Cost Sharing Models in Game Theory

PhD Thesis

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Notations

Cost allocation models

N	$= \{1, 2, \ldots, n\}$ the finite set of players
L	$\subseteq N$ the set of leaves of a tree
i	$\in N$ a given user/player
c_i	the cost of the ith segment/edge
I_i^-	$= \{j \in N \mid j < i\}$ the set of users preceding i
I_i^+	$= \{j \in N \mid i < j\}$ the set of users following i
ξ_i^a	the average cost allocation rule
ξ_i^s	the serial cost allocation rule
ξ_i^{eq}	the allocation rule based on the equal allocation of the non-separable cost
ξ_i^{riu}	the allocation rule based on the ratio of individual use of the non-separable cost
ξ_i^r	the restricted average cost allocation rule

Introduction to cooperative game theory

TU game	cooperative game with transferable utility
$\lvert N \rvert$	the cardinality of N
2^N	the class of all subsets of N
$A \subset B$	$A \subseteq B$, but $A \neq B$
$A \uplus B$	the disjoint union of sets A and B
(N, v)	the cooperative game defined by the set of players N and the characteristic function v
$v(S)$	the value of coalition $S \subseteq N$
$\mathcal{G}^N$	the class of games defined on the set of players N
(N, v_c)	the cost game with the set of players N and cost function v_c

$I^*(N,v)$	the set of preimputations
$I(N,v)$	the set of imputations
$C(N,v); C(v)$	the set of core allocations; the core of a cooperative game
$X^*(N,v)$	$= \{x \in \mathbb{R}^N \mid x(N) \leq v(N)\}$, the set of feasible payoff vectors
$\psi(v)$	$= (\psi_i(v))_{i\in N} \in \mathbb{R}^N$, the single-valued solution of a game v
$v_i'(S)$	$= v(S \cup \{i\}) - v(S)$, the player i's individual marginal contribution to coalition S
$\phi_i(v)$	the Shapley value of player i
$\phi(v)$	$= (\phi_i(v))_{i\in N}$ the Shapley value of a game v
π	an ordering of the players
Π_N	the set of the possible orderings of set of players N

Fixed tree games

$\Gamma(V,E,b,c,N)$	a fixed tree network
$G(V,E)$	directed graph with the sets of nodes V and edges E
r	$\in V$ the root node
c	the non-negative cost function defined on the set of edges
$S_i(G)$	$= \{j \in V : i \leq j\}$ the set of nodes accessible from i via a directed graph
$P_i(G)$	$= \{j \in V : j \leq i\}$ the set of nodes on the unique path connecting i to the root
$\bar{S}$	the trunk of the tree containing the members of coalition S (the union of unique paths connecting nodes of members of coalition S to the root)
u_T	the unanimity game on coalition T
$\bar{u}_T$	the dual of the unanimity game on coalition T

Airport and irrigation games

(G,c)	the cost tree defined by the graph G and cost function c
$\mathcal{G}_I^A$	the class of airport games on the set of players N
$\mathcal{G}_I^N$	the class of irrigation games on the set of players N

$\mathcal{G}_G$	the subclass of irrigation games induced by cost tree problems defined on rooted tree G
Cone $\{v_i\}_{i\in N}$	the convex cone spanned by given v_i games
i_-	$= \{j \in V : \overline{ji} \in E\}$ the player preceding player i
v_{irr}	the irrigation game
$i \sim^v j$	i and j are equivalent players in game v, i.e. $v_i'(S) = v_j'(S)$ for all $S \subseteq N \setminus \{i,j\}$
ξ^{SEC}	"sequential equal contributions" cost allocation rule

Upstream responsibility

$\vec{ij}$	the directed edge pointing from i to j
$\mathcal{E}_i$	the set of edges whose corresponding emissions are the direct or indirect responsibility of i
$\mathcal{E}_S$	the set of edges whose corresponding emissions are the direct or indirect responsibilities of players in S
$\mathcal{G}_{UR}^N$	the class of UR games corresponding to the set of players of N

Shortest path tree games

$\Sigma(V,A,L,S,T)$	the shortest path tree problem
(V,A)	directed, acyclic graph defined by the sets of nodes V and edges A
$L(a)$	the length of the edge a
S	$\subseteq N$ the nonempty set of sources
T	$\subseteq N$ the nonempty set of sinks
P	the path connecting the nodes of $\{x_0,\ldots,x_p\}$
$L(P)$	the length of path P
$o(\{x\})$	function defining the owners of node x
$o(P)$	defines the owners of the nodes on the path P
g	the income from transporting a good from a source to a sink
$\sigma(\Sigma,N,o,g)$	the shortest path cooperative situation
v_σ	the shortest path game

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There is a long journey leading to finishing a doctoral thesis, and if we are lucky, we are not alone on this journey. Our companions are our families, friends, mentors, all playing an important role in shaping how difficult we feel the journey is. When arriving at a crossroads they also undoubtedly influence where we go next.

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I dedicate this thesis to the memories of my parents and grandparents.

Foreword

Dear Reader,

When someone asks me what I do, usually my short answer is: "I'm researching cooperative game theory". Having given this answer, from members of the general public I usually get more questions, so I tend to add that "I attempt to distribute costs fairly". This is usually an attention grabber, because it seems to be an understandable, tangible goal. From this point we continue the discussion and the questions I get (in case the person is still interested, of course) are usually the following. What kind of costs are these? Among whom are we distributing them? What does fair mean? Is there a fair allocation at all? If yes, is it unique? Can it be easily calculated?

The first two questions are interrelated: in a specific economic situation costs arise, if there are participants that perform some activity. This is usually easy to word when we have a concrete example. Those without deeper mathematical knowledge usually stop at these two questions. However, modeling itself is already an important mathematical step. Since we are discussing concrete and (many times) common situations, we do not necessarily think that describing the situation, the participants, and their relationships has in effect given the (practically mathematical) answer to two simple, everyday questions. In other words, modeling does not necessarily require complex mathematical methods. As an example, let us consider an everyday situation, building a water supply network in a village. This network connects to a provider source, and the entire water supply network branches off from this point. It is easy to imagine what this system might look like: from a linear branch further segments branch off providing access to water for all residents. Of course, there are no "circles" in this system, it is sufficient to reach all endpoints via a single path. This system defines a so called tree structure. However, this system must be built, maintained, and so forth. We can already see the costs that the residents of the village will have to pay for, therefore we have now defined the agents of the model as well. The structure of the network describes the relationships among the agents, which is essential for the distribution.

Let us consider the next question, fairness. Not everybody might phrase the question as "what makes a distribution fair". Clearly, the costs must be covered by someone(s). Let us give a distribution for this implicit requirement. Those more interested in the topic may also be curious about the mathematician's expected "complicated" answer to a question that seems so simple. Instead, the mathematician replies with a question: What should the distribution be like? More precisely, what conditions should it satisfy? What do we expect from the distribution? That the entire cost is covered? Is this sufficient? Would it be a good answer that the last house in the village should pay for all the costs? In the end, it is because of this resident that such a long pipeline must be built. What if the resident is (rightfully) displeased with this decision, is not willing to pay, and instead chooses not to join the system? This may delay, or even halt the entire project, for which external monetary funding could be acquired only if everybody joins and the entire village is covered by the network. How about considering another alternative then, let us make everybody pay evenly. This could be an acceptable solution, but after giving it more consideration we might find that this solution is also "unfair" to a certain degree. It is possible that the first resident in the system now pays more than as if it was building the system alone, therefore in practice the resident subsidizes those further back in the system. There is no clear advantage to be gained from such a setup, what's more, it may be downright undesirable for the resident. Therefore, it is yet another rightful expectation that all participants' payments should be proportional to the segments they use in the water supply network. Those closer to the source should pay less, and those further back should pay more, since they use a larger section of the system. Several further aspects could be taken into consideration (e.g. the quantity of water used by the participants, these are the so called weighted solutions). It is the modeler's task to explore the requirements and customize the solution to the specific problem, adding conditions regarding the distribution to the model. These conditions will be given in the form of axioms, and for each solution proposal it must be examined whether it satisfies the axioms and therefore the expectations as well.

The next step is the question of existence. Is it at all possible to give a solution that satisfies all expected properties, or has the definition been perhaps too strict, too demanding? Namely, in certain cases some conditions cannot all be satisfied at the same time. Then we need to do with less, and after removing one (or more) conditions we can look for solutions that satisfy the remaining conditions. Of course, this raises the question, which condition(s) we want to remove, which are removable, if at all. This is not a simple question and always depends on the specific model, the modeler, and the expectations as well. In some fortunate cases, not only one, but multiple solutions satisfy the expected properties. Then we can analyze what further "good" properties these solutions possess, and choose the fittest for our purposes accordingly. Probably the most favorable cases are those for which there is exactly one solution, in other words, the distribution solving the problem is unique. Therefore, there are no further questions, choices, or any aspects to be debated. The answer is given, unambiguous, and therefore (probably) undisputable.

The last question to be raised is just as important, if not more so, than the previous ones. We have created the model, defined the problem, the requirements, and have even found a theoretical solution. But can it be calculated? Can the solution be implemented in practice? What is the use of a solution, if we can never learn, how much the participants really have to pay? It can happen that there are "too many" participants, the model is "too complex", the network is "too large", and we cannot arrive at a precise number for the solution even by leveraging computers and algorithmic methods. Fortunately, in several cases the structure of the network is such that in spite of its "vastness", an efficient algorithm exists for computing a concrete solution. These problems belong to the area of computational complexity.

In the present thesis we will examine these questions for specific types of situations, focusing primarily on the definition of "fairness". We analyze the theory behind models of several economic problems, and give answers applicable in practice to the questions raised regarding distribution. As we progress from question to question in the case of specific situations, the methodology will seemingly become increasingly more abstract, the answers more theoretical. All along, however, we do not lose sight of our main objective: providing "fair" results applicable in practice as a solution. What does cooperative game theory add to all this? It adds methodology, models, axioms, solution proposals, and many more. The details emerge in the following chapters. Thank you for joining me!

Chapter 1

Introduction

In the present thesis we examine economic situations that can be modeled using special directed networks. A special network in Chapters 2, 4, 5, and 6 will translate into rooted trees, while in Chapter 7 will mean directed, acyclic graph. Even though the basic scenarios are different, we will be looking for answers to similar questions in both cases. The difference stems only from the relationships of the agents in the economic situation. Namely, we will be examining problems where economic agents (companies, individuals, cities, states, so forth) collaboratively create (in a wider sense) an "end product", a measurable "result". Our goal is to "fairly" distribute this "result" among the players.

For example, we will be discussing situations, where a group of users are provided a service by a utility (e.g. water supply through an infrastructure). A network (graph) comprises all users, who receive the service through this network. The building, maintenance, and consuming of the service may incur costs. Another example is a problem related to the manufacturing and transportation of a product. A graph describes the relationships among suppliers, companies, and consumers during the manufacturing process. In this case, transportation costs are incurred, or profits and gains are achieved from the sales of the product.

In the case of both examples we will be looking for the answer to the question of "fairly" allocating the arising costs or achieved gains among the participants. This raises further questions, one is what it means for an allocation to be "fair", what properties are expected from a "fair" allocation? A further important question is whether such allocation exists at all, and if it exists, is it unique? If multiple distributions satisfy the criteria, then how should we choose from them? Another important aspect is that we must provide an allocation that can be calculated in the given situation. In the thesis we are going to discuss these questions in the case of concrete models, leveraging the toolset of cooperative game theory.

