weights from the pairwise comparison matrix filled in by the decision maker. Eigenvector method [35] and distance minimizing methods such as Least Squares Method [15] are just two of the basic ideas of the approximation of an inconsistent matrix by a consistent one. All weighting methods provide the same result for consistent matrices but not for inconsistent ones. However, in the paper, the focus is rather on the matrices that can be made consistent by modifying 1, 2 or 3 of their elements than on weighting methods.

In real decision problems consistent matrices are rare but it is crucial to detect *high* inconsistencies. Contradictive responses of the decision maker may result in false outcomes. Nevertheless, the definition of the degree of inconsistency is not unique, there exist different measures and indices for it [35, 26, 9]. An alternative way is presented in the paper for finding the minimal number of elements in the pairwise comparison matrix by the modification of which it can be made consistent. Graph representation of pairwise comparison matrices [21, 25] is used in the paper as an efficient tool for a graphical interpretation of decision maker's preferences.

1 Inconsystency

The low level of inconsistency of a PC matrix is a necessary condition to obtain the right results when scores, weights, or preferences are obtained from the PC matrix. Using the definition of the consistent PC matrix several inconsistency indices can be developed.

1.1 Inconsistency index CR of Saaty

Saaty (1980) proposed to index the inconsistency of pairwise comparison matrix A of size $n \times n$ by a positive linear transformation of its largest eigenvalue $\lambda_{\max}$. The normalized right eigenvector associated to $\lambda_{\max}$ also plays an important role, since it provides the estimation of the weights in the eigenvector method. Saaty (1977) showed that $\lambda_{\max} \geq n$ and $\lambda_{\max} = n$ if and only if A is consistent. Let us generate a large number of random pairwise comparison matrices of size $n \times n$, where each element above the main diagonal are chosen from the ratio scale $1/9, 1/8, 1/7, \ldots, 1/2, 1, 2, ..., 8, 9$ with equal probability. Take the largest eigenvalue of each matrix and let $\overline{\lambda_{\max}}$ denote their average value.Let $RI_n = (\overline{\lambda_{\max}} - n)/(n-1)$.

Saaty defined the inconsistency of matrix A as

$$CR_n(A) = \frac{\frac{\lambda_{\max}(A)-n}{n-1}}{RI_n}$$

being a positive linear transformation of $\lambda_{\max}(A)$. Then $CR_n(A) \geq 0$ and $CR_n(A) = 0$ if and only if A is consistent.
CR is widely used and a threshold value of 0.1 (10%) has been accepted in the practice. One of the drawbacks of the index arises from its construction: having $RI_n n$ in the formula CR could not be investigated and interpreted analytically. It has been computationally verified that RI_n depends not only on n, but as well as on the maximal value of the ratio scale, however, it is ultimately irrelevant whether the ratio scale is discrete or continuous (Bozóki and Rapcsák, 2008).

1.2 Inconsistency index CM of Koczkodaj

The inconsistency index introduced by Koczkodaj (1993) is based on 3×3 submatrices, called *triads*. For the 3×3 pairwise comparison matrix

$$\begin{pmatrix} 1 & a & b \\ 1/a & 1 & c \\ 1/b & 1/c & 1 \end{pmatrix} \tag{II.1}$$

let

$$CM(a,b,c) = \min\left\{\frac{1}{a}\left|a - \frac{b}{c}\right|, \frac{1}{b}\left|b - ac\right|, \frac{1}{c}\left|c - \frac{b}{a}\right|\right\}.$$

CM can be extended to larger sizes (Duszak and Koczkodaj, 1994):

$$CM(A) = \max\left\{CM(a_{ij}, a_{ik}, a_{jk}) \middle|\ 1 \leq i < j < k \leq n\right\}. \tag{II.2}$$

Unlike CR_n, the construction above does not contain any parameter depending on n, so we dispense with the use of the notation CM_n. It is easy to see that CM is an inconsistency index since $CM(A) \geq 0$ for any $A \in \mathcal{P}_n$, and $CM(A) = 0$ if and only if A is consistent.

One of the advantages of CM is that it localizes the 'worst' triad in the PC matrix, an appropriate reconsideration of which (if it is possible) decreases the level of inconsistency of the whole PC matrix. CM always ranges from 0 (in case of consistency) to 1; however, intermediate values are not translated into categories. One of the particular disadvantages of CM is that – up to this point – the threshold for acceptance has yet neither been defined nor validated.

1.3 Inconsistent triads

Let A be an $n \times n$ pairwise comparison matrix, and let

$$\bar{A} = \log A$$

denote the $n \times n$ matrix with

$$\bar{a}_{ij} = \log a_{ij}, \quad i, j = 1, \ldots, n.$$

Then A is consistent if and only if

$$\bar{a}_{ij} + \bar{a}_{jk} + \bar{a}_{ki} = 0, \; \forall \, i, j, k = 1, \ldots, n.$$

III. Analysis of pairwise comparison matrices: an empirical research

1 Introduction

This chapter presents results of an experiment on PC matrices conducted to investigate the characteristics of empirical PC matrices. Although previous research has shown the need for a large number of well-documented matrices obtained from a controlled environment, only a few studies have been analyzed empirical PC matrices. Gass and Standard (2002), for instance, highlighted important differences between randomly generated and experimental matrices. Poesz (2008) collected PC matrices from reported real-world applications to analyze their characteristics and drew conclusions about their inconsistency levels. Linares (2009) investigated the inconsistency of experimental pairwise comparison matrices.

The main goal of our research was to analyze the properties of PC matrices with the help of a database which was derived from our experiments. Prior to designing and running the experiments five research questions were formulated. At first, we describe the experiment, then present the results. This will be followed by drawing conclusions and proposing directions for further research.

2 The experiment

The experiment was conducted in 2010 at Corvinus University of Budapest, Hungary. 227 undergraduate and graduate students participated in the experiments. Subjects were 3rd year Bachelor and 1st year Master students of business and economics majors. The mean age was 22, where a low standard deviation reflected having students as subjects. 39% of the subjects were males, and 61% were females. This skewed gender distribution is consistent with the gender distribution of the students at Corvinus University.

The experiments were conducted in class as previously arranged with the professors. One session lasted approximately 25 minutes. First, the professor introduced the experimenters to the students, and he announced that participation is voluntary and could be discontinued at any time. Note, that no student refused participation in any of the sessions.

Participants received the experimental material in a stapled leaflet with a unique identification number. Each page of the leaflet displayed one comparison, thus each comparison was displayed on a separate page; subjects were not allowed to turn back pages. The first page was a practice task, after which students were encouraged to ask (if any) questions. When they finished working they were asked to wait until everyone was done. Each session was closed with debriefing.

In each session the experiment consisted of two subsequent tasks that were designed to test our hypotheses. The design of the test problems and the experimental setting captured the following four dimensions for future analysis:

- type of the problem,
- size of the PC matrix,
- questioning order,
- completeness.

In order to investigate the impact of the type (nature) of the problem the quality of the applied stimuli was categorized into "subjective" and "objective" groups. For the objective stimuli, our subjects were asked to compare countries by their size. First, subjects had to indicate from a presented pair of countries which country is larger. Then, they had to indicate by how much it is larger on a numerical scale. Thus, if one country was judged 30% larger than the other one, it was indicated to be 1.3 times larger. For the subjective stimuli subjects had to compare summer houses. At first they were asked to reveal which summer house they liked more. Then they indicated how much more they liked the preferred house on a numerical scale. For this latter they were as well given a Saaty-scale (Saaty, 1980). Thus, each comparison consisted of a dichotomous, verbal comparison and a subsequent estimation on a numerical scale. Note, that Bozóki and Rapcsák (2008) showed that using Saaty's inconsistency index it is irrelevant if either discrete or continuous scales was used.

Every subject was presented with one subjective (summer houses task) and one objective (country maps task) set of stimuli and the order of the presentation (that is whether subjective stimuli was given first or second) was randomly assigned to each subject. The countries and the summer houses were projected on the screen, which is a usual classroom practice. Note, that we used an imaginary map with irregular contours of the countries.

The second factor administered in the experiments was the size of the matrices. We applied three sets of matrices with the size 4×4, 6×6 and 8×8. Note, that in one session only one size was applied.

When the elements of the PC matrix are determined one by one, we refer to this as a questioning procedure. The third factor was the impact of three different questioning procedures differing in the order of the questions. In the first procedure subjects compared the summer houses and the countries in a sequential order. Country A, for instance, was first compared to country B, then country A was compared to country C, etc. In the second procedure the subjects compared the summer houses and the countries in a random order. For the third procedure we applied the order proposed by Ross [33] . This procedure satisfies two conditions of an optimally balanced comparison. On one hand, it maximizes the distances for the same items to reappear. On the other hand, for every item the number of the first and the second positions in the comparison should be the closest possible. In contrast to sequential order, where, e.g., country A appears in each of the first five questions and it is always in the first position, Ross order balances both the frequency of reappearing and the first/second positions as much as possible. Table III.1 presents examples of filling out 6×6 matrices in each of the detailed questioning orders.

question \ order	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.
sequential	A-B	A-C	A-D	A-E	A-F	B-C	B-D	B-E	B-F	C-D	C-E	C-F	D-E	D-F	E-F
random	A-F	B-E	A-C	F-E	C-D	B-D	B-F	A-E	C-E	A-D	E-D	C-F	B-C	A-D	B-A
Ross	A-B	F-D	E-A	C-B	E-F	A-C	B-D	F-A	D-C	E-B	A-D	C-E	B-F	D-E	C-F

Table III.1: A completed 6×6 matrix applying the three questioning orders

The experiment was a $2(types) \times 3(sizes) \times 3(questioningorders)$ factorial design determined by the three factors described above. There were 9 sessions run all together, with 25 participants on average in each session. Every subject received a set of objective and subjective type of stimuli. Thus, we ended up having a total number of 454 complete PC matrices (see Table III.2 for details).

	objective	subjective	total
number of alternatives	230	224	454
4×4	68	69	137
6×6	80	77	157
8×8	82	78	160
questioning order	230	224	454
sequential	75	75	150
random	77	74	151
Ross	78	75	153

Table III.2: The number of subjects participated in separate experiments

3 Results

3.1 Inconsistency

The first research question focused on the magnitude of inconsistency regarding different types of the decision problems: Does subjective stimuli yields higher inconsistency level than objective stimuli? We predicted that CR indices will be higher for subjective stimuli (i.e., summer houses task) than for objective stimuli (i.e., maps task).

We computed CR and CM inconsistency indices for all obtained complete PC matrices. Table III.3 presents the CR averages and Table III.4 shows the CM averages for all of the matrices, where the discriminating factor was the type of the problem. In both tables a single cell represents the average of 22-27 matrices.

As we predicted we found that CR indices (disregarding order and size) are higher for summer houses than for maps, Mann-Whitney [29]

	summer house			maps		
	4×4	6×6	8×8	4×4	6×6	8×8
sequential	8,10	10,75	12,46	0,67	0,81	1,31
random	10,38	9,47	13,10	0,78	0,86	2,51
Ross	8,75	10,63	13,31	0,70	0,94	1,73
Total	9,06	10,28	12,96	0,71	0,87	1,86

Table III.3: The average of CR indices for complete matrices

$UTest = 2285.00, p \leq 0.001.$

From the same table one can see that for the summer houses the decision makers were close to Saaty's 10% acceptance rule for each size and for each questioning order.

	summer house			maps		
	4×4	6×6	8×8	4×4	6×6	8×8
sequential	0,62	0,79	0,87	0,29	0,45	0,54
random	0,68	0,77	0,86	0,31	0,46	0,57
Ross	0,60	0,82	0,90	0,28	0,46	0,58
Total	0,63	0,79	0,88	0,29	0,46	0,56

Table III.4: The average of CM indices for complete matrices

Table III.4 presents the average CM indices for the same matrices. The CM values are consistent with the tendencies obtained from Table III.3. Interpretation of the magnitude of the CM values and their order can be interpreted similarly to the CR indices, but we have to note that CM threshold for acceptance has only been defined for a very special case (Koczkodaj, Herman and Orlowski, 1997).

Proposition III.1 *The level of inconsistency for the subjective tasks will be systematically greater than for the objective tasks.*

These considerations led us to another research question: Does increased size predict higher inconsistency level?

