CORVINUS UNIVERSITY OF BUDAPEST
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THESIS BOOK

to the PhD Thesis titled

Cost Sharing Models in Game Theory

written by

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1 Prior research and motivation

The significance of cooperative game theory is beyond dispute. This field of science positioned on the border of mathematics and economics enables us to model and analyze economic situations that emphasize the cooperation among different parties, and achieving their common goal as a result. In such situations we focus primarily on two topics: what cooperating groups (coalitions) are formed, and how the gain stemming from the cooperation can be distributed among the parties. Several theoretical results have been achieved in recent decades in the field of cooperative game theory, but it is important to note that these are not purely theoretical in nature, but additionally provide solutions applicable to and applied in practice.

One example is the Tennessee Valley Authority (TVA) established in 1933 with the purpose of overseeing the economy of Tennessee Valley, which was engaged in the analysis of the area's water management problems. We can find cost distributions among their solutions which correspond to cooperative game theory solution concepts. Results of the TVA's work are discussed from a game theory standpoint in Straffin and Heaney (1981).

We consider situations that can be modeled by fixed trees as known from graph theory nomenclature. There exists a fixed, finite set of participants, who connect to a distinctive node, the root, through a network represented by a fixed tree. Several real-life situations may be modeled using this method. The aforementioned problem of the TVA's water management is defined in Chapter 2, where we examined the maintenance costs of an irrigation ditch.

We present a special class of games, the standard tree games, and provide examples of their applications in the field of water management. Additional game-theoretic applications to water management problems are discussed in Parrachino et al. (2006). Besides fixed tree structures several other graph-theoretic models are applicable as well, for example the class of shortest path games, which we will cover in more detail in Chapter 7.

As a further special case we must mention the class of "airport problems", which can be modeled with a non-branching tree, i.e. a chain. The related *airport games* are a proper subset of the standard fixed tree games. Airport games were introduced by Littlechild and Owen (1973), and the games' characterization will be described in detail in Chapter 5. A summary of further results related to the class is provided in Thomson (2007).

1.1 Maintenance or irrigation games

A widely used application of fixed tree games is the so called maintenance games. These describe situations in which players, a group of users connect to a certain provider (the root of the tree) through the fixed tree network. There is given a maintenance cost for all edges in the network, and the question is how to distribute "fairly" the entire network's maintenance cost (the sum of the costs on the edges) among the users.

A less widely used naming for the same fixed tree games is irrigation games, which are related to the water management problems described in Chapter 2. A group of farmers irrigate their lands using a common ditch, which connects to the main ditch at a distinct point. The costs of the network must be distributed among the farmers. Aadland and Kolpin (1998) have analyzed 25 ditches in the state of Montana, where the local farmers used two different major types of cost allocation methods, variants of the average cost and the serial cost allocations. Moreover, Aadland and Kolpin (2004) also studied the environmental and geographical conditions that influenced the cost allocation principle chosen in the case of different ditches.

1.2 River sharing and river cleaning problems

Let there be given a river, and along the river players that may be states, cities, enterprises, and so forth. For downstream users the quality and quantity of water let on by the state is of concern, and conversely, how upstream users are managing the river water is of concern to the state. In the paper of Ambec and Sprumont (2002) the position of the users (states) along the river defines the quantity of water they have control over, and the welfare they can therefore achieve. Ambec and Ehlers (2008) studied how a river can be distributed efficiently among the connected states. They have shown that cooperation provides a profit for the participants, and have given the method for the allocation of the profit.

In the case of river cleaning problems, the initial structure is similar. There is given a river, the states (enterprises, factories, etc.) along the river, and the amount of pollution emitted by the agents. There are given cleanup costs for each segment of the river as well, therefore the question is how to distribute these costs among the group. Since the pollution of those further upstream influences the pollution and cleanup costs further downstream as well, we get a fixed tree structure with a single path.

Ni and Wang (2007) analyzed the problem of the allocation of cleanup costs.They have shown that there exists an allocation method that is equal to the Shapley value in the corresponding cooperative game. Based on this Gómez-Rúa (2013) studied how the cleanup cost may be distributed taking into consideration certain environmental taxes. The article discusses the expected properties that are prescribed by states in real situations in the taxation strategies, and how these can be implemented for concrete models. Furthermore, the article describes the properties useful for the characterization properties of certain allocation methods, shows that one of the allocation rules is equal to the weighted Shapley value of the associated game.

Khmelnitskaya (2010) discusses problems where the river sharing problem can be represented by a graph comprising a root or a sink. In the latter case the direction of the graph is the opposite of in the case where the graph comprises a root, in other words, the river unifies flows from multiple springs (from their respective the river deltas) in a single point, the sink.

1.3 Airport and irrigation games

The irrigation ditch can be represented by a rooted tree. The root is the head gate, nodes denote users, and the edges represent the segments of the ditch between users. Employing this representation Littlechild and Owen (1973) have shown that the contribution vector (the solution for the cost-sharing problem) given by the "sequential equal contributions rule" (henceforth SEC rule, or Baker-Thompson rule; Baker (1965), Thompson (1971)) is equivalent to the Shapley value (Shapley, 1953). According to this rule, for all segments their respective costs must be distributed evenly among those using the given segment, and for all users the costs of the segments they are using must be summed up. This sum is the cost the user must cover.