The structure of the thesis is as follows. After the introduction, in Chapter 2 we are analyzing a real-life problem and its possible solutions. Aadland and Kolpin (2004) studied cost allocation models applied by farmers for decades in the state of Montana, this work is the origin for the average cost-sharing rule, the serial cost sharing rule, and the restricted average cost-sharing rule. We present two further water management problems that arose during the planning of the economic development of Tennessee Valley. We discuss the two problems and their solution proposals (Straffin and Heaney, 1981). We formulate axioms that are meant to describe the "fairness" of an allocation, and we discuss which axioms are satisfied by the aforementioned allocations.

In Chapter 3 we introduce the fundamental notions and concepts of cooperative game theory, focusing primarily on those areas that are relevant to the present thesis. We discuss in detail the core (Shapley, 1955; Gillies, 1959) and the Shapley value (Shapley, 1953), that play an important role in finding a "fair" allocation.

In Chapter 4 we present the class of fixed-tree games. We provide the representation of a fixed-tree game, and examine what properties the core and the Shapley value of the games possess. We present several application domains stemming from water management problems.

In Chapter 5 we discuss the classes of airport and irrigation games, and the characterizations of these classes. Furthermore, we extend the results of Dubey (1982) and Moulin and Shenker (1992) on axiomatization of the Shapley value on the class of airport games to the class of irrigation games. We compare the axioms used in cost allocation literature with the axioms of TU games, thereby providing two new versions of the results of Shapley (1953) and Young (1985b).

In Chapter 6 we introduce the upstream responsibility games and characterize the game class. We show that Shapley's and Young's characterizations are valid on this class as well.

In Chapter 7 we discuss shortest path games, which are defined on graph structures different from those in the previous chapters. After introducing the notion of a shortest path game, we show that this game class is equal to the class of monotone games. Then we present further axiomatizations of the Shapley value, namely Shapley (1953)'s, Young (1985b)'s, Chun (1989)'s, and van den Brink (2001)'s characterizations, and examine if they are valid in the case of shortest path games.

In Chapter 8 we summarize our results.

We will annotate our own results (lemmata, claims, theorems, and new proofs of known theorems) in the thesis by framing their names. The author of present thesis and the co-authors of the publications have participated equally in the work leading to these results.

Chapter 2

Cost allocation models

This chapter and our entire thesis discusses cost allocation problems. The problem is described as follows. A group of farmers acquire water supply for their land from a ditch that is connected to a main ditch. Operation and maintenance of the ditch incur costs which are jointly covered by the group. The question is how the farmers (henceforth users) may "fairly" divide the aforementioned costs among the group. After the introduction we shall present the basic models and the axioms aiming to define the notion of "fairness". The axioms are based on the work of Aadland and Kolpin (1998).

The basis of our models are solution proposals for real-life problems. In Aadland and Kolpin (2004) cost allocation patterns are examined, utilized for decades in practice by Montana ranchers. In their paper they describe two cost allocation mechanisms, the serial and the average cost share rules. Furthermore, they present the restricted cost share method (Aadland and Kolpin, 1998), which unifies the "advantages" of the previous two methods. Their results were defined on non-branching trees, i.e. chains.

In this chapter we generalize the above models for problems described by tree-structures, and show that they uphold the properties of cost allocations described for chains. These results were presented in our paper (Kovács and Radványi, 2011, in Hungarian).

The basis of two further models we analyzed are also solution proposals for questions arising in connection with water management problems. The Tennessee Valley Authority was established in 1933 with the purpose of creating the plans for the economic development of Tennessee Valley. TVA's work was first presented in economics and game theory literature by Straffin and Heaney (1981), and is the origin for methods based on separable - non-separable costs. The basis for the description of our models is the work of Solymosi (2007) (in Hungarian).

For formal analysis let us first summarize the two most important properties of the ditch in our example. Firstly, all users utilize water first and foremost for the irrigation of their own land, and the water quantity required for and costs related to livestock are negligible. Experience shows that the capacity of ditches is sufficient for the irrigation of all lands. Therefore, the problem we define is not the allocation of the water supply, but the allocation of operation, maintenance, and other costs among users.

2.1 Basic models

Let us represent the problem with a tree, let the root of the tree be the main ditch (denoted by r), and let the nodes of the tree be the users. We denote the set of leaves of the tree by L.

Let $N = \{1, 2, \ldots, n\}$ be the ordered, finite set of users connected to the main ditch ($r \notin N$, $L \subseteq N$). There exists a reflexive, transitive ordering on the set of users, which is not necessarily a total ordering, since not all two users' positions may be comparable. Let us, for example, consider the access order of a depth-first search starting from the root of the graph. In this ordering let i denote a ith user from the flood gate. Let the ith segment of the ditch (i.e. the ith edge of the graph) be the segment through which the ith user connects to the system. For all $i \in N$ let c_i denote the yearly maintenance cost of the ith segment of the ditch, and let the cost vector defined by these costs be denoted by $c = (c_i)_{i\in N} \in \mathbb{R}^N_+$. The sought result is a "fair" allocation of the summed cost $\sum_{i\in N} c_i$.

Remark 2.1 *In special cases the problem can be represented by a single chain, whose first node is the main ditch, while the users are the chain's following nodes, and the ordering is self-evident. The associated problem, representable by a single chain, is the well-known airport problem (Littlechild and Thompson, 1977). We discuss this problem, and the associated cooperative game in detail in Chapter 5.*

In the following we define different cost allocation patterns, and examine their inherent properties. For this end we introduce new notations.

For user i we disambiguate between the sets of *preceding* and *following* users. Let I_i^- denote the set of nodes residing on the unambiguous path connecting i to the root. This is the set of users *preceding* i. Let us consider a direction consisting of edges pointing outwards from the root of the tree as source, and exactly one edge directed to each node. In this tree let I_i^+ denote the set of nodes accessible from i via a directed path. This is the set of users *following* i.

In other words, $I_i^- = \{j \in N | j < i\}$, $I_i^+ = \{j \in N | i < j\}$, where the relation $j < i$ denotes that there exists a directed path from j to i.

In our example we first examine two cost allocation rules, the (a) average and the (b) serial allocations. Costs may be measured per area in acres, in units of consumed water, or per individual user, we will use the latter method. Definition 2.2. and the axioms presented in this section are based on the work of Aadland and Kolpin (1998).

Definition 2.2 *A $\xi : \mathbb{R}_+^N \to \mathbb{R}_+^N$ mapping is a cost allocation rule, if* $\forall c \in \mathbb{R}_+^N$ $\sum\limits_{i \in N} \xi_i(c) = \sum\limits_{i \in N} c_i$, *where* $(\xi_i(c))_{i \in N} = \xi(c)$.

(a) According to the average cost allocation rule the operation and maintenance costs of the ditch are distributed evenly among users, i.e. $\forall i \in N$:

$$\xi_i^a(c) = \sum_{j \in N} \frac{c_j}{n}$$

(b) According to the serial cost allocation rule the costs associated with each segment are distributed evenly among those who utilize the segment, i.e. $\forall i \in N$:

$$\xi_i^s(c) = \sum_{j \in I_i^- \cup \{i\}} \frac{c_j}{|I_j^+| + 1}$$

Remark 2.3 *The latter may be defined as follows for the special case of* chains, *where* $\forall i \in N$:

$$\xi_i^s(c) = \frac{c_1}{n} + \cdots + \frac{c_i}{(n - i + 1)}.$$

We demonstrate the definition through the following two examples.

Example 2.4 *Let us consider the chain in Figure 2.1, where* $N = \{1, 2, 3\}$ *and* $c = \{6, 1, 5\}$. *According to the average cost share principle, the aggregated costs are evenly distributed among users, i.e.* $\xi^a(c) = (4, 4, 4)$. *On the other hand, applying the serial cost share rule the costs of the first segment shall be divided among all three users, the costs of the second segment among 2 and 3, while the third segment's costs are covered by user 3 alone. Therefore, we get the following allocation:* $\xi^s(c) = (2,\ 2.5,\ 7.5)$.

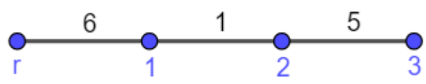

Figure 2.1: Ditch represented by a tree-structure in Example 2.4

Let us examine the result we acquire in the case represented by Figure 2.2.

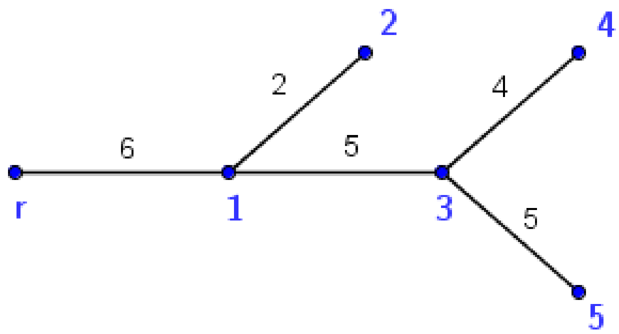

Figure 2.2: Ditch represented by a tree-structure in Example 2.5

Example 2.5 *Figure 2.2 describes a ditch, in which the set of users is* $N = \{1, 2, 3, 4, 5\}$, *and* $c = (6, 2, 5, 4, 5)$ *is the cost vector describing the costs associated with the respective segments. The root of the tree is denoted by* r, *representing the main ditch to which our ditch connects. In this case we get the following results:* $\xi^a(c) = \left(\frac{22}{5}, \frac{22}{5}, \frac{22}{5}, \frac{22}{5}, \frac{22}{5}\right)$, *and* $\xi^s(c) = \left(\frac{6}{5}, \frac{16}{5}, \frac{43}{15}, \frac{103}{15}, \frac{118}{15}\right)$. *In the case of the average cost share rule we divided the overall cost sum (22) into 5 equal parts. In the serial cost share case the cost of segment* c_1 *was divided into 5 parts, since it is utilized by all 5 users. The costs of* c_2 *are covered by the second user alone, while costs of* c_3 *must be divided evenly among users 3, 4, and 5. Costs of*

c_4 are covered by user 4 alone, and similarly, costs of c_5 are covered by user 5. These partial sums must be added up for all segments associated with each user.

In the following we characterize the above cost allocation rules, and introduce axioms which are expected to be fulfilled for proper modeling. Comparison of vectors is always done by comparing each coordinate, i.e. $c \leq c'$, if $\forall i \in N$: $c_i \leq c'_i$.

Axiom 2.6 *A rule ξ is cost monotone, if $\forall c \leq c'$: $\xi(c) \leq \xi(c')$.*

Axiom 2.7 *A rule ξ satisfies ranking, if $\forall c \in \mathbb{R}_+^N$ and $\forall j$ $\forall i \in I_j^- \cup \{j\}$: $\xi_i(c) \leq \xi_j(c)$.*

Remark 2.8 *In the case of chains ξ satisfies ranking, if $\forall i \leq j$: $\xi_i(c) \leq \xi_j(c)$.*

Axiom 2.9 *A rule ξ is subsidy-free, if $\forall c \in \mathbb{R}_+^N$ and $\forall I = \{i_1, i_2, \ldots, i_k\} \subseteq N$:*

$$\sum_{j \in J} \xi_j(c) \leq \sum_{j \in J} c_j,$$

where for the sake of brevity $J := I_{i_1}^- \cup \cdots \cup I_{i_k}^- \cup I$, where J is the sub-tree generated by I.