In Table III.3 we earlier showed that an increase in size is associated with an increase in CR index, and Table III.4 also showed similar pattern for CM index. Regressing the logarithm of CR index on type, order, size and the type× size interaction, we see that the summer house task has a higher mean CR index than the map task, and that increasing matrix sizes lead to the same linear increase in CR index for both tasks (see parameter estimates for this regression in Table III.5). Results of the regression of the logarithm of CM index on the same predictors are similar, with the exception that the type× size interaction is significant. So although both tasks exhibit increasing CM indices with larger matrices, the increase in the summer house task (which again has a higher intercept than the map task) is less severe (See Table III.6 for parameter estimates).

	Estimate	Std error	Wald chi-sq (df)	P-value
type				
map	0,00	n.a.	n.a.	n.a.
summer house	2,34	0,32	53,13 (1)	0,00
order				
random	0,00	n.a.	n.a.	n.a.
sequential	-0,02	0,1	0,05 (1)	0,83
Ross	0,04	0,10	0,15 (1)	0,70
size	0,15	0,04	16,51 (1)	0,00
summer house × size	0,00	0,05	0,00 (1)	0,95

Overall likelihood ratio chi-sq(5) = 477.01, p-value$< 0,001$

Table III.5: Parameter estimates for linear regression of log(CR) index

	Estimate	Std error	Wald chi-sq (df)	P-value
type				
map	0,00	n.a.	n.a.	n.a.
summer house	1,06	0,11	85,09 (1)	0,00
order				
random	0,00	n.a.	n.a.	n.a.
sequential	-0,01	0,04	0,03 (1)	0,86
Ross	0,01	0,04	0,04 (1)	0,84
size	0,16	0,01	162,63 (1)	0,00
summer house x size	-0,07	0,02	16,51 (1)	0,00

Overall likelihood ratio chi-sq(5) = 392.66, p-value $< 0,001$

Table III.6: Parameter estimates for linear regression of log(CM) index

Proposition III.2 *Inconsistency increases as the size of the PC matrix increases.*

We conjectured that one of the questioning orders (e.g. the sequential method) might lead to lower inconsistency than the others. However, data presented in Table III.4 and in Table III.3 do not support our intuition: Table III.5 and Table III.6 further confirm that the questioning order does not have predictive power.

Proposition III.3 *The questioning order does not influence inconsistency.*

3.2 Analyzing the incomplete matrices

Since we recorded every entry that the decision makers made, it is possible to track and analyze the change (if any) in the behavioral consistency of the decision maker. Thus, we can measure/index the inconsistency throughout the procedure and locate the inconsistency. Table 3.2 and Table III.8 present results of such an analysis. The "number of matrix elements" represents the (ordinal) answer number in the sequence. For a 6×6 matrix this range is from 5 to $\frac{n(n-1)}{2} = 15$. The average CR inconsistencies were computed for each stage in the sequence broken down by questioning order. Our research question now: Is the behavior of the decision maker consistent in the course of the entire questioning procedure?

order \ number of matrix elements	5	6	7	8	9	10	11	12	13	14	15
sequential	0,00	0,96	1,82	3,71	4,74	5,66	6,61	7,33	8,35	9,21	10,75
random	0,00	1,38	2,77	3,49	4,42	4,97	6,25	6,91	8,17	8,19	9,47
Ross	0,00	1,37	2,5	3,84	4,93	5,45	6,27	7,24	7,85	9,52	10,63

Table III.7: The average of CR inconsistencies (in %) for 6×6 incomplete matrices: summer houses

order \ number of matrix elements	5	6	7	8	9	10	11	12	13	14	15
sequential	0,00	0,13	0,18	0,25	0,32	0,4	0,48	0,55	0,64	0,72	0,81
random	0,00	0,06	0,11	0,2	0,4	0,5	0,57	0,65	0,71	0,72	0,8
Ross	0,00	0,07	0,14	0,23	0,31	0,37	0,51	0,69	0,73	0,83	0,88

Table III.8: The average of CR inconsistencies (in %) for 6×6 incomplete matrices: maps

Table 3.2 and Table III.8 suggest that from 0 to the final CR value the averages for both types show an almost linear increase as the sequence progresses. To examine the consistency of the CR index during completion, we performed a mixed-effects linear regression of partial CR index on the fixed terms shown in Table III.9, and also random, per-subject slope and intercept terms, to account for the correlations within each subject's responses and model natural variation between subjects. A likelihood ratio chi-squared test of the significance of the random effects shows them to be significant (chi-squared(3) = 2895; p-value < 0.001). Table III.9 shows that as subjects progress through the sequence, the CR index does increase linearly (visual inspection of the data hinted at a possible quadratic component, but this was not statistically significant). This positive association between sequence number and partial CR index is even greater in the summer house task, but it is somewhat dampened by sequential ordering or as matrix dimension increases. We can also see that random ordering gave greater partial CR indices in the map task, but this order effect was largely absent in the summer house task. And, as expected, larger matrices lead to higher partial CR indices – but, as above, this effect is less for the summer house task.

	Estimate	Std. error	T-value (df)	P-value
sequence number	0,02	0,00	10,23 (5463)	0,00
sequence number^2	0,00	0,00	0,32 (5463)	0,75
T=type				
T(map)	0,00	N/A	N/A	N/A
T(summer house)	0,07	0,00	15,63 (5463)	0,00
K=order				
K(random)	0,00	N/A	N/A	N/A
K(sequential)	-0,001	0,00	-2,71 (5463)	0,01
K(Ross)	-0,01	0,00	-5,12 (5463)	0,00
M=size	0,01	0,00	7,04 (5463)	0,00
sequence number × T(summer house)	0,001	0,00	39,74 (5463)	0,00
sequence number × K(Ross)	0,00	0,00	0,86 (5463)	0,40
sequence number × K(sequential)	-0,00	0,00	-2,77 (5463)	0,01
T(summer house) × K(Ross)	0,02	0,00	7,08 (5463)	0,00
T(summer house) × K(sequential)	0,00	0,00	3,50 (5463)	0,00
sequence number × M	-0,00	0,00	-8,66 (5463)	0,00
T(summer house) × M	-0,02	0,00	-20,51 (5463)	0,00

Overall likelihood ratio chi-sq test for the significance of the random effects gave $\chi^2(3) = 2895$, p-érték$< 0,001$

Table III.9: Parameter estimates for mixed effect linear regression of partial CR index

Yet another way to test the behavioral inconsistency is to trace the score vectors and the corresponding ordering as completion progresses (i.e. using the subsequent incomplete matrices).

First we calculated the score vectors for each level of completion in the sequence and compared them to the final score vector calculated from the corresponding complete PC matrix. Two principles were applied for the comparison of two vectors: cardinal and ordinal. Cardinal view (and Table) treats score vectors as elements of the n dimensional Euclidean space, where closeness or similarity of two vectors can be measured by, e.g., Euclidean distance. Ordinal view (Table III.10 and Table 1

type \ number of matrix elements	5	6	7	8	9	10	11	12	13	14	15
summer houses	0,82	0,88	0,90	0,92	0,93	0,94	0,96	0,97	0,97	0,98	1,00
maps	0.99	0.99	0.99	1,00	1,00	1,00	1,00	1,00	1,00	1,00	1,00

Table III.10: Spearman rank correlation coefficients for 6×6 incomplete matrices

Proposition III.4 *Incomplete PC matrices can be used to approximate the final results of all pairwise comparisons.*

(a) Number of comparisons: 10

	1	2	3	4	5	6
1	99	1	0	0	0	0
2	1	99	0	0	0	0
3	0	0	99	1	0	0
4	0	0	1	99	0	0
5	0	0	0	0	99	1
6	0	0	0	0	1	99

(b) Number of comparisons: 14

	1	2	3	4	5	6
1	100	0	0	0	0	0
2	0	100	0	0	0	0
3	0	0	100	0	0	0
4	0	0	0	100	0	0
5	0	0	0	0	100	0
6	0	0	0	0	0	100

Table III.11: Position of the elements of 6×6 complete PC matrices after 10 and 14 comparisons in %, maps

Consider a 6×6 incomplete matrix of a map task after 10 comparisons. The elements of the first row of Table III.11 (a) show that 99% of the alternatives which should be on the first place (according to the known final order) are in fact on the first place after completing 10 pairwise comparisons, and only 1% is on the second place. From the second row we can see that 99% of the alternatives which should be on the second place are in fact in this position after 10 questions, again.

Table III.11 and III.12 suggests that we can use the incomplete matrices for approximation purposes. Executing the entire questioning procedure seems to

(a) Number of comparisons: 10

	1	2	3	4	5	6
1	83	16	0	1	0	0
2	15	65	18	1	0	1
3	1	17	73	9	0	0
4	0	2	8	81	8	1
5	1	0	1	8	73	17
6	0	0	0	0	19	81

(b) Number of comparisons: 14

	1	2	3	4	5	6
1	92	7	1	0	0	0
2	7	84	8	0	0	1
3	1	9	90	0	0	0
4	0	0	1	98	1	0
5	0	0	0	2	95	3
6	0	0	0	0	4	96

Table III.12: Position of the elements of 6×6 complete PC matrices after 10 and 14 comparisons in %, summer houses

be unnecessary: we are able to receive a fairly good estimation of the scores and/or rankings after having a certain number of pairwise comparisons completed. In addition we have also learnt that for objective tasks a significantly fewer pairwise comparisons are required to obtain a good estimation than for subjective tasks. This finding sets clear future research agenda, namely to determine the minimum required number of pairwise comparisons in order to obtain reliable estimations.

Proposition III.5 *Incomplete PC matrices can be used to approximate the final results of all pairwise comparisons.*

Furthermore, we assigned CR values to each of these thirty-eight matrices. Table III.13 shows that most of the matrices with high numbers of intransitive triads (from 3 to 7) have CR values above the 10% threshold – as we expected. On the other hand, ten of the PC matrices with CR values below the 10% threshold have 1 or 2 intransitive triads. It could be in the interest of our further research to analyze the properties of these matrices.

As we mentioned in the introduction highly inconsistent matrices can be corrected. Intransitive triads indicate that the source of inconsistency is the presence of one or more outliers. Bozóki, Fülöp and Poesz [6] proposed a procedure to make these matrices consistent by changing their element(s) and they also determined the number of elements that are necessary to be modified to eliminate inconsistency. In our current database there are 47 of those matrices in which inconsistency can be eliminated by modifying maximum 2 elements (5 of them was consistent without any changes). However, "eliminating the inconsistency" may potentially lead to controversial results, e.g. it

CR (%)	number of intransitive triads							Total
	1	**2**	**3**	**4**	**5**	**6**	**7**	
0 - 5	3	1						4
5 - 10	4	1			1			6
10 - 20	9	2		2				13
20 - 40	4	1	2	1		3		11
- 40		2				1	1	4
Total	20	7	2	3	1	4	1	38

Table III.13: CR values and the number of intransitive triads

is possible that the priorities at the end of this process would distract the DM from the real priorities. One has to be cautious in applying correction methods without confirming its results with the DM. An "automatic" execution of elimination can change the real goal of the decision maker and/or change his real preferences. Thus, correction methods can only be cautiously applied and only with the approval of the decision maker.

IV. Matrices that can be made consistent by the modification of a few elements

Introduce the directed graph $G = \{\mathcal{N}, \mathcal{A}\}$ where $\mathcal{N} = \{1, \ldots, n\}$ is the set of the nodes and $\mathcal{A} = \{(i,j) \mid i,j \in \mathcal{N}, i \neq j\}$ is the set of the edges. Let weights be associated with the edges of graph G, namely, weight $\bar{a}_{ij}$ with the edge (i,j).

Let i, j and k be three different nodes of G. The cycle consisting of the three connecting edges $(i,j),(j,k),(k,i)$ is called a triad, denoted by (i,j,k). The weight $w(i,j,k)$ of triad (i,j,k) is defined by

$$w(i,j,k) = \bar{a}_{ij} + \bar{a}_{jk} + \bar{a}_{ki}.$$

It is clear that

$$w(i,j,k) = w(j,k,i) = w(k,i,j) = -w(k,j,i) = -w(j,i,k) = -w(i,k,j),$$

furthermore, matrix A is consistent exactly when the weight of all triads is zero in the graph G associated with A. A triad is called consistent if its weight is zero, otherwise, it is called inconsistent. We call the graph G also consistent when all of its triads are consistent. In the sequel, when dealing with the number of the inconsistent triads, we consider the triads (i,j,k), (j,k,i), (k,i,j), (k,j,i), (j,i,k), (i,k,j) as identical, and count them only once, since they are based on the same triple of nodes, and they are consistent or inconsistent simultaneously.