In Chapter 2 we have described an empirical and axiomatic analysis of a real cost-sharing problem, an irrigation ditch located in a south-central Montana community (Aadland and Kolpin, 1998).

When considering special rooted trees with no branches (i.e. chains), we arrive at the well-known class of airport games (Littlechild and Thompson, 1977), therefore this class is the proper subset of the class of irrigation games. Thomson (2007) gives an overview on the results for airport games.

Granot et al. (1996) introduce the notion of standard fixed tree games. Irrigation games are equivalent to standard fixed tree games, except for that in irrigation games the tree may vary, while in the approach of Granot et al. (1996) it is fixed. Koster et al. (2001) study the core of fixed tree games. Ni and Wang (2007) characterize the rules meeting properties additivity and independence of irrelevant costs on the class of standard fixed tree games. Fragnelli and Marina (2010) characterize the SEC rule for airport games.

1.4 Upstream responsibility

We consider further cost sharing problems given by rooted trees, called *cost-tree problems*, but we are considering different applications from those so far. We will consider energy supply chains with a motivated dominant leader, who has the power to determine the responsibilities of suppliers for both direct and indirect emissions. The induced games are called *upstream responsibility games* Gopalakrishnan et al. (2017), and henceforth we will refer to it as *UR game*.

We utilize the TU game model of Gopalakrishnan et al. (2017), called GHG Re-spon-si-bil-i-ty-Emissions and Environment (GREEN) game. The Shapley value is used as an allocation method by Gopalakrishnan et al., who also consider some pollution-related properties that an emission allocation rule should meet, and provide several axiomatizations as well.

1.5 Shortest path games

In this chapter we consider the class of *shortest path games*. There are given some agents, a good, and a network. The agents own the nodes of the network and they want to transport the good from certain nodes of the network to others. The transportation cost depends on the chosen path within the network, and the successful transportation of a good generates profit. The problem is not only choosing the shortest path (a path with minimum cost, i.e. with maximum profit), but we also have to divide the profit among the players.

Fragnelli et al. (2000) introduce the notion of shortest path games and they prove that the class of such games coincides with the well-known class of monotone games.

In this chapter we consider further characterizations of the Shapley value: Shapley (1953)'s, Young (1985)'s, Chun (1989)'s, and van den Brink (2001)'s axiomatizations. We analyze their validity on the class of shortest path games, and conclude that all aforementioned axiomatizations are valid on the class.

2 Research methods and notations

Besides basic graph-theoretic notions and models, we apply game-theoretic models in the majority of the thesis. The significance of cooperative game theory is beyond dispute. This field of science positioned on the border of mathematics and economics enables us to model and analyze economic situations that emphasize the cooperation among different parties, and achieving their common goal as a result. In such situations we focus primarily on two topics: what cooperating groups (coalitions) are formed, and how the gain stemming from the cooperation can be distributed among the parties. Several theoretical results have been achieved in recent decades in the field of cooperative game theory, but it is important to note that these are not purely theoretical in nature, but additionally provide solutions applicable to and applied in practice. In the following we present the notations used in the thesis.

Cost allocation models

N	$= \{1, 2, \dots, n\}$ the finite set of players
L	$\subseteq N$ the set of leaves of a tree
i	$\in N$ a given user/player
c_i	the cost of the ith segment/edge
I_i^-	$= \{j \in N \mid j < i\}$ the set of users preceding i
I_i^+	$= \{j \in N \mid i < j\}$ the set of users following i
ξ_i^a	the average cost allocation rule
ξ_i^s	the serial cost allocation rule
ξ_i^{eq}	the allocation rule based on the equal allocation of the non-separable cost
ξ_i^{riu}	the allocation rule based on the ratio of individual use of the non-separable cost
ξ_i^r	the restricted average cost allocation rule

Introduction to cooperative game theory

TU game	cooperative game with transferable utility
$\lvert N \rvert$	the cardinality of N
2^N	the class of all subsets of N

$A \subset B$	$A \subseteq B$, but $A \neq B$
$A \uplus B$	the disjoint union of sets A and B
(N, v)	the cooperative game defined by the set of players N and the characteristic function v
$v(S)$	the value of coalition $S \subseteq N$
$\mathcal{G}^N$	the class of games defined on the set of players N
(N, v_c)	the cost game with the set of players N and cost function v_c
$I^*(N, v)$	the set of preimputations
$I(N, v)$	the set of imputations
$C(N, v); C(v)$	the set of core allocations; the core of a cooperative game
$X^*(N, v)$	$= \{x \in \mathbb{R}^N \mid x(N) \leq v(N)\}$, the set of feasible payoff vectors
$\psi(v)$	$= (\psi_i(v))_{i \in N} \in \mathbb{R}^N$, the single-valued solution of a game v
$v_i'(S)$	$= v(S \cup \{i\}) - v(S)$, the player i's individual marginal contribution to coalition S
$\phi_i(v)$	the Shapley value of player i
$\phi(v)$	$= (\phi_i(v))_{i \in N}$ the Shapley value of a game v
π	an ordering of the players
Π_N	the set of the possible orderings of set of players N