Remark 2.10 *In the case of chains set J will always be the set $I_i^- \cup \{i\}$ associated with I's highest index member i, i.e. $j \in J$ if and only if $j \leq i$. In this case ξ is subsidy-free if $\forall i \in N$ and $c \in \mathbb{R}_+^N$:*

$$\sum_{j \leq i} \xi_j(c) \leq \sum_{j \leq i} c_j.$$

The interpretation of the three axioms is straightforward. Cost monotonicity ensures that in case of increasing costs, the cost of no user shall decrease. Therefore, no user shall profit from any transaction that increases the overall cost.

If ranking is satisfied, cost allocations can be ordered by the ratio of ditch usage of each user. In other words, if $i \in I_j^-$, then by definition user j utilizes more segments of the ditch than i, i.e. the cost for j is at least as much as for i.

The subsidy-free property prevents any group of users having to pay more than the overall cost of the entire group. If this is not satisfied then some groups of users would have to pay for segments they are using, and pay further "subsidy" for users further down in the system. This would prevent us from achieving our goal of providing a "fair" cost allocation. (In a properly defined cooperative game, i.e. in irrigation games, the subsidy-free property ensures that core allocation is achieved as a result, see Radványi (2010) (in Hungarian) and Theorem 5.16 of present thesis.)

Furthermore, it may be easily verified that the serial cost allocation rule satisfies all three axioms, while the average cost allocation rule satisfies only the first two (Aadland and Kolpin, 1998).

In the following we introduce two further allocation concepts and examine which axioms they satisfy. These models are based on the work of Solymosi (2007). Let us introduce the following definitions.

Let $c(N)$ denote the total cost of operations and maintenance of the ditch, i.e. $\sum_{i \in N} c_i$. The sum $s_i = c(N) - c(N \setminus \{i\})$ is the *separable cost*, where $c(N \setminus \{i\})$ denotes the cost of the maintenance of the ditch, if i is not served. (Note that on the leaves, s_i always equals to c_i, otherwise it's 0.) We seek a cost allocation $\xi(c)$ that satisfies the $\xi_i(c) \geq s_i$ inequality for all $i \in N$. A further open question is how much individual users should cover from the remaining *non-separable (or common) cost*

$$k(N) = c(N) - \sum_{i \in N} s_i = c(N) - \sum_{i \in L} c_i.$$

The users do not utilize the ditch to the same degree, they only require its different segments. Therefore, we do not want users to pay more than as if they were using the ditch alone. Let $e(i)$ denote the *individual cost* of user $i \in N$:

$$e(i) = \sum_{j \in I_i^- \cup \{i\}} c_j.$$

"Fairness", therefore, means that the $\xi_i(c) \leq e(i)$ inequality must be satisfied for all $i \in N$. Of course, the aforementioned two conditions can only be met if

$$c(N) \leq c(N \setminus \{i\}) + e(i),$$

for all $i \in N$. In our case this is always true. By rearranging the inequality we get

$$c(N) - c(N \setminus \{i\}) \leq e(i), \text{ i.e. } s_i \leq e(i).$$

If i is not a leaf, then the value of s_i is 0, while $e(i)$ is trivially non-negative. If i is a leaf, then $s_i = c_i$, consequently $c_i \leq e(i)$ must be satisfied, as the definition of $e(i)$ shows. By subtracting the separable costs from the individual costs we get the $(k(i) = e(i) - s_i)_{i \in N}$ vector containing the costs of individual use of common segments.

Based on the above let us consider the below cost allocations:

- The non-separable cost's *equal* allocation:

$$\xi_i^{eq}(c) = s_i + \frac{1}{|N|} k(N) \quad \forall i \in N$$

- The non-separable cost's allocation based on *the ratio of individual use*:

$$\xi_i^{riu}(c) = s_i + \frac{k_i}{\sum\limits_{j \in N} k_j} k(N) \quad \forall i \in N$$

The above defined cost allocation rules are also known in the literature as "equal allocation of non-separable costs" and "alternative cost avoided" methods, respectively (Straffin and Heaney, 1981).

Example 2.11 *Let us consider the ditch in Example 2.5 (Figure 2.2). The previous two allocations can therefore be calculated as follows:* $c(N) = 22$, $s = (0, 2, 0, 4, 5)$, $k(N) = 11$, $e = (6, 8, 11, 15, 16)$, *and* $k = (6, 6, 11, 11, 11)$. *This means*

$$\xi^{eq}(c) = \left(\frac{11}{5}, \frac{21}{5}, \frac{11}{5}, \frac{31}{5}, \frac{36}{5} \right)$$

and

$$\xi^{riu}(c) = \left(\frac{66}{45}, \frac{156}{45}, \frac{121}{45}, \frac{301}{45}, \frac{346}{45} \right).$$

In the following we examine which axioms these cost allocation rules satisfy. Let us first consider the case of even distribution of the non-separable part.

Lemma 2.12 *The rule* ξ^{eq} *is cost monotone.*

Proof: We must show that if $c \leq c'$ then $\xi^{eq}(c) \leq \xi^{eq}(c')$. Vector $s(c)$ comprising separable costs is as follows: for all $i \in L$: $s_i = c_i$, otherwise 0. Accordingly, we consider two different cases, first when the cost increase is not realized on a leaf, and secondly when the cost increases on leaves. In the first case vector s does not change, but $k_c(N) \leq k_{c'}(N)$, since

$$k_c(N) = c(N) - \sum_{i \in L} c_i,$$

$$k_{c'}(N) = c'(N) - \sum_{i \in L} c'_i = c'(N) - \sum_{i \in L} c_i$$

and $c(N) \leq c'(N)$, therefore the claim holds.
However, in the second case for $i \in L$: $s_i(c) \leq s_i(c')$,

$$k_c(N) = \sum_{i \in N \setminus L} c_i = \sum_{i \in N \setminus L} c'_i = k_{c'}(N),$$

hence $k(N)$ does not change, in other words, the value of ξ_n^{eq} could not have decreased. If these two cases hold at the same time, then there is an increase in both vector s and $k(N)$, therefore their sum could not have decreased, either. ∎

Lemma 2.13 *The rule ξ^{eq} satisfies ranking.*

Proof: We must verify that for any i and j, in case of $i \in I_j^-$: $\xi_i^{eq}(c) \leq \xi_j^{eq}(c)$ holds. Since for a fixed c in case of $\xi_i^{eq}(c) = \xi_h^{eq}$ $\forall i, h \in I_j^-$, and $j \in L$: $\xi_i^{eq} < \xi_j^{eq}$ $\forall i \in I_j^-$, the property is satisfied. ∎

Lemma 2.14 *The rule ξ^{eq} (with cost vector $c \neq 0$) is not subsidy-free if and only if the tree representing the problem contains a chain with length of at least 3.*

Proof: If the tree consists of chains with length of 1, the property is trivially satisfied, since for all cases $\xi_i = c_i$.

If the tree comprises chains with length of 2, the property is also satisfied, since for a given chain we know that: $c(N) = c_1 + c_2$, $s = (0, c_2)$, $k(N) = c_1$. Therefore $\xi_1(c) = 0 + \frac{c_1}{2}$, $\xi_2(c) = c_2 + \frac{c_1}{2}$, meaning that $\xi_1(c) \leq c_1$ and $\xi_1(c) + \xi_2(c) \leq c_1 + c_2$, consequently the allocation is subsidy-free if $|N| = 2$.

We must mention one more case, when the tree consists of chains with length of at most 2, but these are not "independent", but "branching". This means that from the first node two further separate nodes branch out (for 3 nodes this is a "Y-shape"). Similarly to the above calculation it can be shown that the property is satisfied in these cases as well, since the c_1 cost is further divided into as many parts as the number of connected nodes. Therefore, for chains with length of at most 2, the subsidy-free property is still satisfied. If this is true for a tree comprising chains of length of at most 2, then the nodes may be chosen in any way, the subsidy-free property will still be satisfied, since for all associated inequalities we know the property is satisfied, hence it will hold for the sums of the inequalities, too.

Let us now consider the case when there is a chain with length of 3. Since the property must be satisfied for all cost structures, it is sufficient to give one example when it does not. For a chain with length of at least 3 (if there are more, then for any of them) we can write the following: $c(N) = c_1 + c_2 + \cdots + c_n$, $s = (0, 0, \cdots 0, c_n)$, $k(N) = c_1 + c_2 + \cdots + c_{n-1}$. Based on this $\xi_1(c) = 0 + \frac{c_1 + \cdots + c_{n-1}}{n}$, meaning that c_1 can be chosen in all cases such that $\xi_1(c) \leq c_1$ condition is not met. ∎

Example 2.15 *Let us demonstrate the above with a counter-example for a chain of length of 3. In this special case,* $\xi_1 = \frac{c_1+c_2}{3}$*, therefore it is sufficient to choose a smaller* c_1*, i.e.* $c_2 > 2c_1$*. For example, let* $c = (3, 9, 1)$ *on the chain, accordingly* $c(N) = 13$*,* $s = (0, 0, 1)$*,* $k(N) = 12$*, therefore* $\xi^{eq}(c) = (0 + \frac{12}{3}, 0 + \frac{12}{3}, 1 + \frac{12}{3}) = (4, 4, 5)$*. Note that the first user must pay more than its entry fee, and by paying more the user "subsidizes" other users further down in the system. This shows that the subsidy-free property is not satisfied, if the tree comprises a chain with length of at least 3.*

Let us now consider the allocation based on the ratio of individual use from non-separable costs.

Lemma 2.16 *The rule* ξ^{riu} *is not cost monotone.*

Proof: We present a simple counter-example: let us consider a chain with length of 4, let $N = \{1, 2, 3, 4\}$, $c = (1, 1, 3, 1)$, $c' = (1, 2, 3, 1)$. Cost is increased in case of $i = 2$. Therefore, cost monotonicity is already falsified for the first coordinates of $\xi^{riu}(c)$ and $\xi^{riu}(c')$, since $\xi_1^{riu}(c) = \dfrac{5}{13} = 0.384$, and $\xi_1^{riu}(c') = \dfrac{6}{16} = 0.375$, accordingly $\xi^{riu}(c) \geq \xi^{riu}(c')$ does not hold. ∎

Lemma **2.17** *The rule ξ^{riu} satisfies ranking.*

Proof: We must show that for $i \in I_j^-$: $\xi_i(c) \leq \xi_j(c)$. Based on the definition:

$$\xi_i(c) = s_i + \frac{k(i)}{\sum\limits_{l \in N} k(l)} k(N),$$

$$\xi_j(c) = s_j + \frac{k(j)}{\sum\limits_{l \in N} k(l)} k(N).$$

We know that for all $i \in I_j^-$: $s_i \leq s_j$, since $i \in N \setminus L$, therefore $s_i = 0$, while s_j is 0, if $j \in N \setminus L$ or c_j, if $j \in L$. We still have to analyze the relationships of $k(i) = e(i) - s_i$ and $k(j) = e(j) - s_j$. We must show that $k(i) \leq k(j)$, for all $i \in I_j^-$, because in this case $\xi_i(c) \leq \xi_j(c)$.