It is evident that for a consistent matrix A, the graph G does not contain any inconsistent triad. However, as shown below, for an inconsistent A, the graph G contains at least $n-2$ inconsistent triads.

Proposition IV.1 *Let (i,j,k) be an inconsistent triad. Then for any $l \in \mathcal{N} \setminus \{i,j,k\}$, at least one of the triads (l,i,j), (l,j,k) and (l,k,i) is inconsistent.*

Proof: It is easy to see, as shown in Figure 1, that

$$w(l,i,j) + w(l,j,k) + w(l,k,i) = w(i,j,k).$$

Note that Figure 1 shows the subgraph of G consisting only of the nodes and edges needed in the proof. Since $w(i,j,k) \neq 0$, at least one of the other three triads must have a nonzero weight.

$\square$

Since the node $l \in \mathcal{N} \setminus \{i,j,k\}$ can be chosen in $n-3$ ways, and (i,j,k) is inconsistent, we obtain directly:

Corollary IV.1 *If A is inconsistent, then G contains at least $n-2$ inconsistent triads.*

Corollary IV.2 *If A is inconsistent, then for any $i \in \mathcal{N}$, G contains an inconsistent triad (i,j,k).*

A direct practical application of Corollary IV.2 is the following: when we want to check whether the pairwise comparison matrix A is consistent or not, instead of checking the consistency of $\binom{n}{3}$ triads, it is enough to do that for $\binom{n-1}{2}$ triads.

Corollary IV.2 has the further meaning that the inconsistency of a triad spreads over, namely, any alternative (or criterion) taking role in the pairwise comparison cannot elude the effect of the inconsistency among any three alternatives (or criterion). This is why it is so difficult to find a cause of the inconsistency in a pairwise comparison matrix.

1 The minimal number of elements to be modified

Two approaches will be proposed. The first one constructs a mixed 0-1 programming problem to answer the question how can an inconsistent pairwise comparison matrix be made consistent by modifying the minimal number of its elements. The second approach is based on elementary, graph theoretic analysis of the graph G.

Assume that an $M \geq 1$ is given serving as an upper bound on the values of the elements in the original and the modified pairwise comparison matrices, i.e. we have

$$a_{ij} \leq M, \; i,j = 1,\dots,n \tag{IV.1}$$

for the elements of A, and we want the modified matrix also with this property. In the practice, this is not a serious restriction since an interval of the reasonable values is usually known. In the Analytic Hierarchy Process M is set to 9, however, it is defined only for the elements given by the decision maker. Let $\bar{M} = \log M$, this is an upper bound on the absolute value of the logarithms of the original and the modified elements. Consider the following optimization problem:

$$\begin{array}{ll} \min & \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} y_{ij} \\ \text{s.t.} & x_{ij} + x_{jk} + x_{ki} = 0, \ \forall \{i,j,k\} \subset \mathcal{N}, \ \mid \{i,j,k\} \mid = 3, \\ & x_{ij} = -x_{ji}, \ i = 1,\dots,n-1, \ j = i+1,\dots,n, \\ & -\bar{M} \le x_{ij} \le \bar{M}, \ i = 1,\dots,n-1, \ j = i+1,\dots,n, \\ & -2\bar{M} y_{ij} \le x_{ij} - \bar{a}_{ij} \le 2\bar{M} y_{ij}, \ i = 1,\dots,n-1, \ j = i+1,\dots,n, \\ & y_{ij} \in \{0,1\}, \ i = 1,\dots,n-1, \ j = i+1,\dots,n, \end{array} \tag{IV.2}$$

where $x_{ij}, i, j = 1,\dots,n, \ i \ne j$, are continuous variables and denote the logarithms of the elements in the modified matrix, $y_{ij}, \ i = 1,\dots,n-1, \ j = i+1,\dots,n$, are binary variables meaning that the modification is allowed in the position (i,j) (then $y_{ij} = 1$) or not (then $y_{ij} = 0$). The following statement is evident:

Proposition IV.2 *The optimal value of problem (IV.2) gives the minimal number of the elements that can be modified to make the pairwise comparison matrix A consistent assuming (IV.1) for A and requiring (IV.1) for the modified matrix.*

If we only want to know whether the matrix A can be made consistent by modifying at most K of its elements, then the constraint

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} y_{ij} \le K \tag{IV.3}$$

is to be added to (IV.2), and it is enough to search only for a feasible solution of (IV.2)-(IV.3).

The practical computational application of the above approach necessitates that an optimization software capable to solve problems (IV.2) or (IV.2)-(IV.3) be callable from the decision support system. An integer programming method of general purpose can solve problems (IV.2) and (IV.2)-(IV.3) with $\binom{n}{2}$ binary variables in an exponential number of steps.

2 Graph theoretic approach

2.1 The case of single modification

Proposition IV.3 *An inconsistent pairwise comparison matrix A can be made consistent by modifying a single element if and only if the corresponding graph G contains exactly $n-2$ inconsistent triads. If $n \geq 4$, then the modification, if any, is unique.*

Proposition IV.4 *Let A be a pairwise comparison matrix obtained from a consistent pairwise comparison matrix by modifying K elements (and their reciprocals). Then the graph G associated with A contains at most $K(n-2)$ inconsistent triads.*

In the light of Propositions IV.3 and IV.4, it may arise the conjecture that an inconsistent pairwise comparison matrix A can be made consistent by modifying at most K elements if and only if the associated graph G contains at most $K(n-2)$ inconsistent triads. This conjecture is however not true as shown in the next example.

Example IV.1 *Let $n = 4$ and*

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 5 \\ 1 & 1/3 & 1 & 7 \\ 1 & 1/5 & 1/7 & 1 \end{pmatrix}.$$

All the four triads of the graph G associated with A are inconsistent, in addition, their weights are different. By Proposition IV.3, it is clear that A cannot be made consistent by modifying a single element. Since for $K = 2$ we have $K(n-2) \leq 4$, if the conjecture was true, then A would be made consistent by modifying two elements. However, after having modified the weight of any of the edges of G, at least three inconsistent triads remain, and they cannot be corrected by modifying the weight of a further edge.

As the example above shows, merely the number of the inconsistent triads does not yield a sufficient condition. The connection of the inconsistent triads, their weights and the relations among them are also to be taken into account. Proposition IV.3 is rephrased in terms of this remark, and its proof comes directly from that of Proposition IV.3.

Proposition IV.5 *An inconsistent pairwise comparison matrix A can be made consistent by modifying a single element if and only if there exists an edge (i, j) in the associated graph G such that the weight of all triads (l, i, j), $l \in \mathcal{N} \setminus \{i, j\}$ is the same nonzero value, and all the other triads are consistent. For the edge (i, j) with this property, the modification is unique. If $n \geq 4$, then there exits at most one edge (i, j) with this property.*

2.2 Modification of two elements

Proposition IV.6 *An inconsistent pairwise comparison matrix A can be made consistent by modifying two elements if and only if exactly one of the following two conditions holds in the graph G associated with A:*

1. *There are two independent edges (i_1, j_1) and (i_2, j_2), and nonzero values α_1 and α_2 such that $w(l, i_1, j_1) = \alpha_1$ for all $l \in \mathcal{N} \setminus \{i_1, j_1\}$, $w(l, i_2, j_2) = \alpha_2$ for all $l \in \mathcal{N} \setminus \{i_2, j_2\}$, and all other triads are consistent.*

2. *There are two connected edges (i, j) and (j, k), and nonzero values α_1 and α_2 such that $w(l, i, j) = \alpha_1$ and $w(l, j, k) = \alpha_2$ for all $l \in \mathcal{N} \setminus \{i, j, k\}$, $w(i, j, k) = \alpha_1 + \alpha_2$, and all other triads are consistent.*

If $n \geq 4$, then for any pair of edges fulfilling conditions 1 or 2, the modification of the weights of the edges that makes the graph G consistent is unique. If $n \geq 5$, then there exists at most one pair of edges fulfilling condition 1. If $n \geq 6$, then there exists at most one pair of edges fulfilling condition 2.

Remark IV.1 *If $n = 4$, then the number of the pairs of edges fulfilling conditions 1 or 2 may not be unique. This means that an inconsistent pairwise comparison matrix A can be altered into different consistent forms by modifying two elements (and their reciprocals). For example, condition 1 holds for the graph G associated with the inconsistent pairwise comparison matrix*

$$\begin{pmatrix} 1 & a & 1 & 1 \\ 1/a & 1 & 1 & 1 \\ 1 & 1 & 1 & 1/a \\ 1 & 1 & a & 1 \end{pmatrix}, \tag{IV.4}$$

where $a \neq 1$, and (IV.4) can be altered into the different consistent form

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & a & a & 1 \\ 1/a & 1 & 1 & 1/a \\ 1/a & 1 & 1 & 1/a \\ 1 & a & a & 1 \end{pmatrix}$$

by modifying two elements (and their reciprocals). Similarly, condition 2 holds for the graph G associated with the inconsistent pairwise comparison matrix

$$\begin{pmatrix} 1 & a & 1 & b \\ 1/a & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1/b & 1 & 1 & 1 \end{pmatrix}, \tag{IV.5}$$

where $a \neq 1$, $b \neq 1$, $a \neq b$, and (IV.5) can be altered into the different consistent forms

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & a & a & a \\ 1/a & 1 & 1 & 1 \\ 1/a & 1 & 1 & 1 \\ 1/a & 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & b & b & b \\ 1/b & 1 & 1 & 1 \\ 1/b & 1 & 1 & 1 \\ 1/b & 1 & 1 & 1 \end{pmatrix}$$

by modifying two elements (and their reciprocals).

There are four inconsistent triads in (IV.4) and three in (IV.5), thus, according to Proposition IV.3, neither (IV.4) or (IV.5) can be made consistent by modifying a single element and its reciprocal. For $n = 4$, it can be shown that the maximal number of the different pairs of edges fulfilling condition 1 is two, and this number is three for condition 2. The proof, based on enumeration of the possible cases and simple arithmetics, is left to the reader.

Remark IV.2 *If $n = 5$, then the number of the pairs of edges fulfilling condition 2 may not be unique. For example, condition 2 holds for the graph G*

associated with the inconsistent pairwise comparison matrix

$$\begin{pmatrix} 1 & a & a & 1 & 1 \\ 1/a & 1 & 1 & 1 & 1 \\ 1/a & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \tag{IV.6}$$

where $a \neq 1$, and (IV.6) can be altered into the different consistent forms

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & a & a & a & a \\ 1/a & 1 & 1 & 1 & 1 \\ 1/a & 1 & 1 & 1 & 1 \\ 1/a & 1 & 1 & 1 & 1 \\ 1/a & 1 & 1 & 1 & 1 \end{pmatrix}$$

by modifying two elements (and their reciprocals). Since there are four inconsistent triads in (IV.6), it cannot be made consistent by modifying a single element and its reciprocal. For $n = 5$, it can be shown that the maximal number of the different pairs of edges fulfilling condition 2 is two. The proof is left again to the reader.

Remark IV.3 *To perform the operations according to Proposition IV.6, we have first to prepare the list of the inconsistent triads. This can be done with $O(n^3)$ operations. If the number of the inconsistent triads is less than $2(n-2)-2$ or greater than $2(n-2)$, then it is sure that there is not any pair of edges fulfilling conditions 1 or 2 of Proposition IV.6. Otherwise, from the list of the edges appearing in the list of the inconsistent triads, we can prepare a list of $O(n^2)$ pairs of edges as candidates to fulfill conditions 1 or 2. For each of these pairs, we can check condition 1 if the two edges are independent, or condition 2 if they are connected with $O(n)$ operations. To check that only triads based on at least one of the two edges can be found in the list of the inconsistent triads, $O(n)$ further operations are needed. Altogether, the pairs of edges fulfilling conditions 1 or 2, if any, and how to modify can be determined with $O(n^3)$ operations. Remember, however, that even if the number of the inconsistent triads is $2(n-2)-2$, $2(n-2)-1$ or $2(n-2)$, it may happen that there is not any pair of edges fulfilling conditions 1 or 2 of*

Proposition IV.6, as shown in Example 2.