Fixed tree games

$\Gamma(V, E, b, c, N)$	a fixed tree network
$G(V, E)$	directed graph with the sets of nodes V and edges E
r	$\in V$ the root node
c	the non-negative cost function defined on the set of edges
$S_i(G)$	$= \{j \in V : i \leq j\}$ the set of nodes accessible from i via a directed graph
$P_i(G)$	$= \{j \in V : j \leq i\}$ the set of nodes on the unique path connecting i to the root
$\bar{S}$	the trunk of the tree containing the members of coalition S (the union of unique paths connecting nodes of members of coalition S to the root)
u_T	the unanimity game on coalition T

$\bar{u}_T$	the dual of the unanimity game on coalition T

Airport and irrigation games

(G, c)	the cost tree defined by the graph G and cost function c
$\mathcal{G}_I^A$	the class of airport games on the set of players N
$\mathcal{G}_I^N$	the class of irrigation games on the set of players N
$\mathcal{G}_G$	the subclass of irrigation games induced by cost tree problems defined on rooted tree G
Cone $\{v_i\}_{i\in N}$	the convex cone spanned by given v_i games
i_-	$= \{j \in V : \overline{ji} \in E\}$ the player preceding player i
v_{irr}	the irrigation game
$i \sim^v j$	i and j are equivalent players in game v, i.e. $v_i'(S) = v_j'(S)$ for all $S \subseteq N \setminus \{i, j\}$
ξ^{SEC}	"sequential equal contributions" cost allocation rule

Upstream responsibility

$\vec{ij}$	the directed edge pointing from i to j
$\mathcal{E}_i$	the set of edges whose corresponding emissions are the direct or indirect responsibility of i
$\mathcal{E}_S$	the set of edges whose corresponding emissions are the direct or indirect responsibilities of players in S
$\mathcal{G}_{UR}^N$	the class of UR games corresponding to the set of players of N

Shortest path tree games

$\Sigma(V, A, L, S, T)$	the shortest path tree problem
(V, A)	directed, acyclic graph defined by the sets of nodes V and edges A
$L(a)$	the length of the edge a
S	$\subseteq N$ the nonempty set of sources
T	$\subseteq N$ the nonempty set of sinks
P	the path connecting the nodes of $\{x_0, \ldots, x_p\}$
$L(P)$	the length of path P
$o(\{x\})$	function defining the owners of node x

$o(P)$	defines the owners of the nodes on the path P
g	the income from transporting a good from a source to a sink
$\sigma(\Sigma, N, o, g)$	the shortest path cooperative situation
v_σ	the shortest path game

3 Scientific results of the thesis

We will annotate our own results (lemmata, claims, theorems, and new proofs of known theorems) in the thesis by $\boxed{\text{framing}}$ their names.

3.1 Cost allocation models

Our entire thesis discusses cost allocation problems. One of the most well-known examples is described as follows. A group of farmers acquire water supply for their land from a ditch that is connected to a main ditch. Operation and maintenance of the ditch incur costs which are jointly covered by the group. The question is how the farmers (henceforth users) may "fairly" divide the aforementioned costs among the group. After the introduction we shall present the basic models and the axioms aiming to define the notion of "fairness". The axioms are based on the work of Aadland and Kolpin (1998).

In this chapter we generalize the above models for problems described by tree-structures, and show that they uphold the properties of cost allocations described for chains. These results were presented in our paper (Kovács and Radványi, 2011, in Hungarian).

Definition 3.1 *A $\xi : \mathbb{R}_+^N \to \mathbb{R}_+^N$ mapping is a cost allocation rule, if $\forall c \in \mathbb{R}_+^N$ $\sum\limits_{i\in N} \xi_i(c) = \sum\limits_{i\in N} c_i$, where $(\xi_i(c))_{i\in N} = \xi(c)$.*

(a) According to the average cost allocation rule the operation and maintenance costs of the ditch are distributed evenly among users, i.e. $\forall i \in N$:

$$\xi_i^a(c) = \sum_{j\in N} \frac{c_j}{n}$$

(b) According to the serial cost allocation rule the costs associated with each segment are distributed evenly among those who utilize the segment, i.e. $\forall i \in N$:

$$\xi_i^s(c) = \sum_{j \in I_i^- \cup \{i\}} \frac{c_j}{|I_j^+| + 1}$$

Axiom 3.2 *A rule ξ is cost monotone, if $\forall c \leq c'$: $\xi(c) \leq \xi(c')$.*

Axiom 3.3 *A rule ξ satisfies ranking, if $\forall c \in \mathbb{R}_+^N$ and $\forall j\ \forall i \in I_j^- \cup \{j\}$: $\xi_i(c) \leq \xi_j(c)$.*

Axiom 3.4 *A rule ξ is subsidy-free, if $\forall c \in \mathbb{R}_+^N$ and $\forall I = \{i_1, i_2, \ldots, i_k\} \subseteq N$:*

$$\sum_{j \in J} \xi_j(c) \leq \sum_{j \in J} c_j,$$

where for the sake of brevity $J := I_{i_1}^- \cup \cdots \cup I_{i_k}^- \cup I$, where J is the sub-tree generated by I.