1. $i \in I_j^-$, $j \notin L$

   Therefore $s_i = s_j = 0$ and $e(i) \leq e(j)$, since $e(i) = \sum\limits_{l \in I_i^- \cup \{i\}} c_l$ and $e(j) = \sum\limits_{l \in I_j^- \cup \{j\}} c_l$, while $I_i^- \cup \{i\} \subset I_j^- \cup \{j\}$, if $i \in I_j^-$. Hence, $k(i) \leq k(j)$.

2. $i \in I_j^-$, $j \in L$

   This means that $0 = s_i < s_j = c_j$.

   In this case:

$$k(i) = e(i) - s_i = \sum_{l \in I_i^- \cup \{i\}} c_l - 0 = \sum_{l \in I_i^- \cup \{i\}} c_l,$$

$$k(j) = e(j) - s_j = \sum_{l \in I_j^- \cup \{j\}} c_l - c_j = \sum_{l \in I_j^-} c_l.$$

   - $i = j - 1$-re $k(i) = k(j)$,
   - $i \in I_{j-1}^-$-re $k(i) \leq k(j)$.

To summarize, for all cases $k(i) \leq k(j)$, proving our lemma. ■

Lemma **2.18** *The rule ξ^{riu} does not satisfy the subsidy-free property for all cases when $c \in \mathbb{R}^+$.*

Proof: Let us consider the following counter-example: for a chain with length of 5, let $N = \{1, 2, 3, 4, 5\}$, $c = (10, 1, 1, 100, 1)$. For the allocation we get the following:

$$\xi^{riu} = \left(\frac{1120}{257}, \frac{1232}{257}, \frac{1344}{257}, \frac{12544}{257}, \frac{12801}{257}\right).$$

The property is not satisfied for $i = 3$, since $\xi_1 + \xi_2 + \xi_3 = \dfrac{3696}{257} \approx 14.38 > 12 = c_1 + c_2 + c_3$. ■

We summarize our results in Table 2.1.

	Cost monotone	Ranking	Subsidy-free
ξ^a	✓	✓	×
ξ^s	✓	✓	✓
ξ^{eq}	✓	✓	×
ξ^{riu}	×	✓	×

Table 2.1: Properties of allocations

Based on the above results we shall analyze whether the axioms are independent. The following claim gives an answer to this question.

Claim 2.19 *In the case of cost allocation problems represented by tree structures, the properties cost monotonicity, ranking, and subsidy-free are independent of each other.*

Let us consider all cases separately:

- The subsidy-free property does not depend on the other two properties. Based on our previous results (see for example Table 2.1), for such cost allocation ξ^a is a good example, since it is cost monotone, satisfies ranking, but is not subsidy-free.

- The ranking property is independent of the other two properties. As an example let us consider the allocation known in the literature as Bird's rule. According to this allocation all users pay for the first segment of the path leading from the root to the user, i.e. $\xi_i^{Bird} = c_i \ \forall i$. It can be shown that this allocation is cost monotone and subsidy-free, but does not satisfy ranking.

- The cost monotone property does not depend on the other two properties. Let us consider the following example with a chain with length of 4, $N = \{1, 2, 3, 4\}$, and $c = \{c_1, c_2, c_3, c_4\}$ weights. Let $c_m = \min\{c_1, c_2, c_3, c_4\}$ and $c^m = \max\{c_1, c_2, c_3, c_4\}$, while $C = c^m - c_m$. Let us now consider the following cost allocation rule:

$$\xi_i(c) = \begin{cases} c_m/C, & \text{if } i \neq 4 \\ \sum_{i=1}^{4} c_i - \sum_{i=1}^{3} \xi(c_i) & \text{if } i = 4. \end{cases}$$

  In other words, user 4 covers the remaining part from the total sum after the first three users. Trivially, this allocation satisfies ranking and is subsidy-free. However, with $c = (1, 2, 3, 4)$ and $c' = (1, 2, 3, 10)$ cost vectors ($c' \geq c$) the allocation is not cost monotone, since if $i \neq 4$ then $\xi_i(c) = 1/3$ and $\xi_i(c') = 1/9$, i.e. $\xi_i(c) > \xi_i(c')$, which contradicts cost monotonicity.

2.2 Restricted average cost allocation

The results of a survey among farmers showed that violating any of the above axioms results in a sense of "unfairness" (Aadland and Kolpin, 2004). At the same time, we may feel that in the case of serial cost allocation the users further down in the system would in some cases have to pay "too much", which may also prevent us from creating a "fair" allocation. We must harmonize these findings with the existence of a seemingly average cost allocation. Therefore, we define a modified rule, which is "as close as possible" to the average cost allocation rule, and satisfies all three axioms at the same time. In this section we are generalizing the notions and results of Aadland and Kolpin (1998) from chain to tree.

Definition 2.20 *A* restricted average cost allocation *is a cost monotone, ranking, subsidy-free cost allocation, where the difference between the highest and lowest distributed costs is the lowest possible, considering all possible allocation principles.*

Naturally, in the case of average cost share the highest and lowest costs are equal. The restricted average cost allocation attempts to make this possible, while preserving the expected criteria, in other words, satisfying the axioms. However, the above definition guarantees neither the existence nor the uniqueness of such an allocation. The question of existence is included in the problem of whether different cost profiles lead to different minimization procedures with respect to the differences in distributed costs. Moreover, uniqueness is an open question too, since the aforementioned minimization does not give direct requirements regarding the "internal" costs. Theorem 2.21 describes the existence and uniqueness of the restricted average cost share rule.

Let us introduce the following. Let there be given sub-trees $H \subset I$ and let

$$P(H, I) = \frac{\sum\limits_{j \in I \setminus H} c_j}{|I| - |H|}.$$

$P(H, I)$ represents the user costs for the $I \setminus H$ ditch segments, distributed among the associated users.

Theorem 2.21 *There exists a restricted average cost share allocation ξ^r and it is unique. The rule can be constructed recursively as follows. Let*

$$\begin{aligned}
&\mu_1 = \min\left\{P(0, J) \,|\, J \text{ sub-tree}\right\}, && J_1 = \max\left\{J | P(0, J) = \mu_1\right\},\\
&\mu_2 = \min\left\{P(J_1, J) \,|\, J_1 \subset J \text{ sub-tree}\right\}, && J_2 = \max\left\{J | P(J_1, J) = \mu_2\right\},\\
&\vdots && \vdots\\
&\mu_j = \min\left\{P(J_{j-1}, J) \,|\, J_{j-1} \subset J \text{ sub-tree}\right\}, && J_j = \max\left\{J | P(J_{j-1}, J) = \mu_j\right\},\\
&\vdots && \vdots
\end{aligned}$$

and $\xi^r_i(c) = \mu_j \; \forall j = 1, \ldots, n'$, $J_1 \subset J_2 \subset \ldots \subset J_{n'} = N$, where $i \in J_j \setminus J_{j-1}$.

The above formula may be interpreted as follows. The value of μ_1 is the lowest possible among the costs for an individual user, while J_1 is the widest sub-tree of ditch segments on which the cost is the lowest possible. The lowest individual user's cost of ditch segments starting from J_1 is μ_2, which is applied to the $J_2 \setminus J_1$ subsystem, and so forth.

Example 2.22 *Let us consider Figure 2.3. In this case the minimum average cost for an individual user is 4, the widest sub-tree on which this is applied is $J_1 = \{1, 2\}$, therefore $\mu_1 = 4$. On the remaining sub-tree the minimum average cost for an individual user is 4.5 on the $J_2 = \{3, 4\}$ sub-tree, therefore $\mu_2 =$*

$\frac{(c_3+c_4)}{2} = 4.5$. *The remaining* c_5 *cost applies to* $J_3 = \{5\}$, *i.e.* $\mu_3 = 5$. *Based on the above* $\xi^r(c) = (4,\ 4,\ 4.5,\ 4.5,\ 5)$.

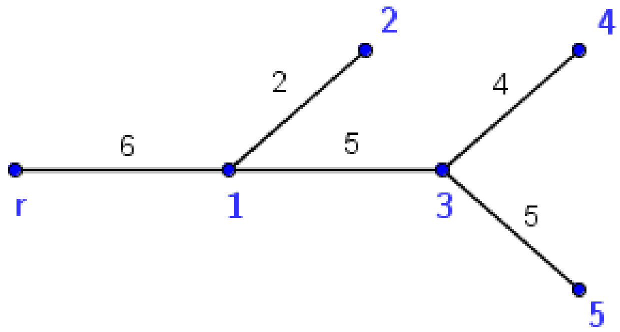

Figure 2.3: Ditch represented by a tree-structure in Example 2.22

Let us now consider the proof of Theorem 2.21.

Proof: Based on the definition of $P(H, I)$ it is easy to see that ξ^r as given by the above construction does satisfy the basic properties. Let us assume that there exists another ξ that is at least as good with respect to the objective function, i.e. besides ξ^r, ξ also satisfies the properties. Let us consider a tree and cost vector c for which ξ provides a different result than ξ^r, and let the value of n' in the construction of ξ^r be minimal. If $\xi(c) \neq \xi^r(c)$, then there exists i for which $\xi_i(c) > \xi_i^r(c)$, among these let us consider the first (i.e. the first node with such properties in the fixed order of nodes). This is $i \in J_k \setminus J_{k-1}$, or $\xi_i^r(c) = \mu_k$. We analyze two cases:

1. $k < n'$:

The construction of ξ^r implies that $\sum_{j \in J_k} \xi_j^r(c) = \sum_{j \in J_k} c_j \geq \sum_{j \in J_k} \xi_j(c)$, with the latter inequality a consequence of the subsidy-free property. Since the inequality holds, it follows that there exists $h \in J_k$, for which $\xi_h^r(c) > \xi_h(c)$.

In addition, let $c' < c$ be as follows: In the case of $j \in J_k$, $c'_j = c_j$, while for $j \notin J_k$ it holds that $\xi_j^r(c') = \mu_k$. The latter decrease is a valid step, since in c for all not in J_k the value of $\xi_j^r(c)$ was larger than μ_k, because of the construction. Because of cost-monotonicity, the selection of h, and the construction of c', the following holds:

$$\xi_h(c') \leq \xi_h(c) < \xi_h^r(c) = \xi_h^r(c') = \mu_k.$$

Consequently, $\xi_h(c) < \xi_h^r(c)$, and in the case of c' the construction consist of only $k < n'$ steps, which contradicts that n' is minimal (if $n' = 1$ then ξ^r equals to the average and is therefore unique).

2. $k = n'$:

In this case $\xi_i(c) > \xi_i^r(c) = \mu_{n'}$, and at the same time $\xi_1(c) \leq \xi_1^r(c)$ (since due to $k = n'$ there is no earlier that is greater). Consequently, in the case of ξ the difference between the distributed minimum and maximum costs is larger than for ξ^r, which contradicts the minimization condition. ■

By definition the restricted average cost allocation was constructed by minimizing the difference between the smallest and largest cost, at the same time preserving the cost monotone, ranking and subsidy-free properties. In their article Dutta and Ray (1989) introduce the so called "egalitarian" allocation, describing a "fair" allocation in connection with Lorenz-maximization. They have shown that for irrigation problems defined on chains (and irrigation games, which we define later) the egalitarian allocation is identical to the restricted average cost allocation. Moreover, for convex games the allocation can be uniquely defined using an algorithm similar to the above.