2.3 Modification of three elements

Proposition IV.7 *An inconsistent pairwise comparison matrix A can be made consistent by modifying three elements if and only if at least one of the following five conditions holds in the graph G associated with A:*

1. *There are independent edges (i_t, j_t) and nonzero values α_t, $t = 1, 2, 3$, such that $w(l, i_t, j_t) = \alpha_t$ for all $l \in \mathcal{N} \setminus \{i_t, j_t\}$, $t = 1, 2, 3$, and all other triads are consistent.*

2. *There are edges $(i_1, j_1), (i_2, j_2), (j_2, k_2)$, where $\mid \{i_1, j_1, i_2, j_2, k_2\} \mid= 5$, and nonzero values $\alpha_1, \alpha_2, \alpha_3$ such that $w(l, i_1, j_1) = \alpha_1$ for all $l \in \mathcal{N} \setminus \{i_1, j_1\}$, $w(l, i_2, j_2) = \alpha_2$ and $w(l, j_2, k_2) = \alpha_3$ for all $l \in \mathcal{N} \setminus \{i_2, j_2, k_2\}$, $w(i_2, j_2, k_2) = \alpha_2 + \alpha_3$, and all other triads are consistent.*

3. *There are edges $(i, j), (j, k), (k, s)$, where $\mid \{i, j, k, s\} \mid= 4$, and nonzero values $\alpha_1, \alpha_2, \alpha_3$ such that $w(l, i, j) = \alpha_1$ for all $l \in \mathcal{N} \setminus \{i, j, k\}$, $w(l, j, k) = \alpha_2$ for all $l \in \mathcal{N} \setminus \{i, j, k, s\}$, $w(l, k, s) = \alpha_3$ for all $l \in \mathcal{N} \setminus \{j, k, s\}$, $w(i, j, k) = \alpha_1 + \alpha_2$, $w(j, k, s) = \alpha_2 + \alpha_3$, and all other triads are consistent.*

4. *There is a triad (i, j, k) and nonzero values $\alpha_1, \alpha_2, \alpha_3$ such that $w(l, i, j) = \alpha_1$, $w(l, j, k) = \alpha_2$ and $w(l, k, i) = \alpha_3$ for all $l \in \mathcal{N} \setminus \{i, j, k\}$, $w(i, j, k) = \alpha_1 + \alpha_2 + \alpha_3$, and all other triads are consistent.*

5. *There are different edges $(i, j), (i, k), (i, s)$ and nonzero values $\alpha_1, \alpha_2, \alpha_3$ such that $w(l, i, j) = \alpha_1$, $w(l, i, k) = \alpha_2$ and $w(l, i, s) = \alpha_3$ for all $l \in \mathcal{N} \setminus \{i, j, k, s\}$, $w(i, j, k) = \alpha_1 - \alpha_2$, $w(i, k, s) = \alpha_2 - \alpha_3$, $w(i, s, j) = \alpha_3 - \alpha_1$, and all other triads are consistent.*

For any triple of edges fulfilling one of the conditions 1 through 5, the modification of the weights of the edges that makes the graph G consistent is unique except for condition 4 in case of $n = 3$, and condition 5 in case of $n = 4$, when there is and there may be, respectively, an infinite number of possible modifications. If $n \geq 6$, then at most one of conditions 1 through 5 *holds, furthermore, there exists at most one triple of edges fulfilling any of conditions 1 through 4. If* $n \geq 8$*, then there exists at most one triple of edges fulfilling condition 5.*

V. Condition of Order Preservation (COP)

1 Introduction

$A = [a_{ij}]$ is a pairwise comparison matrix and in the following we use the logarithm of this matrix which is defined as $\tilde{a}_{ij} = log(a_{ij}), \forall i, j$. Let us introduce the $z^{em} = log(w^{em})$, where the w^{em} is weight vector derived from the EM method. Furthermore M is a scalar with great value, for instance $M = log(10^6)$.

Consider the following Mixed Integer Programing model:

$$\begin{array}{lrcl}
\min & \sum \xi_j + \sum Y_{ijkl} & & \\
\text{s.t.} & \tilde{a_{kl}} & \leq & MY_{ijkl} + \tilde{a_{ij}}, \quad \forall i, j, k, l \\
& z_j^{em} - z_j^* & \leq & \xi_j, \quad \forall j \\
& z_j^{em} - z_j^* & \geq & -\xi_j, \quad \forall j \\
& z_i^* - z_j^* & \leq & z_k^* - z_l^* + M(1 - Y_{ijkl}), \quad \forall i, j, k, l \\
& z_i^* - z_j^* & \geq & z_k^* - z_l^* - MY_{ijkl}, \quad \forall i, j, k, l \\
& Y_{ijkl} & = & \{0, 1\} \quad \forall i, j, k, l
\end{array} \tag{V.1}$$

where $z_j^*,\ i = 1 \dots n$ are continuous variable and denote the logarithms of the elements in the weight vector, $Y_{ijkl},\ i = 1 \dots n$ are a binary variables meaning that $a_{ij} \leq a_{kl}$.

It is possible to rewrite the first problem into a LP-based form:

$$\begin{array}{lrcl}
\min & \sum \xi_j + \sum Y_{ijkl} & & \\
\text{s.t.} & z_j^{em} - z_j^* & \leq & \xi_j, \quad \forall j \\
& z_j^{em} - z_j^* & \geq & -\xi_j, \quad \forall j \\
& z_i^* - z_j^* & \leq & z_k^* - z_l^* + M(1 - C_{ijkl}) + Y_{ijkl}, \quad \forall i, j, k, l \\
& z_i^* - z_j^* & \geq & z_k^* - z_l^* - MC_{ijkl} - Y_{ijkl}, \quad \forall i, j, k, l \\
& Y_{ijkl} & \geq & 0 \quad \forall i, j, k, l
\end{array} \tag{V.2}$$

where C_{ijkl} is defined as,

$$C_{ijkl} = \begin{cases} 0, & if\ \tilde{a_{ij}} \geq \tilde{a_{kl}}, \\ 1, & otherwise \end{cases}$$

The NLP-based model with average values in objective function.

$$\bar{\xi_j} = \sum \xi_j \frac{1}{n}$$
$$\bar{Y_{ijkl}} = \sum Y_{ijkl} \frac{2}{v(v-1)}$$
$$v = n(n-1)/2$$

$$\begin{array}{lrcl}
\min & (1-\alpha)\ \bar{\xi_j} + \alpha\ \bar{Y_{ijkl}} & & \\
\text{s.t.} & z_j^{em} - z_j^* & \leq & \xi_j, \quad \forall j \\
& z_j^{em} - z_j^* & \geq & -\xi_j, \quad \forall j \\
& z_i^* - z_j^* & \leq & z_k^* - z_l^* + M(1 - C_{ijkl}) + Y_{ijkl}, \quad \forall i,j,k,l \\
& z_i^* - z_j^* & \geq & z_k^* - z_l^* - MC_{ijkl} - Y_{ijkl}, \quad \forall i,j,k,l \\
& exp(\sum z_j^*) & = & 1, \quad \forall j \\
& Y_{ijkl} & \geq & 0 \quad \forall i,j,k,l
\end{array} \tag{V.3}$$

1.1 Logarithmic Least Squares Method (LLS) + COP

$$\begin{array}{lrcl}
\min & \sum_{i=1}^{n} \sum_{j=1}^{n} \Delta_{ij}^2 & & \\
\text{f.h.} & \bar{a}_{ij} - (z_i - z_j) & = & \Delta_{ij},\ \forall i,j \\
& \sum_{i=1}^{n} z_i & = & 0
\end{array} \tag{V.4}$$

$$COP: \quad C_{ijkl}(-z_i + z_j + z_k - z_l) \ < \ 0, \ \forall i,j,k,l$$

1.2 Logarithmic Least Absolute Error (LLAE)+ COP

$$
\begin{array}{llcl}
\min & \sum_{i=1}^{n}\sum_{j=1}^{n}\Delta_{ij} & & \\
\text{f.h.} & -\bar{a}_{ij}+(z_i-z_j) & \leq & \Delta_{ij},\ \forall i,j \\
 & \bar{a}_{ij}-(z_i-z_j) & \leq & \Delta_{ij},\ \forall i,j \\
 & \sum_{i=1}^{n} z_i & = & 0 \\
\hline
COP: & C_{ijkl}(-z_i+z_j+z_k-z_l) & < & 0,\ \ \forall i,j,k,l
\end{array}
\tag{V.5}
$$

1.3 $EM + COP$

$$
\begin{array}{llcl}
\min\ \ \lambda & & & \\
\text{f.h.} & \sum_{j=1}^{n} e^{\bar{a}_{ij}+z_j-z_i} & \leq & \lambda,\ \forall i \\
\hline
COP & C_{ijkl}(-z_i+z_j+z_k-z_l) & < & 0,\ \ \forall i,j,k,l
\end{array}
\tag{V.6}
$$

The simulation is made up by the following steps:

- First of all it is necessary to generate fully consistent matrices which elements come from the AHP scale. This ratio scale of measurement consists of the integer numbers from 1 to 9 and there reciprocal values.
- The next step is the introduction of the uncertainty. In this step we create from the consistent matrices an inconsistent one. But it is very crucial to save the original weight of alternatives.
- The sample which consist of 1000 matrices in each matrix size (4,5,6) is generated by using the above mentioned steps.
- Each matrix is evaluated by the help of EM method as well as EM with additional constrains (the Condition of order preservation) and then I

calculated the norm of the deviation vector and the rank reversal. In this case the deviation vector is defined as the difference between the original (originated from a fully consistent matrix) and the calculated weight vectors.

- According to the simulation we have studied whether the EM with COP condition could provide better solution than the EM method. Unfortunately, there are just some matrices in which the EM method made rank reversal but the EM+COP did not. Therefore I am not sure that it is worth or not to follow the research in these theme (low percentage means 0.1% in case of 4 and 5 dimensional matrices and 0% in 6 dimension).

In this section I examine whether there is a strong link between the uncertainty parameter and the CR inconsistency. The matrix element with uncertainty is defined by the uncertainty parameter (ϵ_{ij}) as the following multiplicative way $a_{ij} = a^o_{ij}\epsilon_{ij}$, where a^o_{ij} is the element of the original consistent matrix. Furthermore the reciprocal values are obtained $a_{ji} = a^o_{ji}\frac{1}{\epsilon_{ij}} = \frac{1}{a_{ij}}$ to ensure the reciprocity feature of the pairwise comparison matrices. The uncertainty parameter follows lognormal distribution, $log(\epsilon_{ij}) \sim N(0, \sigma^2)$. So we can generate it by the help of one normal distributed variable,

$$\begin{aligned} \epsilon_{ij} &= exp(v_{ij}),\ \forall i, j \\ v_{ij} &\sim N(0, \sigma^2) \end{aligned}$$

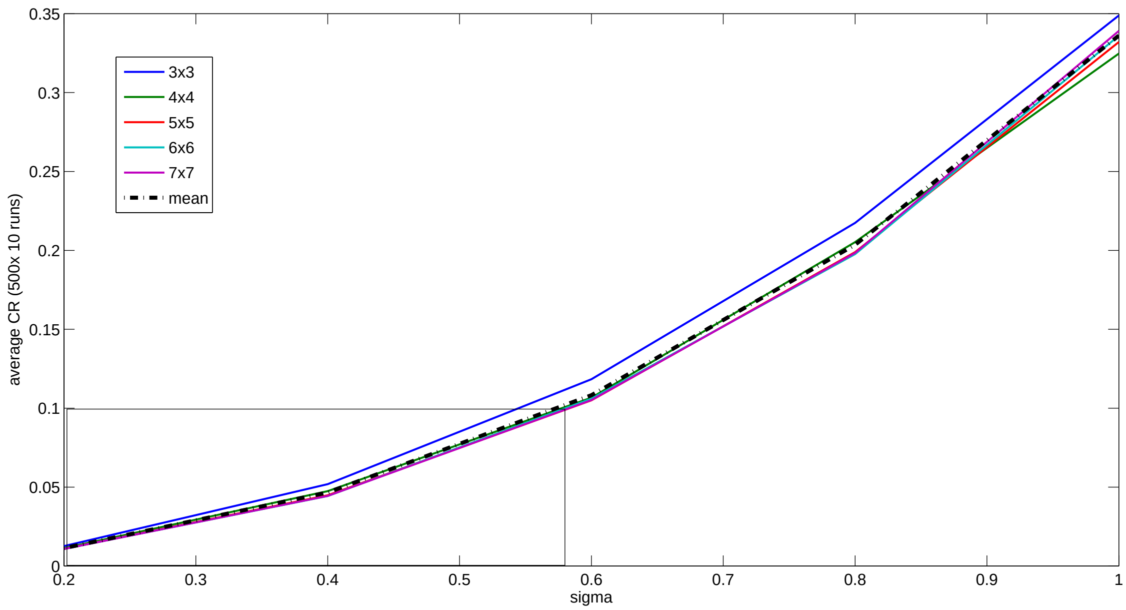

Figure: Uncertainty levels and the average CR inconsistency in case of 3-7 dimensional matrices.