Definition 3.5
- *The non-separable cost's* equal *allocation:*

$$\xi_i^{eq}(c) = s_i + \frac{1}{|N|} k(N) \quad \forall i \in N$$

- *The non-separable cost's allocation based on* the ratio of individual use*:*

$$\xi_i^{riu}(c) = s_i + \frac{k_i}{\sum_{j \in N} k_j} k(N) \quad \forall i \in N$$

We summarize our results in the following table.

	Cost monotone	Ranking	Subsidy-free
ξ^a	✓	✓	×
ξ^s	✓	✓	✓
ξ^{eq}	✓	✓	×
ξ^{riu}	×	✓	×

Properties of allocations

$\boxed{\textbf{Claim}}$ **3.6** *In the case of cost allocation problems represented by tree structures, the properties cost monotonicity, ranking, and subsidy-free are independent of each other.*

Definition 3.7 *A* restricted average cost allocation *is a cost monotone, ranking, subsidy-free cost allocation, where the difference between the highest and lowest distributed costs is the lowest possible, considering all possible allocation principles.*

$\boxed{\textbf{Theorem}}$ **3.8** *There exists a restricted average cost share allocation ξ^r and it is unique. The rule can be constructed recursively*

$\boxed{\textbf{Theorem}}$ **3.9** *The restricted average cost rule is the only cost monotone, ranking, subsidy-free method providing maximal Rawlsian welfare.*

Axiom 3.10 *A rule ξ satisfies the reciprocity axiom, if $\forall i$ the points*

(a) $\sum_{h \leq i} \xi_h(c) \leq \sum_{h \leq i} c_h$

(b) $c' \geq c$ *and*

(c) $\sum_{h \leq i} (c_h - \xi_h(c)) \geq \sum_{j > i} (c'_j - c_j)$

imply that the following is not true: $\xi_h(c') - \xi_h(c) < \xi_j(c') - \xi_j(c) \quad \forall h \leq i$ *and* $j > i$.

Axiom 3.11 *A rule ξ is semi-marginal, if $\forall i \in N \setminus L$: $\xi_{i+1}(c) \leq \xi_i(c) + c_{i+1}$, where $i+1$ denotes a direct successor of i in I_i^+.*

Axiom 3.12 *A rule ξ is incremental subsidy-free, if $\forall i \in N$ and $c \leq c'$:*

$$\sum_{h \in I_i^- \cup \{i\}} (\xi_h(c') - \xi_h(c)) \leq \sum_{h \in I_i^- \cup \{i\}} (c'_h - c_h).$$

$\boxed{\textbf{Theorem}}$ **3.13** *Cost-sharing rule ξ is cost monotone, ranking, semi-marginal, and incremental subsidy-free if and only if $\xi = \xi^s$, i.e. it is the serial cost allocation rule.*

$\boxed{\textbf{Theorem}}$ **3.14** *The serial cost-sharing rule is the unique cost monotone, ranking, and incremental subsidy-free method that ensures maximal Rawlsian welfare.*

$\boxed{\textbf{Theorem}}$ **3.15** *The serial cost-sharing rule is the single cost monotone, ranking, semi-marginal method ensuring minimal Rawlsian welfare.*

3.2 Airport and irrigation games

In this chapter we introduce irrigation games and characterize their class. We show that the class of irrigation games is a non-convex cone which is a proper subset of the finite convex cone spanned by the duals of the unanimity games, therefore every irrigation game is concave. Furthermore, as a corollary we show that the class of airport games has the same characteristics as that of irrigation games.

In addition to the previously listed results, we extend the results of Dubey (1982) and Moulin and Shenker (1992) to the class of irrigation games. Furthermore, we “translate” the axioms used in the cost-sharing literature (see e.g. Thomson, 2007) to the language of transferable utility cooperative games, and show that the results of Dubey (1982) and Moulin and Shenker (1992) can be deduced directly from Shapley (1953)'s and Young (1985)'s results. Namely, we present two new variants of Shapley (1953)'s and Young (1985)'s results, and we provide Dubey (1982)'s, Moulin and Shenker (1992)'s and our characterizations as direct corollaries of the two new variants.

In our characterization results we relate the TU games terminologies to the cost sharing terminologies, therefore we bridge between the two fields.

Up to our knowledge these are the first results in the literature which provide a precise characterization of the class of irrigation games, and extend Shapley's and Young's axiomatizations of the Shapley value to this class of games. We conclude that applying the Shapley value to cost-tree problems is theoretically well-founded, therefore, since the Shapley value behaves well from the viewpoint of computational complexity theory (Megiddo, 1978), the Shapley value is a desirable tool for solving cost-tree problems. The results of present chapter have been published in Márkus, Pintér és Radványi (2011).