According to our next theorem the same result can be achieved by simply minimizing the largest value calculated as a result of the cost allocation. If we measure usefulness as a negative cost, the problem becomes equivalent to maximizing Rawlsian welfare (according to Rawlsian welfare the increase of society's welfare can be achieved by increasing the welfare of those worst-off). Namely, in our case maximizing Rawlsian welfare is equivalent to minimizing the costs of the nth user. Accordingly, the restricted average cost allocation can be perceived as a collective aspiration towards maximizing societal welfare, on the basis of equity.

Theorem 2.23 *The restricted average cost rule is the only cost monotone, ranking, subsidy-free method providing maximal Rawlsian welfare.*

Proof: The proof is analogous to that of Theorem 2.21. The $\xi_i(c) > \xi_i^r(c) = \mu_{n'}$ result in the last step of the aforementioned proof also means that in the case of ξ allocation Rawlsian welfare cannot be maximal, leading to a contradiction. ■

In the following we discuss an additional property of the restricted average cost rule, which helps expressing "fairness" of cost allocations in a more subtle way. We introduce an axiom that enforces a group of users that has so far been "subsidized" to partake in paying for increased total costs, should such an increase be introduced in the system. This axiom and Theorem 2.25 as shown by Aadland and Kolpin (1998) are as follows:

Axiom 2.24 *A rule ξ satisfies the reciprocity axiom, if $\forall i$ the points*

(a) $\sum_{h \leq i} \xi_h(c) \leq \sum_{h \leq i} c_h$

(b) $c' \geq c$ *and*

(c) $\sum_{h \leq i} (c_h - \xi_h(c)) \geq \sum_{j > i} (c'_j - c_j)$

imply that the following is not true: $\xi_h(c') - \xi_h(c) < \xi_j(c') - \xi_j(c) \quad \forall h \leq i$ *and* $j > i$.

The reciprocity axiom describes that if (a) users $\{1, ..., i\}$ receive (even a small) subsidy, (b) costs increase from c to c', and (c) in case the additional costs are higher on the segments after i than the subsidy received by group $\{1, ..., i\}$, then it would be inequitable if the members of the subsidized group had to incur less additional cost than the $\{i + 1, ..., n\}$ segment subsidizing them. Intuitively, as long as the cost increase of users $\{i + 1, ..., n\}$ is no greater than the subsidy given to users $\{1, ..., i\}$, the subsidized group is indebted towards the subsidizing group (even if only to a small degree). The reciprocity axiom ensures that for at least one member of the subsidizing group the cost increase is no greater than the highest cost increase in the subsidized group.

Theorem 2.25 (Aadland and Kolpin, 1998) *The restricted average cost allocation, when applied to chains meets cost monotonicity, ranking, subsidy-free and reciprocity.*

2.3 Additional properties of serial cost allocation

The serial cost allocation rule is also of high importance with respect to our original problem statement. In this section we present further properties of this allocation. The axioms and theorems are based on the article of Aadland and Kolpin (1998).

Axiom 2.26 *A rule ξ is semi-marginal, if $\forall i \in N \setminus L$: $\xi_{i+1}(c) \leq \xi_i(c) + c_{i+1}$, where $i+1$ denotes a direct successor of i in I_i^+.*

Axiom 2.27 *A rule ξ is incremental subsidy-free, if $\forall i \in N$ and $c \leq c'$:*

$$\sum_{h \in I_i^- \cup \{i\}} (\xi_h(c') - \xi_h(c)) \leq \sum_{h \in I_i^- \cup \{i\}} (c'_h - c_h).$$

Semi-marginality expresses that if $\xi_i(c)$ is a "fair" allocation on $I_i^- \cup \{i\}$, then user $i+1$ must not pay more than $\xi_i(c) + c_{i+1}$. Increasing subsidy-free ensures that starting from $\xi(c)$ in case of a cost increase no group of users shall pay more than the total additional cost.

For sake of completeness we note that increasing subsidy-free does not imply subsidy-free as defined earlier (see Axiom 2.9). We demonstrate this with the following counter-example.

Example 2.28 *Let us consider an example with 3 users, and defined by the cost vector $c = (c_1, c_2, c_3)$ and Figure 2.4.*

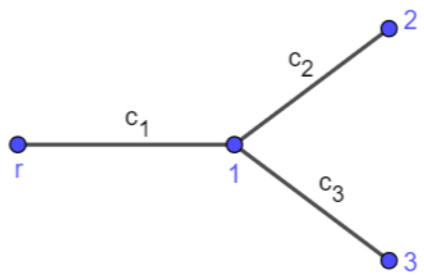

Figure 2.4: Tree structure in Example 2.28

Let the cost allocation in question be the following: $\xi(c) = (0, c_2-1, c_1+c_3+1)$. This allocation is incremental subsidy-free with arbitrary $c' = (c'_1, c'_2, c'_3)$, given that $c \leq c'$. This can be demonstrated with a simple calculation:

- *$i = 1$ implies the $0 \leq c'_1 - c_1$ inequality,*
- *$i = 2$ implies the $c'_2 - c_2 \leq c'_2 - c_2 + c'_1 - c_1$ inequality,*
- *$i = 3$ implies the $c'_3 - c_3 + c'_1 - c_1 \leq c'_3 - c_3 + c'_1 - c_1$ inequality.*

The above all hold because $c \leq c'$. However, the aforementioned cost allocation is not subsidy-free, since e.g. for $I = \{3\}$ we get the $0 + c_1 + c_3 + 1 \leq c_1 + c_3$ inequality, contradicting the subsidy-free axiom.

The following theorems characterize the serial cost allocation.

Theorem 2.29 *Cost-sharing rule ξ is cost monotone, ranking, semi-marginal, and incremental subsidy-free if and only if $\xi = \xi^s$, i.e. it is the serial cost allocation rule.*

Proof: Based on the construction of the serial cost-sharing rule it can be easily proven that the rule possesses the above properties. Let us now assume that there exists a cost share rule ξ that also satisfies these properties. We will demonstrate that in this case $\xi^s = \xi$. Let J be a sub-tree, and let c^J denote the following cost vector: $c^J_j = c_j$, if $j \in J$, and 0 otherwise.

1. We will first show that $\xi(c^0) = \xi^s(c^0)$, where *0* denotes the tree consisting of a single root node. For two $i < j$ neighboring succeeding nodes it holds that $\xi_i(c^0) \leq \xi_j(c^0)$ due to the ranking property, while $\xi_i(c^0) + c^0_j \geq \xi_j(c^0)$ due to being subsidy-free. Moreover, $c^0_j = 0$. This implies that $\xi(c^0)$ is equal everywhere, in other words, it is equal to $\xi^s(c^0)$.

2. In the next step we show that if for some sub-tree J: $\xi(c^J) = \xi^s(c^J)$, then extending the sub-tree with j, for which $J \cup \{j\}$ is also a sub-tree, we can also see that $\xi(c^{J\cup\{j\}}) = \xi^s(c^{J\cup\{j\}})$. Through induction we reach the $c^N = c$ case, resulting in $\xi(c) = \xi^s(c)$.

Therefore, we must prove that $\xi(c^{J\cup\{j\}}) = \xi^s(c^{J\cup\{j\}})$. Monotonicity implies that $\xi_h(c^J) \leq \xi_h(c^{J\cup\{j\}})$ holds everywhere. Let us now apply the increasing subsidy-free property to the $H = N \backslash j \backslash I^+_j$ set. Now $\sum_{h\in H} \left(\xi_h(c^{J\cup\{j\}}) - \xi_h(c^J)\right) \leq \sum_{h\in H} \left(c^{J\cup\{j\}} - c^J\right)$. However, $c^{J\cup\{j\}} = c^J$ holds on set H, therefore the right side of the inequality is equal to 0. Using the cost monotone property we get that $\xi_h(c^J) = \xi_h(c^{J\cup\{j\}})$, $\forall h \in H$. However, on this set ξ^s has not changed either, therefore $\xi_h(c^{J\cup\{j\}}) = \xi^s_h(c^{J\cup\{j\}})$.

Applying the results from point 1 on set $\{j\} \cup I^+_j$, and using the ranking and semi-marginality properties, it follows that ξ is equal everywhere on this set, i.e. equal to the average. Therefore, it is equal to ξ^s on this set, too. ∎

Theorem 2.30 *The serial cost-sharing rule is the unique cost monotone, ranking, and incremental subsidy-free method that ensures maximal Rawlsian welfare.*

Proof: It is easy to verify that the serial cost-sharing rule satisfies the desired properties. Let us now assume that besides ξ^s the properties also hold for a different ξ. Let us now consider cost c for which $\exists i$, such that $\xi_i(c) > \xi_i^s$, and among these costs let c be such that the number of components where $c_i \neq 0$ is minimal. Let i be such that in the tree $\xi_i(c) > \xi_i^s$.

We decrease cost c as follows: we search for a cost $c_j \neq 0$, for which $j \notin I_i^- \cup \{i\}$, i.e. j is not from the chain preceding i.

1. If it exists, then we decrease c_j to 0, and examine the resulting c'. Similarly to point 2 in Theorem 2.29, on the chain $H = I_i^- \cup \{i\}$ due to cost-monotonicity $\xi_h(c') \leq \xi_h(c)$. Moreover, there is no change on the chain due to being incremental subsidy-free. Therefore $\xi_i(c') = \xi_i(c) > \xi_i^s(c) = \xi_i^s(c')$, i.e. in c the number of not 0 values was not minimal, contradicting the choice of c.

2. This means that the counter-example with minimal not 0 values belongs to a cost where for all outside $I_i^- \cup \{i\}$: $c_j = 0$. Due to ranking $\forall j \in I_i^+$: $\xi_j(c) \geq \xi_i(c) > \xi_i^s(c) = \xi_j^s(c)$. The latter equation is a consequence of the construction of the serial cost share rule, since $c_j = 0$ everywhere on I_i^+. In a tree where c_j differs from 0 only in i and I_i^-, $\xi_i^s(c)$ will be the largest serial cost allocation. Maximizing Rawlsian welfare is equivalent to minimizing the distributed cost, therefore allocation ξ cannot satisfy this property, since $\xi_i^s(c) < \xi_i(c)$. ∎

Consequently, according to the theorem the serial cost allocation is the implementation of a maximization of welfare.

Theorem 2.31 *The serial cost-sharing rule is the single cost monotone, ranking, semi-marginal method ensuring minimal Rawlsian welfare.*

Proof: It can be easily proven that the serial cost share rule satisfies the preconditions of the theorem. Let us now suppose that this also holds for ξ different from ξ^s. Moreover, let us consider the cost where $\exists i$ such that $\xi_i(c) < \xi_i^s(c)$, and c is such that the number of costs where $c_j \neq 0$ is minimal.

We decrease cost c as follows: we search for a component $c_j \neq 0$ for which $j \notin I_i^- \cup \{i\}$, and decrease c_j to 0. For the resulting cost c': $\xi_i^s(c') = \xi_i^s(c) >$

$\xi_i(c) \geq \xi_i(c')$, the latter inequality holds due to cost monotonicity. This implies $\xi_i^s(c') > \xi_i(c)$, contradicting the choice of c. Therefore such c_j does not exist, $c_j \neq 0$ may only be true for those in $I_i^- \cup \{i\}$.