According to the Figure, we have decided to analyse the matrices with following σ *levels* $0.2,\ 0.4,\ 0.6,\ 0.8$ in the next section.

2 Analytical results

In this section we comparison different methods using for estimating the priority vector such as the Saaty's Eigenvector methods (EM), the Least Square Technique (LLS) and the Logarithmic Least Absolute Error (LLAE). For analyses we use random generated inconsistent matrices, which originated from a consistent one, therefore the original priority vectors are known.

Considering that the notable part of the empirical pairwise comparison matrices belongs to $R^{n\times n}$ $n = 3,\ldots,7$, our analyses focuses on these matrix sizes. Furthermore, in order to show the effect of the uncertainty (inconsistency) we use four different uncertainty level ($\sigma = 0.2,\ 0.4,\ 0.6, 0.8$). Since the sample of the matrices is divided to $4\times 5 = 20$ parts and every part consists of 250 matrices, we have to generate, store and study 5000 matrices.

3×3	EM	LLS	LLAE	EM+COP	LLS+COP	LLAE+COP
$\sigma = 0.2$	0.0369	0.0369	0.0369	0.0387	0.0387	0.0408
$\sigma = 0.4$	0.0814	0.0814	0.0814	0.0856	0.0845	0.0893
$\sigma = 0.6$	0.1208	0.1208	0.1208	0.1304	0.1293	0.1425
$\sigma = 0.8$	0.1505	0.1505	0.1505	0.1563	0.1564	0.1728

5×5	EM	LLS	LLAE	EM+COP	LLS+COP	LLAE+COP
$\sigma = 0.2$	0.0328	0.0327	0.0350	0.0594	0.1040	0.0570
$\sigma = 0.4$	0.0650	0.0654	0.0703	0.1050	0.1368	0.1067
$\sigma = 0.6$	0.0971	0.0953	0.0992	0.1362	0.1574	0.1476
$\sigma = 0.8$	0.1440	0.1418	0.1391	0.1618	0.1848	0.1748

7×7	EM	LLS	LLAE	EM+COP	LLS+COP	LLAE+COP
$\sigma = 0.2$	0.0258	0.0254	0.0266	0.1010	0.1265	0.0874
$\sigma = 0.4$	0.0680	0.0704	0.0775	0.1161	0.1343	0.1295
$\sigma = 0.6$	0.1232	0.1226	0.1327	0.1150	0.1622	0.1930
$\sigma = 0.8$	0.1864	0.1968	0.2419	0.1356	0.3183	0.2177

Table V.1: The norm of the deviation from the original rank vector

The norm of the deviation

Rank reversal

3×3	EM	LLS	LLAE	EM+COP	LLS+COP	LLAE+COP
$\sigma = 0.2$	0.0120	0.0120	0.0120	0.0480	0.0480	0.0480
$\sigma = 0.4$	0.0850	0.0850	0.0850	0.1660	0.1660	0.1660
$\sigma = 0.6$	0.1624	0.1624	0.1624	0.2906	0.2906	0.2906
$\sigma = 0.8$	0.2133	0.2133	0.2133	0.3200	0.3200	0.3200

5×5	EM	LLS	LLAE	EM+COP	LLS+COP	LLAE+COP
$\sigma = 0.2$	0.0642	0.0588	0.0802	0.1551	0.7914	0.7540
$\sigma = 0.4$	0.3861	0.3663	0.4653	0.7030	1.5941	1.4158
$\sigma = 0.6$	0.9375	0.8542	0.8333	1.0417	2.2708	2.0000
$\sigma = 0.8$	0.7037	0.7778	0.7778	1.0000	2.2222	1.8519

7×7	EM	LLS	LLAE	EM+COP	LLS+COP	LLAE+COP
$\sigma = 0.2$	0.2727	0.3182	0.4091	1.3182	5.1818	0.5455
$\sigma = 0.4$	1.3750	1.3750	1.7500	2.3750	7.7500	2.0000
$\sigma = 0.6$	4.0000	4.0000	5.0000	5.0000	2.0000	10.0000
$\sigma = 0.8$	3.0000	2.0000	1.0000	2.0000	8.0000	3.0000

Table V.2: The average number of rank reversals

VI. On reducing inconsistency of pairwise comparison matrices below an acceptance threshold

1 Introduction

Pairwise comparison matrices (Saaty, 1977) are used in multi-attribute decision problems, where relative importance of the criteria, the evaluations of the alternatives with respect to each criterion are to be quantified. The method of pairwise comparison is also applied for determining voting powers in group decision making. One of the advantages of pairwise comparison matrices is that the decision maker is faced to a sequence of elementary questions concerning the comparison of two criteria/alternatives at a time, instead of a complex task of providing the weights of the whole set of them.

A real $n \times n$ matrix A is a *pairwise comparison matrix* if it is positive and reciprocal, i.e.,

$$a_{ij} > 0, \tag{VI.1}$$

$$a_{ij} = \frac{1}{a_{ji}} \tag{VI.2}$$

for all $i, j = 1, \dots, n$. A is *consistent* if the transitivity property

$$a_{ij} a_{jk} = a_{ik} \tag{VI.3}$$

holds for all $i, j, k = 1, 2, \dots, n$; otherwise it is called *inconsistent*.

For a positive $n \times n$ matrix A, let $\bar{A} = \log A$ denote the $n \times n$ matrix with the elements

$$\bar{a}_{ij} = \log a_{ij}, \quad i, j = 1, \dots, n.$$

Then A is consistent if and only if

$$\bar{a}_{ij} + \bar{a}_{jk} + \bar{a}_{ki} = 0, \ \forall\, i, j, k = 1, \dots, n \tag{VI.4}$$

holds. Matrices $\bar{A}$ fulfilling the homogenous linear system (VI.4) constitute a linear subspace in $\mathbb{R}^{n \times n}$.

Let $\mathcal{P}_n$ denote the set of the $n \times n$ pairwise comparison matrices, and $\mathcal{C}_n \subset \mathcal{P}_n$ the set of the consistent matrices. Since the reciprocity constraint (VI.2) corresponds to $\bar{a}_{ij} = -\bar{a}_{ji}$ in the logarithmized space, the set $\log \mathcal{P}_n = \{\log A \mid A \in \mathcal{P}_n\}$ is the set of $n \times n$ skew-symmetric matrices, an $n(n-1)/2$-dimensional linear subspace of $\mathbb{R}^{n \times n}$. The set $\log \mathcal{C}_n = \{\log A \mid A \in \mathcal{C}_n\}$ is the set of matrices fulfilling (VI.4), and as pointed out in Chu (1997), is an $(n-1)$-dimensional linear subspace of $\mathbb{R}^{n \times n}$. Clearly, $\log \mathcal{C}_n \subset \log \mathcal{P}_n$.

In decision problems of real life, the pairwise comparison matrices are rarely consistent. Nevertheless, decision makers are interested in the level of inconsistency of their judgements, which somehow expresses the goodness or "quality" of pairwise comparisons totally, because conflicting judgements may lead to senseless decisions. Therefore, some index is needed to measure the possible contradictions and inconsistencies of the pairwise comparison matrix.

A function $\phi_n : \mathcal{P}_n \to R$ is called an *inconsistency index* if $\phi_n(A) = 0$ for every consistent and $\phi_n(A) > 0$ for every inconsistent pairwise comparison matrix A. The inconsistency indices used in the practice are continuous, and the value of $\phi_n(A) > 0$ indicates, more or less, how much an inconsistent matrix differs from a consistent one.

Since in the practice the consistency of a pairwise comparison matrix is not easy to assure, certain level of inconsistency is usually accepted by the decision makers. This works in the practice in such a way that for a given inconsistency index ϕ_n an acceptance threshold $\alpha_n \geq 0$ is chosen, and a matrix $A \in \mathcal{P}_n$ is kept for further use only if $\phi_n(A) \leq \alpha_n$ holds; otherwise, it is rejected or the pairwise comparisons are carried out again. The carrying out of all pairwise comparisons for filling-in the matrix is often a time-consuming task. Therefore, before the total rejection of a pairwise comparison matrix with an inconsistency level above a prescribes acceptance threshold, it may be worth investigating whether it is possible to improve the inconsistency of the matrix to an acceptable level by performing fewer pairwise comparisons.

The paper will concentrate on the following problem: for a given $A \in \mathcal{P}_n$, inconsistency index ϕ_n and acceptance level α_n, what is the minimal number of the elements of matrix A that by modifying these elements, and of course their reciprocals, the pairwise comparison matrix can be made acceptable. We shall show that under a slight boundedness assumption, this can be achieved by solving a nonlinear mixed 0-1 optimization problem. If it comes out that the matrix can be turned into an acceptable one by modifying relatively few elements, then it may be a case when a more-or-less consistent evaluator was less attentive at these few elements, or a data-recording error happened. So it may be worth re-evaluating these elements. Of course, if the the evaluator insists on the previous values, or the acceptable inconsistency threshold cannot be reached with the new values, then this approach was unsuccessful: all pairwise comparisons are to be evaluated again. If however after the revision of the critical elements, the inconsistency level of the modified matrix is already acceptable, then we can continue the decision process with it.

Concerning the investigations above, when solving the nonlinear mixed 0-1 programming problems, it is very beneficial if the nonlinear optimization problems obtained after the relaxation of the 0-1 variables are convex optimization problems. In the convex case several sophisticated methods and softwares are available, while in the nonconvex case methodological and implementation difficulties may arise. Since $\log \mathcal{C}_n$ is a linear subspace, $\mathcal{C}_n$ is a nonconvex manifold in $\mathbb{R}^{n\times n}$. One can immediately conclude that it is better to investigate the convexity issues in the logarithmized space.

Several proposals of inconsistency indices are known, see the overviews of Brunelli and Fedrizzi (2011, 2013a) and Brunelli et al. (2013b) for detailed lists and properties. This paper focuses on three well-known inconsistency indices. They are CR proposed by Saaty (1980), CM proposed by Koczkodaj and Duszak (Koczkodaj 1993; Duszak and Koczkodaj 1994), and CI proposed by Peláez and Lamata (2003). The properties and relationship of the fundamental indices CR and CM were also studied in Bozóki and Rapcsák (2008). In this paper we point out that for the inconsistency indices in our focus, the nonlinear mixed 0-1 optimization problems mentioned above can be formulated in the logarithmized space, and appropriate convexity properties hold on them. We show that CR and CI are convex function in the logarithmized space, and CM is quasiconvex, but can be transformed into a convex function by applying a suitable strictly monotone univariate function on it.

This paper is in a close relation to an earlier paper of the authors (Bozóki et al. 2011b). In the latter paper we investigated the special case when the acceptance threshold α_n is 0, i.e. the modified pairwise comparison matrix must be consistent. No inconsistency indices were needed for this investigation, simple graph theoretic ideas were applied. Unfortunately, the technique applied for $\alpha_n = 0$ cannot be extended to the general case, therefore, a new approach is proposed in this paper.