In this section we build on the duals of unanimity games.The *dual of the unanimity game* for all $T \in 2^N \setminus \{\emptyset\}$ and $S \subseteq N$ is:

$$\bar{u}_T(S) = \begin{cases} 1, & \text{if } T \cap S \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3.16 (Irrigation game) *For all cost trees (G, c) and player set $N = V \setminus \{r\}$, and coalition S let*

$$v_{(G,c)}(S) = \sum_{e \in \bar{S}} c_e \ ,$$

where the value of the empty sum is 0.

Definition 3.17 (Airport games I.) *For an airport problem let $N = N_1 \uplus \cdots \uplus N_k$ be the set of players, and let there be given $c \in \mathbb{R}_+^k$, such that $c_1 < \ldots < c_k \in \mathbb{R}_+$. Then the airport game $v_{(N,c)} \in \mathcal{G}^N$ can be defined as $v_{(N,c)}(\emptyset) = 0$, and for all non-empty coalitions $S \subseteq N$:*

$$v_{(N,c)}(S) = \max_{i: N_i \cap S \neq \emptyset} c_i \ .$$

Definition 3.18 (Airport games II.) *For an airport problem let* $N = N_1 \uplus \cdots \uplus N_k$ *be the set of players, and* $c = c_1 < \ldots < c_k \in \mathbb{R}_+$. *Let* $G = (V, E)$ *be a chain such that* $V = N \cup \{r\}$, *and* $E = \{\overline{r1}, \overline{12}, \ldots, \overline{(|N|-1)|N|}\}$, $N_1 = \{1, \ldots, |N_1|\}, \ldots, N_k = \{|N| - |N_k| + 1, \ldots, |N|\}$. *Furthermore, for all* $\overline{ij} \in E$ *let* $c(\overline{ij}) = c_{N(j)} - c_{N(i)}$, *where* $N(i) = \{N^* \in \{N_1, \ldots, N_k\} : i \in N^*\}$.

For a cost tree (G, c), *an airport game* $v_{(N,c)} \in \mathcal{G}^N$ *can be defined as follows. Let* $N = V \setminus \{r\}$ *be the set of players, then for each coalition* S *(the empty sum is* 0*)*

$$v_{(N,c)}(S) = \sum_{e \in \bar{S}} c_e \ .$$

Clearly, both definitions above define the same games.

$\boxed{\textbf{Lemma}}$ **3.19** *For an arbitrary coalition* $\emptyset \neq T \subseteq N$, *for which* $T = S_i(G)$, $i \in N$, *there exists chain* G, *such that* $\bar{u}_T \in \mathcal{G}_G$. *Therefore* $\{\bar{u}_T\}_{T \in 2^N \setminus \{\emptyset\}} \subset \mathcal{G}_A^N \subset \mathcal{G}_I^N$.

$\boxed{\textbf{Lemma}}$ **3.20** *For all rooted trees* G: $\mathcal{G}_G \subset \mathrm{Cone}\,\{\bar{u}_{S_i(G)}\}_{i \in N}$. *Therefore,* $\mathcal{G}_A^N \subset \mathcal{G}_I^N \subset \mathrm{Cone}\,\{\bar{u}_T\}_{T \in 2^N \setminus \{\emptyset\}}$.

$\boxed{\textbf{Lemma}}$ **3.21** *Neither* $\mathcal{G}_A^N$ *nor* $\mathcal{G}_I^N$ *is convex.*

$\boxed{\textbf{Lemma}}$ **3.22** *For all rooted trees* G *and* $v = \sum_{i \in N} \alpha_{S_i(G)} \bar{u}_{S_i(G)} \in \mathcal{G}_G$, *and for all* $i^* \in N$: $\sum_{i \in N \setminus \{i^*\}} \alpha_{S_i(G)} \bar{u}_{S_i(G)} \in \mathcal{G}_G$. *Then for all airport games* $v = \sum_{T \in 2^N \setminus \{\emptyset\}} \alpha_T \bar{u}_T$ *and coalition* $T^* \in 2^N \setminus \{\emptyset\}$: $\sum_{T \in 2^N \setminus \{\emptyset, T^*\}} \alpha_T \bar{u}_T \in \mathcal{G}_A^N$, *and for all irrigation games* $v = \sum_{T \in 2^N \setminus \{\emptyset\}} \alpha_T \bar{u}_T$ *and* $T^* \in 2^N \setminus \{\emptyset\}$: $\sum_{T \in 2^N \setminus \{\emptyset, T^*\}} \alpha_T \bar{u}_T \in \mathcal{G}_I^N$.