In this case due to ranking in chain $I_i^- \cup \{i\}$ the greatest ξ_j value belongs to i. Since outside the chain $\forall c_j = 0$, from ranking and semi-marginality (based on part 1 of Theorem 2.29) it follows that outside the chain $\xi_j(c)$ is equal everywhere to $\xi_h(c)$, where h is the last node preceding j in the chain. Due to the construction of serial cost-sharing, $\xi_i^s(c)$ is also the greatest for c. Minimizing Rawlsian welfare is equivalent to maximizing the largest distributed cost, therefore due to $\xi_i(c) < \xi_i^s(c)$, ξ cannot be a Rawlsian minimum. ∎

Naturally, the restricted average cost rule is semi-marginal, while the serial cost share rule is subsidy-free. To summarize, both methods are cost monotone, ranking, subsidy-free, semi-marginal, and while the restricted average cost rule maximizes Rawlsian welfare, contrarily, the serial cost share rule minimizes welfare. Consequently, the restricted average cost rule is advantageous for those downstream on the main ditch, while the serial cost share rule is desirable for upstream users.

2.4 Weighted cost allocations

A practical question based on real-life problems related to cost allocation is how we can take into account for example the amount of water consumed by an individual user. The models discussed so far only took into account the maintenance costs (i.e. we assigned weights only to edges of the graphs). To be able to examine situations such as when we have to incorporate into our calculations water consumption given in acres for each user, we require so called weighted models, where we assign values to individual users (i.e. the nodes of the graph). These versions can be described analogously to the case involving an individual user, and are not separately discussed in present thesis. Incorporating weights into the restricted average cost rule and serial cost share rule is discussed by Aadland and Kolpin (1998), who also describe related results on chains. Additional results on weighted cost allocations are discussed by Bjørndal, Koster and Tijs (2004).

Chapter 3

Introduction to cooperative game theory

The significance of cooperative game theory is beyond dispute. This field of science positioned on the border of mathematics and economics enables us to model and analyze economic situations that emphasize the cooperation among different parties, and achieving their common goal as a result. In such situations we focus primarily on two topics: what cooperating groups (coalitions) are formed, and how the gain stemming from the cooperation can be distributed among the parties. Several theoretical results have been achieved in recent decades in the field of cooperative game theory, but it is important to note that these are not purely theoretical in nature, but additionally provide solutions applicable to and applied in practice. One example is the Tennessee Valley Authority (TVA) established in 1933 with the purpose of overseeing the economy of Tennessee Valley, which was engaged in the analysis of the area's water management problems. We can find cost distributions among their solutions which correspond to cooperative game theory solution concepts. Results of the TVA's work are discussed from a game theory standpoint in Straffin and Heaney (1981).

In the following we present the most important notions and concepts of cooperative game theory. Our definitions and notations follow the conventions of the book by Peleg and Sudhölter (2007) and manuscript by Solymosi (2007).

3.1 TU games

We call a game cooperative if it comprises coalitions and enforceable contracts. This means that the players may agree on the distribution of payments or the chosen strategy, even if these agreements are not governed by the rules of the game. Contracting and entering into agreements are pervading concepts in economics, for example all buyer-seller transactions are agreements. Moreover, the same can be stated for multi-step transactions as well. In general, we consider an agreement to be valid and current if its violation incurs a fine (even a high, financial fine), withholding players from violating it.

Cooperative games are classified into the following two groups: transferable and non-transferable utility games. In the case of transferable utility games we assume that the individual preferences of players are comparable by using a mediation tool (e.g. money). Therefore members of a specific coalitional may freely distribute among themselves the payoff achieved by the coalition. Following the widespread naming, we will refer to these games as TU games (transferable utility games). In the case of non-transferable utility games, i.e. NTU games, this mediation tool is either missing, or even if this good exists enabling compensation, the players do not judge it evenly. In the present thesis we will discuss TU games.

3.1.1 Coalitional games

Let N be a non-empty, finite set of players, and coalition S a subset of N. Let $|N|$ denote the cardinality of N, and 2^N the class of all subsets of N. $A \subset B$ denotes that $A \subseteq B$, but $A \neq B$. $A \uplus B$ denotes the union of disjoint sets A and B.

Definition 3.1 *A transferable utility cooperative game is an (N, v) pair, where N is the non-empty, finite set of players, and v is a function mapping a $v(S)$ real number to all S subsets of N. In all cases we assume that $v(\emptyset) = 0$.*

Remark 3.2 *A (N, v) cooperative game is commonly abbreviated as a v game. N is the set of players, v is the coalitional or characteristic function, and S is a subset of N. If coalition S is formed in game v, then the members of the coalition are assigned value $v(S)$, called the* value of the coalition.

The class of games defined on the set of players N is denoted by $\mathcal{G}^N$. It is worth noting that $\mathcal{G}^N$ is isomorphic with $\mathbb{R}^{2^{|N|}-1}$, therefore we assume that there exists a fixed isomorphism[1] between the two spaces, i.e. $\mathcal{G}^N$ and $\mathbb{R}^{2^{|N|}-1}$ are identical.

Remark 3.3 *A coalition S may distribute $v(S)$ among its members at will. An $x \in \mathbb{R}^S$ payoff vector is* feasible, *if it satisfies the $\sum_{i \in S} x_i \leq v(S)$ inequality. In fact, transferable utility means that coalition S can achieve all feasible payoff vectors, and that the sum of utilities is the utility of the coalition.*

In most applications, players in cooperative games are individuals or groups of individuals (for example trade unions, cities, nations). In some interesting economic game theory models, however, the players are not individuals, but goals of economics projects, manufacturing factors, or variables of other situations.

Definition 3.4 *A game (N, v) is superadditive, if*

$$v(S \cup T) \geq v(S) + v(T),$$

for all $S, T \subseteq N$ and $S \cap T = \emptyset$. In the case of an opposite relation the game is subadditive.

If the $S \cup T$ coalition is formed then its members may behave as if S and T had been created separately, in which case they achieve payoff $v(S) + v(T)$. However, the superadditivity property is many times not satisfied. There are anti-trust laws that would decrease the profit of coalition $S \cup T$ if it formed.

Definition 3.5 *A game (N, v) is convex if*

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T),$$

for all $S, T \subseteq N$. The game is concave, if $S, T \subseteq N$, $v(S) + v(T) \geq v(S \cup T) + v(S \cap T)$, i.e. $-v$ is convex.

[1]The isomorphism can be given by defining a total ordering on N, i.e. $N = \{1, \ldots, |N|\}$. Consequently, for all $v \in \mathcal{G}^N$ games let $v = (v(\{1\}), \ldots, v(\{|N|\}), v(\{1,2\}), \ldots, v(\{|N|-1, |N|\}), \ldots, v(N)) \in \mathbb{R}^{2^{|N|}-1}$.

The *dual* of a game $v \in \mathcal{G}^N$ is the game $\bar{v} \in \mathcal{G}^N$, where for all $S \subseteq N$: $\bar{v}(S) = v(N) - v(N \setminus S)$. The dual of a convex game is concave, and the opposite is true as well.

A convex game is clearly superadditive as well. The following equivalent characterization exists: a game (N, v) is convex if and only if $\forall i \in N$ and $\forall S \subseteq T \subseteq N \setminus \{i\}$ it holds that

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T).$$

In other words, a game v is convex if and only if the *marginal contribution* of players in coalition S, defined as $v_i^{'}(S) = v(S \cup \{i\}) - v(S)$ is monotonically increasing with respect to the inclusion of coalitions. Examples of convex games are the savings games (related to concave cost games), which we will discuss later.

Definition 3.6 *A game (N, v) is essential, if $v(N) > \sum_{i \in N} v(\{i\})$.*

Definition 3.7 *A game (N, v) is additive, if $\forall S \subseteq N$ it holds that $v(S) = \sum_{i \in S} v(\{i\})$.*

Additive games are not essential, from a game theory perspective they are regarded as trivial, since if the expected payoff of all $i \in N$ players is at least $v(\{i\})$, then the distribution of $v(N)$ is the only "reasonable" distribution.

Remark 3.8 *Let N be the set of players. If $x \in \mathbb{R}^N$ and $S \subseteq N$, then $x(S) = \sum_{i \in S} x_i$.*

Remark 3.9 *Let N be the set of players, $x \in \mathbb{R}^N$, and based on the above let x be the coalitional function. In the form of (N, x) there is now given an additive game, where $x(S) = \sum_{i \in S} x_i$ for all $S \subseteq N$.*

Definition 3.10 *An (N, v) game is 0-normalized, if $v(\{i\}) = 0 \ \forall i \in N$.*

3.1.2 Cost allocation games

Let N be the set of players. The basis of the cost allocation problem is a game (N, v_c), where N is the set of players and the v_c coalitional function is the cost function assigned to the problem. Intuitively, N may be the set of users of a public utility or public facility. All users are served to a certain degree, or not at all. Let $S \subseteq N$, then $v_c(S)$ represents the minimal cost required for serving members of S. The game (N, v_c) is called a *cost game*. Our goal is to provide a cost distribution among users that can be regarded as "fair" from a certain standpoint.

Cost game (N, v_c) can be associated with the so called *saving game* from coalitional games (N, v), where $v_s(S) = \sum_{i \in S} v_c(\{i\}) - v_c(S)$, for all $S \subseteq N$. Several applications can be associated with well-known subclasses of TU games, for example cost games are equivalent to non-negative, subadditive games, while saving games with 0-normalized, non-negative, superadditive games, see Driessen (1988).

Let (N, v_c) be a cost game, and (N, v_s) the associated saving game. Then (N, v_c) is

- *subadditive*, i.e.

  $$v_c(S) + v_c(T) \geq v_c(S \cup T),$$

  for all $S, T \subseteq N$ and $S \cap T = \emptyset$ if and only if (N, v_s) is superadditive.

- *concave*, i.e.

  $$v_c(S) + v_c(T) \geq v_c(S \cup T) + v_c(S \cap T),$$

  for all $S, T \subseteq N$ if and only if (N, v_s) is convex.

In most applications cost games are subadditive (and monotone), see the papers of Lucas (1981), Young (1985a), and Tijs and Driessen (1986) regarding cost games.

Let us consider the following situation as an example. A group of cities (i.e. a county), denoted by N have the opportunity to build a common water supply system. Each city has its own demand for the minimum amount of water, satisfied by its own distribution system, or by a common system with some other, maybe all other cities. The $S \subseteq N$ coalition's alternative or individual cost $v_c(S)$ is the minimal cost needed to satisfy the requirements of the members of S most efficiently. Given that a set $S \subseteq N$ may be served by several different subsystems, we arrive at a subadditive cost game. Such games are discussed by, among others, Suzuki and Nakayama (1976), and Young, Okada and Hashimoto (1982).

3.2 Payoffs and the core

In the case of real life problems, besides examining whether a specific coalition will form, or which coalitions form, it is also relevant whether the members of a given coalition can agree on how the total gain achieved by the coalition is distributed among the members. This allocation is called a solution of the game.

Frequently, the players achieve the highest payoff if the grand coalition is formed. For example, for cases that can be modeled with superadditive games, it is beneficial for all coalitions with no common members to be unified, since this way they achieve greater total gain. This results in all players deciding to form the grand coalition. However, this is not always so straightforward. For example in cases that can be only be modeled with 0-monotone or essential games it may happen that for a proper subset of users it is more advantageous to choose a coalition comprising them instead of the grand coalition.