We also mention that some of the issues investigated in this paper were already considered, in Hungarian, in Bozóki et al. (2012).

Since inconsistent matrices are in the focus of this paper, and for $n = 1$ and $n = 2$ the pairwise comparison matrices are consistent, we shall assume in the sequel, without loss of generality, that $n \geq 3$.

In Section 2, the optimization problems to be solved are presented in a general form. The general issues are specialized and investigated for the inconsistency indices CR of Saaty, CM of Koczkodaj and Duszak, and CI proposed by Peláez and Lamata in Sections 3 through 5, respectively. A numerical example is presented in Section 6.

2 The general form of the optimization problems

Let ϕ_n be an inconsistency index and α_n be an acceptance threshold, and let

$$\mathcal{A}_n(\phi_n, \alpha_n) = \{A \in \mathcal{P}_n \mid \phi_n(A) \leq \alpha_n\} \tag{VI.5}$$

denote the set of $n \times n$ pairwise comparison matrices with inconsistency ϕ_n not exceeding threshold α_n. Let $A, \hat{A} \in \mathcal{P}_n$ and

$$d(A, \hat{A}) = |\{(i,j) : 1 \leq i < j \leq n, a_{ij} \neq \hat{a}_{ij}\}| \tag{VI.6}$$

denote the number of matrix elements above the main diagonal, where matrices A and $\hat{A}$ differ from each other. By reciprocity, the number of different elements is the same as in positions below the main diagonal.

Consider pairwise comparison matrix $A \in \mathcal{P}_n$ with $\phi_n(A) > \alpha_n$ as it is not acceptable in terms of inconsistency. We want to calculate the minimal number of matrix elements above the main diagonal to be modified in order to make matrix acceptable (elements below the main diagonal are determined by the elements above the main diagonal). That is to solve the optimization problem

$$\begin{array}{ll} \min & d(A, \hat{A}) \\ \text{s.t.} & \hat{A} \in \mathcal{A}_n(\phi_n, \alpha_n), \end{array} \tag{VI.7}$$

where the elements above the main diagonal of $\hat{A}$ are variables.

We could also ask the minimal inconsistency of $A \in \mathcal{P}_n$ matrix can be reached by modifying at most K elements and their reciprocals. The optimization problem is

$$\begin{array}{ll} \min & \alpha \\ \text{s.t.} & d(A, \hat{A}) \leq K, \\ & \hat{A} \in \mathcal{A}_n(\phi_n, \alpha), \end{array} \tag{VI.8}$$

where α and the elements above the main diagonal of $\hat{A}$ are variables.

Problems (VI.7) and (VI.8) can be formulated in logarithmic space:

$$\log \mathcal{A}_n(\phi_n, \alpha_n) = \{X \in \log \mathcal{P}_n \mid \phi_n(\exp X) \leq \alpha_n\}, \tag{VI.9}$$

therefore (VI.7) is equivalent to

$$\begin{array}{ll} \min & d(\log A, X) \\ \text{s.t.} & X \in \log \mathcal{P}_n, \\ & \phi_n(\exp X) \leq \alpha_n, \end{array} \tag{VI.10}$$

where elements above the main diagonal of X are variables. The first constraint in (VI.10) means that X belongs to the subspace of skew-symmetric matrices. In this paper we show that the second, nonlinear inequality is a convex constraint in case of inconsistency indices CR (Saaty 1980), CM (Koczkodaj 1993; Duszak and Koczkodaj 1994) and CI (Peláez and Lamata, 2003).

Problem (VI.8) can be rewritten in the same way as above:

$$\begin{array}{ll} \min & \alpha \\ \text{s.t.} & d(\log A, X) \leq K, \\ & X \in \log \mathcal{P}_n, \\ & \phi_n(\exp X) \leq \alpha, \end{array} \tag{VI.11}$$

where α and elements above the main diagonal of X are variables.

The objective function d can be replaced by using the well-known "Big M" method. Assume that $M \geq 1$ is given as an upper bound of the values of the elements in $A \in \mathcal{P}_n$ and the computed $\hat{A} \in \mathcal{P}_n$ matrices, which is determined as the optimal solution of problems (VI.7) and (VI.8), i.e.,

$$1/M \leq a_{ij} \leq M, \;\; 1/M \leq \hat{a}_{ij} \leq M, \;\; i, j = 1, \ldots, n. \tag{VI.12}$$

We can find such an upper bound M if we get a bounded interval by knowing the actual level of ϕ_n, which contains at least one optimal solution of problems (VI.7), and (VI.8).

On the other hand, if a theoretical upper bound M is not given, then a reasonable bound M is usually determined on the values of the pairwise comparison matrices in every specific problem. Constraint (VI.12) can be described as

$$A, \hat{A} \in [1/M, M]^{n\times n} \tag{VI.13}$$

in matrix form, and if the condition (VI.13) associated with $\hat{A}$ is attached to problems (VI.7) and also (VI.8), we get

$$\begin{aligned} \min \quad & d(A, \hat{A}) \\ \text{s.t.} \quad & \hat{A} \in \mathcal{A}_n(\phi_n, \alpha_n) \cap [1/M, M]^{n\times n}, \end{aligned} \tag{VI.14}$$

and, respectively,

$$\begin{aligned} \min \quad & \alpha \\ \text{s.t.} \quad & d(A, \hat{A}) \le K, \\ & \hat{A} \in \mathcal{A}_n(\phi_n, \alpha) \cap [1/M, M]^{n\times n}. \end{aligned} \tag{VI.15}$$

Introduce $\bar{M} = \log M$, problems (VI.14) and (VI.15) become equivalent to

$$\begin{aligned} \min \quad & d(\log A, X) \\ \text{s.t.} \quad & X \in \log \mathcal{P}_n \cap [-\bar{M}, \bar{M}]^{n\times n}, \\ & \phi_n(\exp X) \le \alpha_n, \end{aligned} \tag{VI.16}$$

and

$$\begin{aligned} \min \quad & \alpha \\ \text{s.t.} \quad & d(\log A, X) \le K, \\ & X \in \log \mathcal{P}_n \cap [-\bar{M}, \bar{M}]^{n\times n}, \\ & \phi_n(\exp X) \le \alpha. \end{aligned} \tag{VI.17}$$

in the logarithmic space.

The "Big M" method can be applied for (VI.16) and (VI.17). Let $\bar{A} = \log A$, and introduce binary variables $y_{ij} \in \{0, 1\},\ 1 \le i < j \le n$. Problem (VI.16) can be altered by using $\bar{A} \in [-\bar{M}, \bar{M}]^{n\times n}$ into the following mixed 0-1 programming problem:

$$
\begin{array}{lll}
\min & \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} y_{ij} & \\
\text{s.t.} & \phi_n(\exp X) \le \alpha_n, & \\
& x_{ij} = -x_{ji}, & 1 \le i \le j \le n, \\
& -\bar{M} \le x_{ij} \le \bar{M}, & 1 \le i < j \le n, \\
& -2\bar{M} y_{ij} \le x_{ij} - \bar{a}_{ij} \le 2\bar{M} y_{ij}, & 1 \le i < j \le n, \\
& y_{ij} \in \{0,1\}, & 1 \le i < j \le n.
\end{array}
\tag{VI.18}
$$

The optimal value of (VI.18) gives the minimal number of the matrix elements above the main diagonal to be modified in order to achieve $\phi_n \le \alpha_n$. In the optimal solution, $y_{ij} = 1$ indicates the matrix elements that (and their reciprocal pairs) are modified, and $\exp x_{ij}$ gives a feasible value of these elements.

Problem (VI.18) may have multiple optimal solutions with respect to the binary variables. If all of them are of interest, we list them one by one as follows. Assume that L^* is the optimum value of the problem (VI.18), y^*_{ij}, $1 \le i < j \le n$, is an optimal solution and $I^*_0 = \{(i,j) \mid y^*_{ij} = 0, 1 \le i < j \le n\}$. By adding the constraint

$$
\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} y_{ij} = L^* \tag{VI.19}
$$

to (VI.18) we can ensure, that the optimal solutions of (VI.18) can only be the feasible solutions of (VI.18)-(VI.19).

The addition of constraint

$$
\sum_{(i,j)\in I^*_0} y_{ij} \ge 1 \tag{VI.20}
$$

excludes the already known solution from further search. If problem (VI.18)-(VI.19)-(VI.20) has no feasible solution, then all optimal solutions of (VI.18) have been found. Otherwise, each recently found optimal solution brings a constraint as (VI.20), and resolve (VI.18)-(VI.19)-(VI.20). The algorithm stops in a finite number of steps, resulting in all optimal solutions through binary variables (VI.18).

Problem (VI.17) can also be rewritten as in (VI.18):

$$\begin{aligned}
\min \quad & \alpha \\
\text{s.t.} \quad & \phi_n(\exp X) \leq \alpha, \\
& \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} y_{ij} \leq K, \\
& x_{ij} = -x_{ji}, & 1 \leq i \leq j \leq n, \\
& -\bar{M} \leq x_{ij} \leq \bar{M}, & 1 \leq i < j \leq n, \\
& -2\bar{M} y_{ij} \leq x_{ij} - \bar{a}_{ij} \leq 2\bar{M} y_{ij}, & 1 \leq i < j \leq n, \\
& y_{ij} \in \{0, 1\}, & 1 \leq i < j \leq n.
\end{aligned} \tag{VI.21}$$

If $\phi_n(\exp X)$ is a convex function of the elements (above the main diagonal) of X, then the relaxations of (VI.18) and (VI.21) are convex optimization problems, consequently, (VI.18) and (VI.21) are mixed 0-1 convex problems.

3 Inconsistency index CR of Saaty

Saaty (1980) proposed to index the inconsistency of pairwise comparison matrix A of size $n \times n$ by a positive linear transformation of its largest eigenvalue $\lambda_{\max}$. The normalized right eigenvector associated to $\lambda_{\max}$ also plays an important role, since it provides the estimation of the weights in the eigenvector method. However, in this paper weighting methods are not discussed. Saaty (1977) showed that $\lambda_{\max} \geq n$ and $\lambda_{\max} = n$ if and only if A is consistent. Let us generate a large number of random pairwise comparison matrices of size $n \times n$, where each element above the main diagonal are chosen from the ratio scale $1/9, 1/8, 1/7, \ldots, 1/2, 1, 2, ..., 8, 9$ with equal probability. Take the largest eigenvalue of each matrix and let $\overline{\lambda_{\max}}$ denote their average value. Let $RI_n = (\overline{\lambda_{\max}} - n)/(n-1)$. Saaty defined the inconsistency of matrix A as

$$CR_n(A) = \frac{\frac{\lambda_{\max}(A) - n}{n-1}}{RI_n}$$

being a positive linear transformation of $\lambda_{\max}(A)$. Then $CR_n(A) \geq 0$ and $CR_n(A) = 0$ if and only if A is consistent. The heuristic rule of acceptance is $CR_n \leq 0.1$ for all sizes, also known as the ten percent rule (Saaty, 1980), supported by Vargas' (1982) statistical analysis. However, some refinements are also known: $CR_3 \leq 0.05$ for 3×3 matrices $CR_4 \leq 0.08$ for 4×4 matrices (Saaty, 1994). Note that any rule of acceptance is somehow heuristic.

Now we apply the results of Section 2 by setting $\phi_n = CR_n$. Let $X \in \log \mathcal{P}_n$ and let $\lambda_{\max}(\exp X)$ denote the largest eigenvalue of $A = \exp X$. Then

$$\phi_n(\exp X) = \frac{\lambda_{\max}(\exp X) - n}{RI_n(n-1)}. \tag{VI.22}$$

Bozóki et al. (2010) showed that $\lambda_{\max}(\exp X)$ is a convex function of the elements of X, therefore, through (VI.22), $\phi_n(\exp X)$ is a convex function of the elements of X, too.

It is proven that (VI.22) implies that both (VI.18) and (VI.21) are mixed 0-1 convex optimization problems. However, they are still challenging from numerical computational point of view, since $\phi_n(\exp X)$ cannot be given in an explicit form as $\lambda_{\max}$ values are themselves computed by iterative methods (Saaty, 1980). We will show that $\lambda_{\max}$ is not only a limit of an iterative process, but an optimal solution of a convex optimization problem as well. The embedded convex optimization problem can be considered together the embedding optimization problem.