$\boxed{\textbf{Lemma}}$ **3.23** *All irrigation games are concave.*

$\boxed{\textbf{Corollary}}$ **3.24** *In the case of a fixed set of players the class of airport games is a union of finitely many convex cones, but the class itself is not convex. Moreover, the class of airport games is a proper subset of the class of irrigation games. The class of irrigation games is also a union of finitely many convex cones, but is not convex, either. Furthermore, the class of irrigation games is a proper subset of the finite convex cone spanned by the duals of the unanimity games, therefore every irrigation game is concave, and consequently every airport game is concave too.*

Definition 3.25 *Let $v \in \mathcal{G}^N$ and*

$$p^i_{Sh}(S) = \begin{cases} \dfrac{|S|!(|(N \setminus S)| - 1)!}{|N|!} \ , & \text{if } i \notin S, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\phi_i(v)$, the Shapley value *of player i in game v is the p^i_{Sh}-weighted expected value of all v'_i. In other words:*

$$\phi_i(v) = \sum_{S \subseteq N} v'_i(S) \ p^i_{Sh}(S) \ . \tag{1}$$

the core of v is

$$C(v) = \left\{ x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N), \text{ and for all } S \subseteq N : \sum_{i \in S} x_i \leq v(S) \right\} \ .$$

Definition 3.26 *Value ψ in game class $A \subseteq \mathcal{G}^N$ is* core compatible, *if for all $v \in A$ it holds that $\psi(v) \in C(v)$.*

Theorem 3.27 *A cost allocation rule ξ on the cost tree (G, c) is subsidy-free, if and only if the value generated by the cost allocation rule ξ on the irrigation game $v_{(G,c)}$ induced by the cost tree is core compatible.*

Definition 3.28 *A single-valued solution ψ on $A \subseteq \mathcal{G}^N$ is / satisfies*

- *Pareto optimal (PO), if for all $v \in A$, $\sum\limits_{i \in N} \psi_i(v) = v(N)$,*
- *null-player property (NP), if for all $v \in A$, $i \in N$, $v'_i = 0$ implies $\psi_i(v) = 0$,*
- *equal treatment property (ETP), if for all $v \in A$, $i, j \in N$, $i \sim^v j$ implies $\psi_i(v) = \psi_j(v)$,*
- *additive (ADD), if for all $v, w \in A$ such that $v + w \in A$, $\psi(v + w) = \psi(v) + \psi(w)$,*
- *marginal (M), if for all $v, w \in A$, $i \in N$, $v'_i = w'_i$ implies $\psi_i(v) = \psi_i(w)$.*

Theorem 3.29 (Shapley's axiomatization) *For all rooted trees G a value ψ is PO, NP, ETP and ADD on $\mathcal{G}_G$ if and only if $\psi = \phi$, i.e. the value is the Shapley value. In other words, a value ψ is PO, NP, ETP and ADD on the class of airport and irrigation games if and only if $\psi = \phi$.*

$\boxed{\textbf{Theorem}}$ **3.30 (Young's axiomatization)** *For any rooted tree G, a single-valued solution ψ on $\mathcal{G}_G$ is PO, ETP and M if and only if $\psi = \phi$, i.e. it is the Shapley value. Therefore, a single-valued solution ψ on the class of airport games is PO, ETP and M if and only if $\psi = \phi$, and a single-valued solution ψ on the class of irrigation games is PO, ETP and M if and only if $\psi = \phi$.*

$\boxed{\textbf{Corollary}}$ **3.31** *For any irrigation game v, $\phi(v) \in C(v)$, i.e. the Shapley value is in the core. Moreover, since every airport game is an irrigation game, for any airport game v: $\phi(v) \in C(v)$.*

Definition 3.32 (SEC rule) *For all cost trees (G, c) and for all players i the distribution according to the* SEC rule *is given as follows:*

$$\xi_i^{SEC}(G, c) = \sum_{j \in P_i(G) \setminus \{r\}} \frac{c_{\overline{j_- j}}}{|S_j(G)|} \ .$$

Definition 3.33 *Let $G = (V, A)$ be a rooted tree. Rule χ defined on the set of cost trees denoted by G satisfies*

- *non-negativity, if for each cost function c, $\chi(G, c) \geq 0$,*
- *cost boundedness, if for each cost function c, $\chi(G, c) \leq \left(\sum_{e \in A_{P_i(G)}} c_e \right)_{i \in N}$,*
- *efficiency, if for each cost function c, $\sum_{i \in N} \chi_i(G, c) = \sum_{e \in A} c_e$,*
- *equal treatment of equals, if for each cost function c and pair of players $i, j \in N$, $\sum_{e \in A_{P_i(G)}} c_e = \sum_{e \in A_{P_j(G)}} c_e$ implies $\chi_i(G, c) = \chi_j(G, c)$,*
- *conditional cost additivity, if for any pair of cost functions c, c', $\chi(G, c + c') = \chi(G, c) + \chi(G, c')$,*
- *independence of at-least-as-large costs, if for any pair of cost functions c, c' and player $i \in N$ such that for each $j \in P_i(G)$, $\sum_{e \in A_{P_j(G)}} c_e = \sum_{e \in A_{P_j(G)}} c'_e$, $\chi_i(G, c) = \chi_i(G, c')$.*

$\boxed{\textbf{Claim}}$ **3.34** *Let G be a rooted tree, χ be defined on cost trees (G, c), solution ψ be defined on $\mathcal{G}_G$ such that $\chi(G, c) = \psi(v_{(G,c)})$ for any cost function c. Then, if χ satisfies*