Henceforth, let us suppose that the grand coalition does form, i.e. it is more beneficial for all players to form a single coalition. This means that the objective is the allocation of the achieved maximal gain such that it is "satisfactory" for all parties.

Definition 3.11 *The* payoff *of player* $i \in N$ *in a given game* (N, v) *is the value* $x_i \in \mathbb{R}$ *calculated by distributing* $v(N)$*. A possible solution of the game* (N, v) *is characterized by* payoff vector $x = (x_1, \dots, x_n) \in \mathbb{R}^N$.

Definition 3.12 *A payoff vector* $x = (x_1, \dots, x_n)$ *in a game* (N, v) *is*

- feasible *for coalition* S*, if* $\sum\limits_{i \in S} x_i \leq v(S)$,
- acceptable *for coalition* S*, if* $\sum\limits_{i \in S} x_i \geq v(S)$,
- preferred *by* S *to the payoff vector* $y = (y_1, \dots, y_n)$*, if* $\forall i \in S$*:* $x_i > y_i$,
- dominating through coalition S *the payoff vector* $y = (y_1, \dots, y_n)$ *if* x *is feasible for* S*, and at the same time preferred to* y *(we will denote this with* $x \ \mathrm{dom}_S \ y$*),*
- not dominated through coalition S*, if there is no feasible payoff vector* z *such that* $z \ \mathrm{dom}_S \ x$*,*

- dominating the payoff vector y, *if there exists coalition S for which* x dom$_S$ y *(we will denote this with* x dom y*),*

- not dominated, *if it is not dominated through any coalition S.*

The concepts of *feasibility, acceptability* and *preference* simply mirror the natural expectations that in the course of the distribution of $v(S)$, members of the coalition can only receive payoffs that do not exceed the total gain achieved by the coalition. Furthermore, all players strive to maximize their own gains, and choose the payoff more advantageous for them.

Remark 3.13 *The following relationships hold:*

1. *A payoff vector x is acceptable for coalition S if and only if x is not dominated through S.*

2. *For all $S \subseteq N$: dom$_S$ is an asymmetric, irreflexive, and transitive relation.*

3. *Relation* dom *is irreflexive, not necessarily asymmetric or transitive, even in the case of superadditive games.*

The core (Shapley, 1955; Gillies, 1959) is one of the most fundamental notions of cooperative game theory. It helps understanding which allocations of a game will be accepted as a solution by members of a specific coalition. Let us consider the following definition.

Definition 3.14 *A payoff vector $x = (x_1, \ldots, x_n)$ in a game (N, v) is a(n)*

- preimputation, *if $\sum_{i \in N} x_i = v(N)$, i.e. it is feasible and acceptable for coalition N,*

- imputation, *if $\sum_{i \in N} x_i = v(N)$ and $x_i \geq v(\{i\})$ $\forall i \in N$, i.e. a preimputation that is acceptable for all coalitions consisting of one player (i.e. for all individual players), i.e. it is individually rational,*

- core allocation, *if $\sum_{i \in N} x_i = v(N)$ and $\sum_{i \in S} x_i \geq v(S)$ $\forall S \subseteq N$, i.e. an allocation acceptable for all coalitions, i.e. coalitionally rational.*

In a game (N, v) we denote the set of preimputations by $I^*(N, v)$, the set of imputations by $I(N, v)$, and the set of core allocations by $C(N, v)$. The latter set $C(N, v)$ is commonly abbreviated as the *core*.

Therefore, the core expresses which allocations are deemed by members of different coalitions "stable" enough that they cannot block.

Remark 3.15 *The following claims hold:*

1. *For any (N, v) game the set of preimputations $I^*(N, v)$ is a hyperplane, therefore never empty.*

2. *In a (N, v) game the set of imputations $I(N, v)$ is non-empty if and only if $v(N) \geq \sum_{i \in N} v(\{i\})$.*

We demonstrate the above through the following example.

Example 3.16 *Let (N, v) be a (0,1)-normalized game with 3 players (that is, a 0-normalized game where $v(N) = 1$). The set of preimputations $I^*(N, v)$ is the hyperplane consisting of solution vectors of the $x_1 + x_2 + x_3 = 1$ equation. The set of imputations $I(N, v)$ is the unit simplex residing on the hyperplane, defined by vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$.*
However, the question of the core's nonemptiness is not straightforward. Let us take the following two specific cases:

1.

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	0	2/3	2/3	2/3	1

2.

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	0	1	1	1	1

Considering case 1., we find that the core consists of a single $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ payoff. However, examining case 2., we find that for coalitions with two members an $x = (x_1, x_2, x_3)$ payoff will be acceptable only if $x_1 + x_2 \geq 1$, $x_1 + x_3 \geq 1$, $x_2 + x_3 \geq 1$ all hold, i.e. $x_1 + x_2 + x_3 \geq \frac{3}{2}$. This is not feasible for the grand coalition, however. Consequently in this case the core of the game is empty.

In the second case of the previous example the reason the core is empty is that the value of the grand coalition is not "sufficiently large".

Beyond the above it is important to recognize that in some cases the core comprises multiple elements. Let us consider the following example with three players, known as the horse market game (for a more detailed description see Solymosi, 2007, pp. 13., Example 1.6):

Example 3.17 (The horse market game) *There are three players in a market,* A *possesses a horse to sell,* B *and* C *are considering purchasing the horse. Player* A *wants to receive at least 200 coins for the horse, while* B *is willing to pay at most 280,* C *at most 300 coins. Of course, they do not know these pieces of information about each other. Simplifying the model, if we only take into account the gain resulting from the cooperation in certain cases, we arrive at the game described in Table 3.1.*

S	A	B	C	AB	AC	BC	ABC
$v(S)$	0	0	0	80	100	0	100

Table 3.1: The horse market game

It is easy to calculate (see for example Solymosi, 2007, pp. 37., Example 2.5), that the core of the game is the following set containing payoffs:

$$\{(x_A,\ x_B = 0,\ x_C = 100 - x_A) |\ 80 \leq x_A \leq 100\}.$$

The core of an arbitrary game is usually either empty or contains multiple elements. In the latter case, we can ask how to choose one from the multiple core allocations, since in principle all of them provide an acceptable solution both on an individual and on a coalitional level. When comparing two core-allocations, sometimes one of them may be more advantageous for a specific coalition, and therefore they would prefer this allocation. However, the same may not be advantageous for another coalition, and so forth. It is plausible to choose the one solution which ensures that the coalition that is worst-off still receives the maximal possible gains. The nucleolus (Schmeidler, 1969) provides such a solution as a result of a lexicographic minimization of a non-increasing ordering of increases feasible for coalitions. This allocation will be a special element of the core (assuming that the set of allocations is nonempty, otherwise it is not defined).

Even though the nucleolus (if it is defined) consists of a single element, it is difficult to calculate. For special cases there exist algorithms that "decrease" this computational complexity in some way. The following publications provide further details and specific examples about the topic: Solymosi and Raghavan (1994), Aarts, Driessen and Solymosi (1998), Aarts, Kuipers and Solymosi (2000), Fleiner, Solymosi and Sziklai (2017). Solymosi and Sziklai (2016) present cases where special sets are leveraged for calculating the nucleolus.

3.3 The Shapley value

The core allocations presented so far may provide a solution for a specific allocation problem (assuming the core is non-empty), yet it is not always easy to choose a solution, should there be more than one. In certain cases the nucleolus is a good choice, but more often than not, it is difficult to determine. We would like to present another allocation rule, Shapley's famous solution concept, the *Shapley value* (Shapley, 1953).

In this section we define the Shapley value and examine its properties. Shapley studied the "value" gained by a player due to joining a coalition. In other words, what "metric" defines the value of the player's role in the game. We introduce the notions *solution* and *value*, and present some axioms that uniquely define the Shapley value on the sub-class in question.

Let $\mathcal{G}^N$ denote the set of TU games with the set of players N. Let $X^*(N,v)$ denote the set of feasible payoff vectors, i.e. $X^*(N,v) = \left\{x \in \mathbb{R}^N | \; x(N) \leq v(N)\right\}$. Using the above notations, consider the following definition.

Definition 3.18 *A* solution *on the set of games $\mathcal{G}^N$ is a set-valued mapping σ, that assigns to each game $v \in \mathcal{G}^N$ a subset $\sigma(v)$ of $X^*(N,v)$. A* single-valued solution *(henceforth,* value*) is defined as a function $\psi : \mathcal{G}^N \to \mathbb{R}^N$ that maps to all $v \in \mathcal{G}^N$ the vector $\psi(v) = (\psi_i(v))_{i \in N} \in \mathbb{R}^N$, in other words, it gives the* value *of the players in any game $v \in \mathcal{G}^N$.*

In the following we will be focusing primarily on single-valued solutions, i.e. values.

Definition 3.19 *In game* (N, v) *player* i*'s* individual marginal contribution *to coalition* S *is* $v_i'(S) = v(S \cup \{i\}) - v(S)$. *Player* i*'s* marginal contribution vector *is* $v_i' = (v_i'(S))_{S \subseteq N}$.

Definition 3.20 *We say that value* ψ *on class of games* $A \subseteq \mathcal{G}^N$ *is / satisfies*

- efficient (Pareto-optimal), *if* $\sum_{i \in N} \psi_i(v) = v(N)$,
- individually acceptable, *if* $\psi_i(v) \geq v(\{i\})$ *for all* $i \in N$,
- equal treatment property, *if* $\forall i, j \in N$, $\forall S \subseteq N \setminus \{i, j\}$ *and* $v(S \cup \{i\}) = v(S \cup \{j\})$ *implies* $\psi_i(v) = \psi_j(v)$,
- null-player property, *if for all* $i \in N$ *such that* $v_i' = 0$ *implies that* $\psi_i(v) = 0$, *where* $v_i'(S) = v(S \cup \{i\}) - v(S)$ *is the individual contribution of* i *to* S,
- dummy, *if* $\psi_i(v) = v(\{i\})$, *where* $i \in N$ *is a dummy player in* v, *i.e.* $v(S \cup \{i\}) - v(S) = v(\{i\})$ *for all* $S \subseteq N \setminus \{i\}$,
- additive, *if* $\psi(v + w) = \psi(v) + \psi(w)$ *for all* $v, w \in A$, *where* $(v + w)(S) = v(S) + w(S)$ *for all* $S \subseteq N$,
- homogeneous, *if* $\psi(\alpha v) = \alpha\psi(v)$ *for all* $\alpha \in \mathbb{R}$, *where* $(\alpha v)(S) = \alpha v(S)$ *for all* $S \subseteq N$,
- covariant, *if* $\psi(\alpha v + \beta) = \alpha\psi(v) + b$ *for all* $\alpha > 0$ *and* $b \in \mathbb{R}^N$, *where* β *is the additive game generated by vector* b,

given that the above conditions hold for all $v, w \in A$ *for all properties.*

Efficiency implies that we get an allocation as result. Individual acceptability represents that each player "is worth" at least as much as a coalition consisting of only the single player. The equal treatment property describes that the payoff of a player depends solely on the role played in the game, in the sense that players with identical roles receive identical payoffs. The null-player property ensures that a player whose contribution is 0 to all coalitions does not receive payoff from the grand coalition. The dummy property expresses that a player that neither increases nor decreases the value of any coalition by a value other than the player's own, shall be assigned the same value as this constant contribution. The dummy property also implies the null-player property.