Harker (1987) described the derivatives of $\lambda_{\max}$ with respect to a matrix element and recommended to change the element with the largest decrease in $\lambda_{\max}$. The theorems in this section, based on other tools, can be considered as some extensions of Harker's idea. Reducing CR, being equivalent to decreasing $\lambda_{\max}$, is in the focus of Xu and Wei (1999) and Cao et al. (2008).

A special case of Frobenius theorem is applied (Saaty, 1977; Sekitani and Yamaki, 1999):

Proposition VI.1 Let A be an $n \times n$ irreducibile nonnegative matrix and $\lambda_{\max}(A)$ denote the maximal eigenvalue of A. Then the following equalities hold

$$\max_{w>0} \min_{i=1,\dots,n} \frac{\sum_{j=1}^{n} a_{ij} w_j}{w_i} = \lambda_{\max}(A) = \min_{w>0} \max_{i=1,\dots,n} \frac{\sum_{j=1}^{n} a_{ij} w_j}{w_i}. \tag{VI.23}$$

Since the pairwise comparison matrices are positive, Theorem VI.1 can be applied. In order to rewrite the right-hand side of (VI.23), $\bar{a}_{ij} = \log a_{ij},\ i,j =$

$1, \ldots, n$, and $z_i = \log w_i,\ i = 1, \ldots, n$ are used:

$$\lambda_{\max}(A) = \min_{z} \max_{i=1,\ldots,n} \sum_{j=1}^{n} e^{\bar{a}_{ij}+z_j-z_i} \tag{VI.24}$$

The sum of convex exponential functions in the right-hand side (VI.24), furthermore, their maximum are also convex. Thus, $\lambda_{\max}$ can be determined as the optimum value of a convex optimization problem, and the form (VI.24) is equivalent to the optimization problem

$$\min\ \lambda \quad \text{s.t.} \quad \sum_{j=1}^{n} e^{\bar{a}_{ij}+z_j-z_i} \leq \lambda,\ i = 1, \ldots, n, \tag{VI.25}$$

where λ and $z_i, i = 1,\ \ldots, n$ are variables.

Let α_n be given as a threshold of inconsistency index $\phi_n = CR_n$. Then the constraint

$$\phi_n(\exp X) \leq \alpha_n \tag{VI.26}$$

from problem (VI.18) can be transformed by using (VI.22) as

$$\lambda_{\max}(\exp X) \leq n + RI_n(n-1)\alpha_n. \tag{VI.27}$$

Denote $\alpha_n^* = n + RI_n(n-1)\alpha_n$. Hence, the formula (VI.24), substituting $x_{ij} = \bar{a}_{ij}$, implies an equivalent form

$$\sum_{j=1}^{n} e^{x_{ij}+z_j-z_i} \leq \alpha_n^*,\ i = 1, \ldots, n. \tag{VI.28}$$

Let us replace formula (VI.26) by (VI.28) in problem (VI.18). We get a mixed 0-1 convex programming problem:

$$\begin{array}{lll}
\min & \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} y_{ij} & \\
\text{s.t.} & \sum_{j=1}^{n} e^{x_{ij}+z_j-z_i} \leq \alpha_n^*, & i = 1, \ldots, n, \\
& x_{ij} = -x_{ji}, & 1 \leq i \leq j \leq n, \\
& -\bar{M} \leq x_{ij} \leq \bar{M}, & 1 \leq i < j \leq n, \\
& -2\bar{M}y_{ij} \leq x_{ij} - \bar{a}_{ij} \leq 2\bar{M}y_{ij}, & 1 \leq i < j \leq n, \\
& y_{ij} \in \{0,1\}, & 1 \leq i < j \leq n.
\end{array} \tag{VI.29}$$

Proposition VI.2 Let α_n denote the acceptance threshold of inconsistency and let $\alpha_n^* = n + RI_n(n-1)\alpha_n$. Then the optimum value of (VI.29) gives the minimal number of the elements to be modified above the main diagonal in A (and their reciprocals) in order to achieve that $CR_n \leq \alpha_n$.

Problem (VI.21) can also be rewritten in case of $\phi_n = CR_n$. In the light of (VI.22), the minimization of ϕ_n is equivalent to the minimization of $\lambda_{\max}$. Furthermore, program (VI.25) depending on $\lambda_{\max}$ is used to obtain:

$$
\begin{array}{lll}
\min & \lambda & \\
\text{s.t.} & \sum\limits_{j=1}^{n} e^{x_{ij}+z_j-z_i} \leq \lambda, & i=1,\dots,n, \\
& \sum\limits_{i=1}^{n-1} \sum\limits_{j=i+1}^{n} y_{ij} \leq K, & \\
& x_{ij} = -x_{ji}, & 1 \leq i \leq j \leq n, \\
& -\bar{M} \leq x_{ij} \leq \bar{M}, & 1 \leq i < j \leq n, \\
& -2\bar{M}y_{ij} \leq x_{ij} - \bar{a}_{ij} \leq 2\bar{M}y_{ij}, & 1 \leq i < j \leq n, \\
& y_{ij} \in \{0,1\}, & 1 \leq i < j \leq n.
\end{array}
\tag{VI.30}
$$

Proposition VI.3 Denote the optimum value of (VI.30) by λ_{opt}, and let $\alpha_{\text{opt}} = \frac{\lambda_{\text{opt}}-n}{RI_n(n-1)}$. Then α_{opt} is the minimal value of inconsistency CR_n which can be obtained by the modification of at most K elements above the main diagonal of A (and their reciprocals).

4 Inconsistency index CM of Koczkodaj and Duszak

The inconsistency index introduced by Koczkodaj (1993) is based on 3×3 submatrices, called *triads*. For the 3×3 pairwise comparison matrix

$$
\begin{pmatrix} 1 & a & b \\ 1/a & 1 & c \\ 1/b & 1/c & 1 \end{pmatrix}
\tag{VI.31}
$$

let

$$
CM(a,b,c) = \min\left\{ \frac{1}{a}\left|a - \frac{b}{c}\right|, \frac{1}{b}\left|b - ac\right|, \frac{1}{c}\left|c - \frac{b}{a}\right| \right\}.
$$

CM can be extended to larger sizes (Duszak and Koczkodaj, 1994):

$$CM(A) = \max\left\{CM(a_{ij}, a_{ik}, a_{jk}) \mid 1 \le i < j < k \le n\right\}. \tag{VI.32}$$

Unlike CR_n, the construction above does not contain any parameter depending on n, so we dispense with the use of the notation CM_n. It is easy to see that CM is an inconsistency index since $CM(A) \ge 0$ for any $A \in \mathcal{P}_n$, and $CM(A) = 0$ if and only if A is consistent.

For a general triad (a, b, c) let

$$T(a,b,c) = \max\left\{\frac{ac}{b}, \frac{b}{ac}\right\}. \tag{VI.33}$$

It can be shown (Bozóki and Rapcsák, 2008) that there exists a direct relation between CM and T:

$$CM(a,b,c) = 1 - \frac{1}{T(a,b,c)}, \quad T(a,b,c) = \frac{1}{1 - CM(a,b,c)}. \tag{VI.34}$$

Since $T(a,b,c) \ge 1$, we get $0 \le CM(a,b,c) < 1$, so $0 \le CM(A) < 1$.

Let $(\bar{a}, \bar{b}, \bar{c})$ denote the logarithmized values of the triad (a, b, c), and let

$$\bar{T}(\bar{a}, \bar{b}, \bar{c}) = \max\left\{\bar{a} + \bar{c} - \bar{b},\ -(\bar{a} + \bar{c} - \bar{b})\right\}.$$

Then

$$T(a,b,c) = \exp(\bar{T}(\bar{a}, \bar{b}, \bar{c})), \tag{VI.35}$$

$$CM(a,b,c) = 1 - \frac{1}{\exp(\bar{T}(\bar{a}, \bar{b}, \bar{c}))}. \tag{VI.36}$$

It is easy to check that even for triads, CM is not a convex function of the logarithmized matrix elements, thus, if we choose the inconsistency index $\phi_n = CM$, then $\phi_n(\exp X)$ appearing in (VI.18) and (VI.21) is not a convex function of the element of matrix X. We show however that by using the univariate function

$$f(t) = \frac{1}{1-t} \tag{VI.37}$$

being strictly monotone increasing on the interval $(-\infty, 1)$, $f(\phi_n(\exp X)) = f(CM(\exp X))$ is already a convex function of the elements of matrix X.

Then we can change the constraint

$$\phi_n(\exp X) \leq \alpha_n$$

of problem (VI.18) to the convex constraint

$$f(\phi_n(\exp X)) \leq f(\alpha_n).$$

Also, instead of function $\phi_n(\exp X)$ appearing in problem (VI.21) we can write $f(\phi_n(\exp X))$ directly, and the value $f^{-1}(\alpha^*)$ computed from the optimal value α^* of the modified problem is the optimal value of the original problem (VI.21).

To show the statement above, extend the index T defined in (VI.33) for arbitrary $n \times n$ pairwise comparison matrix A:

$$T(A) = \max \left\{ T(a_{ij}, a_{ik}, a_{jk}) | \; 1 \leq i < j < k \leq n \right\}. \tag{VI.38}$$

According to (VI.34), used there for triads, there is a strictly monotone increasing functional relationship between CM and T. Consequently,

$$CM(A) = 1 - \frac{1}{T(A)} = f^{-1}(T(A)), \quad T(A) = \frac{1}{1 - CM(A)} = f(CM(A)), \tag{VI.39}$$

where f is the function defined in (VI.37).

By expressing T in the logarithmized space, we get

$$T(\exp X) = \max \Big\{ \max\{e^{x_{ij}+x_{jk}+x_{ki}}, e^{-x_{ij}-x_{jk}-x_{ki}}\} \mid 1 \leq i < j < k \leq n \Big\}. \tag{VI.40}$$

Since on the right-hand-side of (VI.40) the maximum of convex functions is taken, $T(\exp X)$ is convex function of the elements of matrix X. Consequently, if we choose the inconsistency index $\phi_n = CM$, then $f(\phi_n(\exp X))$ is already a convex function, and the problems (VI.18) and (VI.21) modified as shown above are already convex mixed 0-1 optimization problems.

Although $CM(\exp X)$ is not convex, it is quasiconvex. To prove it, we show that the lower level sets of $CM(\exp X)$ are convex. Let $\beta \in [0, 1)$ an arbitrary possible value of $CM(\exp X)$. Since f is strictly monotone increasing, we have

$$\{X \in \mathbb{R}^{n\times n} \mid CM(\exp X) \le \beta\} = \{X \in \mathbb{R}^{n\times n} \mid f(CM(\exp X)) \le f(\beta)\}.$$

Due to the convexity of $T(\exp X) = f(CM(\exp X))$ the above level set are convex, and this implies the quasiconvexity of $CM(\exp X)$.

Proposition VI.4 $CM(\exp X)$ is quasiconvex on the set of the $n\times n$ matrices, and $T(\exp X) = f(CM(\exp X))$ is convex, where f is defined in (VI.37).

In the following we show that problems (VI.18) and (VI.21) can be solved in an easier way, namely, by solving appropriate linear mixed 0-1 optimization problems. By exploiting the strictly monotone increasing property of the exponential function, (VI.40) can also be written in the following form:

$$T(\exp X) = e^{\max\left\{\max\{x_{ij}+x_{jk}+x_{ki}, -x_{ij}-x_{jk}-x_{ik}\} \mid 1\le i<j<k\le n\right\}}. \tag{VI.41}$$

Now, (VI.41) also means that $CM(A)$ can be obtained by determining the maximum of linear expressions of the elements of matrix $\bar{A} = \log A$ and by applying the exponential function and function f once.

Proposition VI.5 *(Bozóki et al. 2011a)* For any $n \times n$ pairwise comparison matrix A, inconsistency index CM can be obtained from the optimal solution of the following univariate linear program:

$$\begin{array}{lll} \min & z & \\ \text{s.t.} & \bar{a}_{ij} + \bar{a}_{jk} + \bar{a}_{ki} \le z, & 1 \le i < j < k \le n, \\ & -(\bar{a}_{ij} + \bar{a}_{jk} + \bar{a}_{ki}) \le z & 1 \le i < j < k \le n. \end{array} \tag{VI.42}$$

Let z_{opt} be the optimal value of (VI.42). Then $CM(A) = 1 - \frac{1}{\exp(z_{\text{opt}})}$.