- *non-negativity and cost boundedness, then ψ is NP,*
- *efficiency, then ψ is PO,*
- *equal treatment of equals, then ψ is ETP,*
- *conditional cost additivity, then ψ is ADD,*
- *independence of at-least-as-large costs, then ψ is M.*

$\boxed{\textbf{Theorem}}$ **3.35** *A rule χ on cost-tree problems satisfies non-negativity, cost boundedness, efficiency, equal treatment of equals and conditional cost additivity, if and only if $\chi = \xi$, i.e. χ is the SEC rule.*

$\boxed{\textbf{Theorem}}$ **3.36** *Rule χ on cost-tree problems satisfies efficiency, equal treatment of equals and independence of at-least-as-large costs, if and only if $\chi = \xi$, i.e. χ is the SEC rule.*

3.3 Upstream responsibility

In this chapter we consider further cost sharing problems given by rooted trees, called *cost-tree problems*, but we are considering different applications from those so far. We will consider energy supply chains with a motivated dominant leader, who has the power to determine the responsibilities of suppliers for both direct and indirect emissions. The induced games are called *upstream responsibility games* Gopalakrishnan et al. (2017), and henceforth we will refer to it as *UR game*.

The results on the class of UR games presented in the following have been published in working paper Radványi (2018).

Definition 3.37 (UR game) *For all cost trees (G, c), player set $N = V \setminus \{r\}$, and coalition S let the UR game be defined as follows.*

$$v_{(G,c)}(S) = \sum_{j \in \mathcal{E}_S} c_j \ ,$$

where the value of the empty set is 0.

$\boxed{\textbf{Lemma}}$ **3.38** *For any chain G, $T \subseteq N$ such that $T = P_i(G)$, $i \in N$, $\bar{u}_T \in \mathcal{G}_G$ it holds that $\{\bar{u}_T\}_{T \in 2^N \setminus \{\emptyset\}} \subset \mathcal{G}_A^N \subset \mathcal{G}_{UR}^N$.*

Lemma **3.39** *For all rooted trees G: $\mathcal{G}_G \subset$ Cone $\{\bar{u}_{P_i(G)}\}_{i \in N}$. Therefore, $\mathcal{G}_A \subset \mathcal{G}_{UR}^N \subset$ Cone $\{\bar{u}_T\}_{T \in 2^N \setminus \{\emptyset\}}$.*

Lemma **3.40** *Neither $\mathcal{G}_A^N$ nor $\mathcal{G}_{UR}^N$ is convex.*

Lemma **3.41** *For any rooted tree G and $v = \sum_{i \in N} \alpha_{P_i(G)} \bar{u}_{P_i(G)} \in \mathcal{G}_G$, and for each $i^* \in N$ it holds that $\sum_{i \in N \setminus \{i^*\}} \alpha_{P_i(G)} \bar{u}_{P_i(G)} \in \mathcal{G}_G$. Therefore, for any airport game $v = \sum_{T \in 2^N \setminus \{\emptyset\}} \alpha_T \bar{u}_T$ and $T^* \in 2^N \setminus \{\emptyset\}$ it holds that $\sum_{T \in 2^N \setminus \{\emptyset, T^*\}} \alpha_T \bar{u}_T \in \mathcal{G}_A^N$, and for any upstream responsibility game $v = \sum_{T \in 2^N \setminus \{\emptyset\}} \alpha_T \bar{u}_T$ and $T^* \in 2^N \setminus \{\emptyset\}$ it holds that $\sum_{T \in 2^N \setminus \{\emptyset, T^*\}} \alpha_T \bar{u}_T \in \mathcal{G}_{UR}^N$.*

Lemma **3.42** *All UR games are concave.*

Let us summarize our results in the following.

Corollary **3.43** *For a fixed player set the class of airport games is a union of finitely many convex cones, but the class is not convex. Moreover, the class of airport games is a proper subset of the class of upstream responsibility games. The class of upstream responsibility games is also a union of finitely many convex cones, but is not convex either. Finally, the class of UR games is a proper subset of the finite convex cone spanned by the duals of the unanimity games, therefore every upstream responsibility game is concave, so every airport game is concave too.*

Theorem **3.44 (Shapley's axiomatization)** *For all rooted trees G a value ψ is PO, NP, ETP and ADD on $\mathcal{G}_G$, if and only if $\psi = \phi$, i.e. the value is the Shapley value. In other words, value ψ on the class of UR games is PO, NP, ETP and ADD, if and only if $\psi = \phi$.*

Theorem **3.45 (Young's axiomatization)** *For any rooted tree G, a single-valued solution ψ on $\mathcal{G}_G$ is PO, ETP and M if and only if $\psi = \phi$, i.e. it is the Shapley value. Therefore, a value ψ on the class of UR games is PO, ETP and M if and only if $\psi = \phi$.*

Corollary **3.46** *For all UR games v it holds that $\phi(v) \in C(v)$ i.e. the Shapley value is in the core.*

Corollary **3.47** *The Shapley value on the class of the upstream responsibility games can be calculated in polynomial time.*

3.4 Shortest path games

In this chapter we consider the class of *shortest path games*. There are given some agents, a good, and a network. The agents own the nodes of the network and they want to transport the good from certain nodes of the network to others. The transportation cost depends on the chosen path within the network, and the successful transportation of a good generates profit. The problem is not only choosing the shortest path (a path with minimum cost, i.e. with maximum profit), but we also have to divide the profit among the players.

Fragnelli et al. (2000) introduce the notion of shortest path games and they prove that the class of such games coincides with the well-known class of monotone games.

In this chapter we consider further characterizations of the Shapley value: Shapley (1953)'s, Young (1985)'s, Chun (1989)'s, and van den Brink (2001)'s axiomatizations. We analyze their validity on the class of shortest path games, and conclude that all aforementioned axiomatizations are valid on the class.

Our results are different from Fragnelli et al. (2000) in two aspects. Firstly, Fragnelli et al. (2000) give a new axiomatization of the Shapley value, conversely, we consider four well-known characterizations. Secondly, Fragnelli et al. (2000)'s axioms are based on the graph behind the problem, in this chapter we consider game-theoretic axioms only. Namely, while Fragnelli et al. (2000) consider a fixed-graph problem, we consider all shortest path problems, and examine them from the viewpoint of an abstract decision maker who focuses rather on the abstract problem, instead of the concrete situations. The following results have been published in Pintér and Radványi (2013).

Definition 3.48 *A* shortest path cooperative situation σ *is a tuple* (Σ, N, o, g)*. We can identify* σ *with the corresponding cooperative TU game* v_σ *given by, for each* $S \subseteq N$*:*

$$v_\sigma(S) = \begin{cases} g - L_S, & \text{if } S \text{ owns a path in } \Sigma \text{ and } L_S < g, \\ 0 & \text{otherwise,} \end{cases}$$

where L_S *is the length of the shortest path owned by* S*.*

Definition 3.49 *A* shortest path game v_σ *is a game associated with a shortest path cooperative situation* σ*. Let* SPG *denote the class of shortest path games.*

$\boxed{\textbf{Theorem}}$ **3.50** *Let there be given* $A \subseteq \mathcal{G}^N$ *such that the* Cone $\{u_T\}_{T\subseteq N,\ T\neq\emptyset} \subseteq A$*. Then value* ψ *on* A *is PO, NP, ETP and ADD, if and only if* $\psi = \phi$*.*

$\boxed{\textbf{Corollary}}$ **3.51 (Shapley's axiomatization)** *A value ψ is PO, NP, ETP and ADD on the class of monotone games, if and only if $\psi = \phi$, i.e. it is the Shapley value.*

Definition 3.52 *On set $A \subseteq \mathcal{G}^N$ value ψ is / satisfies*

- *fairness property (FP), if for all games $v, w \in A$ and players $i, j \in N$ for which $v + w \in A$ and $i \sim^w j$: $\psi_i(v + w) - \psi_i(v) = \psi_j(v + w) - \psi_j(v)$,*
- *coalitional strategic equivalence (CSE), if for all games $v \in A$, player $i \in N$, coalition $T \subseteq N$, and $\alpha > 0$: $i \notin T$ and $v + \alpha u_T \in A$ implies $\psi_i(v) = \psi_i(v + \alpha u_T)$.*

$\boxed{\textbf{Theorem}}$ **3.53 (van den Brink's axiomatization)** *A value ψ is PO, NP, and FP on the class of monotone games, if and only if $\psi = \phi$, i.e. it is the Shapley value.*

$\boxed{\textbf{Lemma}}$ **3.54** *On the class of monotone games M and CSE are equivalent.*

$\boxed{\textbf{Corollary}}$ **3.55 (Young's and Chun's axiomatization)** *A value ψ is PO, ETP, and CSE on the class of monotone games, if and only if $\psi = \phi$, i.e. it is the Shapley value.*

4 Publications

4.1 In English

4.1.1 Refereed journal

Pintér M, Radványi AR (2013) The Shapley value for shortest path games: a non-graph based approach. Central European Journal of Operations Research 21(4):769–781

4.1.2 Working paper

Pintér M, Radványi AR (2019a) Axiomatizations of the Shapley Value for Upstream Responsibility Games. Corvinus Economics Working Papers, ID: 4090, Corvinus University Budapest

Pintér M, Radványi AR (2019b) Upstream responsibility games – the non-tree case. Corvinus Economics Working Papers, ID: 4325, Corvinus University Budapest

Radványi AR (2018) The Shapley Value for Upstream Responsibility Games. Corvinus Economics Working Papers, ID: 3779, Corvinus University of Budapest

Márkus J, Pintér M, Radványi AR (2011) The Shapley value for airport and irrigation games. MPRA, Working Paper

4.2 In Hungarian

4.2.1 Refereed journal

Radványi AR (2019) Kooperatív sztenderd fixfa játékok és alkalmazásuk a vízgazdálkodásban. Alkalmazott Matematikai Lapok 36:83–105

Biró P, Csóka P, Kóczy L, Radványi AR, Sziklai B (2013) K ozgazdasági Nobel-emlékdíj 2012: Alvin E. Roth és Lloyd S. Shapley. Magyar Tudomány 2:190–199

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