In the following let α_n denote the acceptance threshold associated with the inconsistency index $\phi_n = CM$, and let

$$\alpha_n^* = \log\left(\frac{1}{1-\alpha_n}\right). \tag{VI.43}$$

Consider the linear mixed 0-1 optimization problem

$$
\begin{array}{lll}
\min & \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} y_{ij} & \\
\text{s.t.} & x_{ij} + x_{jk} + x_{ki} \leq \alpha_n^*, & 1 \leq i < j < k \leq n, \\
& -(x_{ij} + x_{jk} + x_{ki}) \leq \alpha_n^*, & 1 \leq i < j < k \leq n, \\
& x_{ij} = -x_{ji}, & 1 \leq i \leq j \leq n, \\
& -\bar{M} \leq x_{ij} \leq \bar{M}, & 1 \leq i < j \leq n, \\
& -2\bar{M} y_{ij} \leq x_{ij} - \bar{a}_{ij} \leq 2\bar{M} y_{ij}, & 1 \leq i < j \leq n, \\
& y_{ij} \in \{0, 1\}, & 1 \leq i < j \leq n.
\end{array} \tag{VI.44}
$$

Based on the findings above, the following two theorems follow.

Proposition VI.6 Let α_n denote the acceptance threshold of inconsistency and let $\alpha_n^* = \log(\frac{1}{1-\alpha_n})$. Then the optimum value of (VI.44) gives the minimal number of the elements to be modified above the main diagonal in A (and their reciprocals) in order to achieve that $CM \leq \alpha_n$.

By some alterations in (VI.44), the following linear mixed 0-1 optimization problem can be written:

$$
\begin{array}{lll}
\min & \alpha & \\
\text{s.t.} & x_{ij} + x_{jk} + x_{ki} \leq \alpha, & 1 \leq i < j < k \leq n, \\
& -(x_{ij} + x_{jk} + x_{ki}) \leq \alpha, & 1 \leq i < j < k \leq n, \\
& \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} y_{ij} \leq K, & \\
& x_{ij} = -x_{ji}, & 1 \leq i \leq j \leq n, \\
& -\bar{M} \leq x_{ij} \leq \bar{M}, & 1 \leq i < j \leq n, \\
& -2\bar{M} y_{ij} \leq x_{ij} - \bar{a}_{ij} \leq 2\bar{M} y_{ij}, & 1 \leq i < j \leq n, \\
& y_{ij} \in \{0, 1\}, & 1 \leq i < j \leq n.
\end{array} \tag{VI.45}
$$

Proposition VI.7 Let α_{opt} denote the optimum value of (VI.45). Then $1 - \frac{1}{\exp(\alpha_{\text{opt}})}$ is the minimal value of inconsistency CM which can be obtained by the modification of at most K elements above the main diagonal of A (and their reciprocals).

5 Inconsistency index CI of Peláez and Lamata

Similarly to CM, the inconsistency index CI proposed by Peláez and Lamata (2003) is also based on triads of form (VI.31). It is easy to see that the determinant of the triad (VI.31) is nonnegative, and it is zero if and only if the triad is consistent. Based on this interesting property, Peláez and Lamata (2003) proposed to characterize the inconsistency of a pairwise comparison matrix $A \in \mathcal{P}_n$ by the average of the determinants of the triads of matrix A:

$$CI_n(A) = \begin{cases} \det(A), & \text{for } n = 3, \\ \frac{1}{NT(n)} \sum\limits_{i=1}^{NT(n)} \det(\Gamma_i), & \text{for } n > 3, \end{cases} \tag{VI.46}$$

where Γ_i, $i = 1, \ldots, NT(n)$ denote the triads of matrix A, and $NT(n) = \binom{n}{3}$ is the number of triads in A.

We show that CI is a convex function of the logarithmized matrix elements, thus if the inconsistency index $\phi_n = CI_n$ is chosen, then $\phi_n(\exp X)$ appearing in problems (VI.18) and (VI.21) is a convex function of the elements of matrix X.

The determinant of triad $\Gamma \in \mathcal{P}_3$ comparing objects (i, j, k) can be written as

$$\det(\Gamma) = \frac{a_{ik}}{a_{ij}a_{jk}} + \frac{a_{ij}a_{jk}}{a_{ik}} - 2. \tag{VI.47}$$

Let $X = \log \Gamma \in \log\ \mathcal{P}_3$, i.e., $\Gamma = \exp X$. Equation (VI.47) can be reformulated as a convex function of the elements of X:

$$\det(\exp X) = e^{x_{ik} - x_{ij} - x_{jk}} + e^{x_{ij} + x_{jk} - x_{ik}} - 2. \tag{VI.48}$$

Let α_n be a given acceptance threshold for the inconsistency index $\phi_n = CI_n$. According to (VI.46) and (VI.48), the constraint

$$\phi_n(\exp X) \leq \alpha_n \tag{VI.49}$$

appearing in (VI.18) can be expressed as

$$\frac{1}{\binom{n}{3}} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \left(e^{x_{ik} - x_{ij} - x_{jk}} + e^{x_{ij} + x_{jk} - x_{ik}} - 2 \right) \leq \alpha_n. \tag{VI.50}$$

By denoting $\alpha_n^* = (\alpha_n + 2)\binom{n}{3}$, (VI.50) can be simplified as

$$\sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \left(e^{x_{ik} - x_{ij} - x_{jk}} + e^{x_{ij} + x_{jk} - x_{ik}} \right) \leq \alpha_n^*, \tag{VI.51}$$

and inserting it into (VI.18), we get the mixed 0-1 convex optimization problem

$$
\begin{array}{lll}
\min & \sum\limits_{i=1}^{n-1}\sum\limits_{j=i+1}^{n} y_{ij} & \\
\text{s.t.} & \sum\limits_{i=1}^{n-2}\sum\limits_{j=i+1}^{n-1}\sum\limits_{k=j+1}^{n} \left(e^{x_{ik}-x_{ij}-x_{jk}} + e^{x_{ij}+x_{jk}-x_{ik}}\right) \leq \alpha_n^*, & \\
& x_{ij} = -x_{ji}, & 1 \leq i < j \leq n, \\
& -\bar{M} \leq x_{ij} \leq \bar{M}, & 1 \leq i < j \leq n, \\
& -2\bar{M}y_{ij} \leq x_{ij} - \bar{a}_{ij} \leq 2\bar{M}y_{ij}, & 1 \leq i < j \leq n, \\
& y_{ij} \in \{0,1\}, & 1 \leq i < j \leq n.
\end{array}
\tag{VI.52}
$$

Proposition VI.8 Let α_n denote the acceptance threshold of inconsistency and let $\alpha_n^* = (\alpha_n + 2)\binom{n}{3}$. Then the optimum value of (VI.52) gives the minimal number of the elements to be modified above the main diagonal in A (and their reciprocals) in order to achieve that $CI \leq \alpha_n$.

In the same way as for other inconsistency indices, the following mixed 0-1 convex optimization problem can also be considered:

$$
\begin{array}{lll}
\min & \alpha & \\
\text{s.t.} & \sum\limits_{i=1}^{n-2}\sum\limits_{j=i+1}^{n-1}\sum\limits_{k=j+1}^{n} \left(e^{x_{ik}-x_{ij}-x_{jk}} + e^{x_{ij}+x_{jk}-x_{ik}}\right) \leq \alpha, & \\
& x_{ij} = -x_{ji}, & 1 \leq i < j \leq n, \\
& -\bar{M} \leq x_{ij} \leq \bar{M}, & 1 \leq i < j \leq n, \\
& -2\bar{M}y_{ij} \leq x_{ij} - \bar{a}_{ij} \leq 2\bar{M}y_{ij}, & 1 \leq i < j \leq n, \\
& y_{ij} \in \{0,1\}, & 1 \leq i < j \leq n, \\
& \sum\limits_{i=1}^{n-1}\sum\limits_{j=i+1}^{n} y_{ij} \leq K. &
\end{array}
\tag{VI.53}
$$

Proposition VI.9 Let α_{opt} denote the optimum value of (VI.53). Then $\frac{\alpha_{\text{opt}}}{\binom{n}{3}} - 2$ is the minimal value of inconsistency CI which can be obtained by the modification of at most K elements above the main diagonal of A (and their reciprocals).

6 A numerical example

Our approach is also presented on a classic numerical example from the book of Saaty (1980), for the inconsistency index CR. Table 1 contains pairwise comparison values of six cities concerning their distances from Philadelphia. As an example, the evaluator judged that the distance between London and Philadelphia is five times greater than that between Chicago and Philadelphia.

	Cairo	Tokyo	Chicago	San Francisco	London	Montreal
Cairo	1	1/3	8	3	3	7
Tokyo	3	1	9	3	3	9
Chicago	1/8	1/9	1	1/6	1/5	2
San Francisco	1/3	1/3	6	1	1/3	6
London	1/3	1/3	5	3	1	6
Montreal	1/7	1/9	1/2	1/6	1/6	1

Table VI.1: Comparison of distances of cities from Philadelphia

Let A denote the pairwise comparison matrix concerning Table VI.1. We get that $\lambda_{\max}(A) = 6.4536$, and from $RI_6 = 1.24$, also $CR(A) = 0.0732$. Since the value of $CR(A)$ is significantly below the 10% threshold, we can consider the inconsistency of A acceptable.

Let $A^{(1)}$ denote the matrix obtained from A by exchanging the elements $a_{1,2}$ (and $a_{2,1}$). This is a typical mistake at filling-in a pairwise comparison matrix. For the matrix $A^{(1)}$, we get $CR(A^{(1)}) = 0.0811$. Therefore, although the level of inconsistency of $A^{(1)}$ has increased as consequence of the data-recording error, it is still below the acceptance level of 10%. In this case the proposed methodology is not able to detect the mistake, and $A^{(1)}$ is still accepted.

Consider now the case when $a_{1,3}$ and $a_{3,1}$ are exchanged, say by accident, in the matrix A. Let $A^{(2)}$ denote the matrix obtained in this way. Then $CR(A^{(2)}) = 0.5800$, which is well over the acceptance level of 10%, and it refers to a rough inconsistency in the matrix. By solving the corresponding problem (VI.29), we obtain that the inconsistency of $A^{(2)}$ can be pushed below the critical 10% by modifying a single element (and its reciprocal). This element is just in the spoilt position $a_{1,3}$. It can also be shown that this is the single optimal solution to problem (VI.29) considering the 0-1 variables. Consequently, the proposed methodology has detected the single possible element for the case of correcting in a single position (and in its reciprocal). It also turned out that this single position is just the one of the values exchanged by accident.

In the previous example the spoilt matrix caused a rough increase of the inconsistency. In this view, it is not surprising that the proposed method offers a unique way of repairing. However, at smaller increase of inconsistency the situation is not that obvious.

Assume now that the element $a_{1,3}$ of matrix A is changed to 2 instead of the value 1/8 of the previous example. This is a smaller difference in relation to the original value 8, the increase of the inconsistency of the modified matrix, denoted by $A^{(3)}$, is also less: $CR(A^{(3)}) = 0.1078$. The inconsistency of $A^{(3)}$ barely exceeds the critical level 10%, therefore, one would expect that by the modification of a single element can make the inconsistency decrease below 10%, and also that several positions are eligible for this purpose. Indeed, the optimal value of the relating problem (VI.29) is 1, and by resolving the problem after adding the constraints (VI.19) and (VI.20) we find that problem (VI.29) has 6 different optimal solutions according to the binary variables. Namely, the inconsistency of matrix $A^{(3)}$ decreases below 10% not only by modifying $a_{1,3}$, but also by modifying any single element of $\{a_{1,4}, a_{1,5}, a_{2,6}, a_{3,4}, a_{4,5}\}$. In the ideal case, the evaluator spots the data-recording error in position $a_{1,3}$ immediately. If not, then s/he may have to reconsider the evaluation of each of the 6 positions, but it is still fewer than the 15 possible positions in the upper triangular part of the matrix.

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