Covariance ensures that a potential change in scale is "properly" reflected in the valuation as well. The additive and homogeneous properties, on the other hand, are not straightforward to satisfy. In case both are satisfied, the linearity property holds, which is much stronger than covariance.
Let us now consider the definition of the solution of Shapley (1953).

Definition 3.21 *In a game* (N, v) *the* Shapley value *of player* $i \in N$ *is*

$$\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N \setminus S| - 1)!}{|N|!} v_i'(S),$$

and the Shapley value *of the game is*

$$\phi(v) = (\phi_i(v))_{i \in N} \in \mathbb{R}^N.$$

Claim 3.22 *The Shapley value is / satisfies all the above defined properties, namely: efficient, null-player property, equal treatment property, dummy, additive, homogeneous, and covariance.*

It is important to note that the Shapley value is not necessarily a core allocation. Let us consider the previously discussed horse market game (Section 3.17).

Example 3.23 *The horse market game has been defined as seen in Table 3.2.*

S	A	B	C	AB	AC	BC	ABC
$v(S)$	0	0	0	80	100	0	100

Table 3.2: The horse market game and its Shapley value

The Shapley value for the individual players can be calculated as follows

$$\phi_A = \frac{0!2!}{3!}(0-0) + \frac{1!1!}{3!}(80-0) + \frac{1!1!}{3!}(100-0) + \frac{0!2!}{3!}(100-0) = 63\frac{1}{3},$$

$$\phi_B = \frac{1}{3}(0-0) + \frac{1}{6}(80-0) + \frac{1}{6}(0-0) + \frac{1}{3}(100-100) = 13\frac{1}{3},$$

$$\phi_C = \frac{1}{3}(0-0) + \frac{1}{6}(100-0) + \frac{1}{6}(0-0) + \frac{1}{3}(100-80) = 23\frac{1}{3}.$$

The core of the game was the set containing the following payoffs:

$$\{(x_A,\ x_B = 0,\ x_C = 100 - x_A) |\ 80 \leq x_A \leq 100\}.$$

However, since the core contains only payoffs where the value of player B *is* 0*, clearly, in this case the Shapley value is not a core allocation.*

The following claim describes one of the possible characterizations of the Shapley value.

Claim 3.24 (Shapley, 1953) *A value ψ on a class of games $\mathcal{G}^N$ is efficient, dummy, additive and satisfies the equal treatment property, if and only if $\psi = \phi$, i.e. it is the Shapley value.*

Further axiomatization possibilities are discussed in Section 5.2, for details see the papers of Pintér (2007, 2009) (both in Hungarian), and Pintér (2015).

With $|N| = n$ let $\pi : N \to \{1, \ldots, n\}$ an ordering of the players, and let Π_N be the set of all possible orderings of the players. Let us assume that a coalition is formed by its members joining successively according to the ordering $\pi \in \Pi_N$. Then $\pi(i)$ denotes the position of player i in ordering π, and $P_i^\pi = \{j \in N \mid \pi(j) \leq \pi(i)\}$ denotes the players preceding i. The *marginal contribution* of player i in the game v where the ordering π is given is $x_i^\pi(v) = v(P_i^\pi \cup \{i\}) - v(P_i^\pi)$. The payoff vector $x^\pi(v) = (x_i^\pi(v))_{i \in N} \in \mathbb{R}^N$ defined by these components is called the *marginal contribution vector* by ordering π.

It can be shown that in a generalized form (see for example Solymosi, 2007), the Shapley value can be given as:

$$\phi_i(v) = \frac{1}{|N|!} \sum_{\pi \in \Pi_N} x_i^\pi(v).$$

Consequently, the Shapley value is the mean of the marginal contribution vectors.

Remark 3.25 *In an arbitrary convex game the core is never empty (Shapley, 1971), and the Shapley value always provides a core allocation. Namely, based on the Shapley-Ichiishi theorem (Shapley, 1971; Ichiishi, 1981), we know, that the marginal contribution vectors are extremal points of the core if and only if the game is convex, while the Shapley value is the mean of the marginal contribution vectors.*

Chapter 4

Fixed tree games

In this chapter we present a special class of games, the standard tree games, and provide examples of their applications in the field of water management. Additional game-theoretic applications to water management problems are discussed in Parrachino, Zara and Patrone (2006). Besides fixed tree structures several other graph-theoretic models are applicable as well, for example the class of shortest path games, which we will cover in more detail in Chapter 7. For a recent summary of cost allocation problems on networks (e.g. fixed trees, among others) we recommend the book of Hougaard (2018).

4.1 Introduction to fixed tree games

Fixed tree games, due to their structure, assist the modeling of several practical applications. As we will see in this section, these are such special cost games that there always exists a core-allocation, nucleolus, and that the Shapley value is always a core allocation.

Accordingly, this section covers situations that can be modeled by fixed trees as known from graph theory nomenclature. There exists a fixed, finite set of participants, who connect to a distinctive node, the root, through a network represented by a fixed tree. Several real-life situations may be modeled using this method. Let us consider as an example the problem defined in Chapter 2, where we examined the maintenance costs of an irrigation ditch. The users of the ditch are represented by the nodes in the network, while the edges are segments of the ditch, and the weights of the edges are considered to be the maintenance costs. This problem can be represented with a cooperative fixed tree game, where the cost of certain groups or coalitions is given by the minimal cost associated with the edges connecting the members of the group to the root. The model of fixed tree games originates from the article of Megiddo (1978), who proved that for these games there exists an efficient algorithm for calculating the Shapley value and the nucleolus, and the complexity of such algorithm is $O(n^3)$ for calculating the nucleolus, and $O(n)$ for the Shapley value.

As a special case we must mention the class of "airport problems", which can be modeled with a non-branching tree, i.e. a chain. The related *airport games* are a proper subset of the standard fixed tree games. Airport games were introduced by Littlechild and Owen (1973), and the games' characterization will be described in detail in Chapter 5. A summary of further results related to the class is provided in Thomson (2007). The summary paper this section is based on is the work of Radványi (2019) (in Hungarian).

Let us now examine fixed tree games in further detail. Granot, Maschler, Owen and Zhu (1996) were studying cost games that stem from the fixed tree networks $\Gamma(V, E, b, c, N)$. In this list the (V, E) pair defines the directed tree, V is the set of vertices (nodes), and E is the set of edges. Node r in V plays a distinct role and we will refer to it as root. (The tree may be undirected, should the model require, this makes no difference from the game's standpoint). Moreover, there is given a cost function $c : E \to \mathbb{R}$ on the set of edges, where c_e denotes the entire (building, maintenance, so forth) cost associated with edge e. Similarly, there is given a cost function $b : V \to \mathbb{R}$ on the set of nodes. N is the set of users, henceforth players, and all $i \in N$ is assigned to a $v_i \in V$ node. A node v is occupied, if at least one player is assigned to it. We will denote by N_T the coalition consisting of players associated with nodes $T \subseteq V$. Moreover, we will require a (partial) ordering of nodes: for two nodes $i, j \in V$ such that $i \leq j$, if the unique path leading from the root to j passes through i. We will denote by $S_i(G)$ the set $\{j \in V : i \leq j\}$, i.e. the set of nodes accessible from i via a directed path. (Note that for all $i \in V$: $i \in S_i(G)$.) For all $i \in V$ let $P_i(G) = \{j \in V : j \leq i\}$, i.e. the set of nodes on the unique path connecting i to the root, and note that for all $i \in V$ it holds that $i \in P_i(G)$. Furthermore, for all $V' \subseteq V$ let $(P_{V'}(G), E_{V'})$ be the sub-tree of (V, E) containing a root, where $P_{V'}(G) = \cup_{i \in V'} P_i(G)$ and $E_{V'} = \{\overline{ij} \in E : i, j \in P_{V'}(G)\}$.

Definition 4.1 *A fixed tree network $\Gamma(V, E, b, c, N)$ is standard, if the following properties are satisfied:*

- *Cost function c assigns non-negative numbers to the edges.*
- *Costs associated with the edges are zero, i.e. $b(i) = 0$ for all $i \in V$.*
- *There is no player assigned to the root.*
- *At least one player is assigned to each leaf (i.e. node from which no further node can be reached), $i \in V$*
- *If no user is assigned to $i \in V$, then there exists at least two nodes $j \neq k$, for which $(i, j), (i, k) \in E$.*
- *There is exactly one $i \in V$, for which $(r, i) \in E$.*

Note that in present thesis we only examine standard fixed tree networks in which all nodes (except for the root) are occupied by exactly one player.

Graph (V, E) will be henceforth denoted by G, and a $\Gamma(V, E, b, c, N)$ standard fixed tree network by $\Gamma(G, c, N)$. The objective of the players is to be connected to the root via the network. Let us consider the above example, where a group of farmers wish to irrigate their land using a common ditch. This ditch is connected to the main ditch at a single point, from where water is supplied, this is the root node. The users are the nodes of the graph, and all edges in the network are directed outwards from the root towards the users. The users must cover the cost of the path leading to their nodes. Similarly, a group or coalition of users are connected to the root if all members of the coalition are accessible from the root via a directed path (even if the path passes through users who are not members of the coalition). In this case the group collectively covers the cost associated with the subgraph. Cooperative game theory assigns the *cost game* (N, v_c) to such situations (see Subsection 3.1.2), which for brevity we will henceforth refer to as the ordered pair (N, c).

We will call the union of unique paths connecting nodes of members of coalition $S \subseteq N$ in the graph the trunk of the tree, and will denote by $\bar{S}$. The proof of the following claim is described in the paper of Koster, Molina, Sprumont and Tijs (2001).

Claim 4.2 *In the case of a standard fixed tree network $\Gamma(G, c, N)$ the following equation holds: $c(S) = \sum_{i \in \bar{S}} c(e_i)$, for all $S \subseteq N$, where e_i denotes the edge which is connected to node i and the node preceding it on the path from i to the root.*

The proof contains the duals of unanimity games, which provide the basis for the representation of the related cost games. Let us consider the following two definitions.

Definition 4.3 *On a set of players N for all $T \in 2^N \setminus \{\emptyset\}$, and $S \subseteq N$ let*

$$u_T(S) = \begin{cases} 1, & \text{if } T \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

The game u_T is called the unanimity game *on coalition T.*

Definition 4.4 *On the set of players N for all $T \in 2^N \setminus \{\emptyset\}$, and $S \subseteq N$ let*

$$\bar{u}_T(S) = \begin{cases} 1, & \text{if } T \cap S \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

The game $\bar{u}_T$ is called the dual of the unanimity game *on coalition T.*

All $\bar{u}_T$ games can be represented with a non-branching standard fixed tree network. As an example let us consider the standard fixed tree representations of the duals of unanimity games associated with the set of players $N = \{1, 2, 3\}$ in Figure 4.1.

In Chapter 5 we will show (Lemma 5.7) that the dual of the unanimity game is equal to a corresponding airport game. Given the above, it is easy to see that a cost game associated with a standard fixed tree problem can be given as follows:

Claim 4.5 *Let $\Gamma(G, c, N)$ bet a standard fixed tree network. Then the respective game (N, c) can be represented as follows: $c = \sum_{i \in V \setminus \{r\}} c(e_i)\, \bar{u}_{S_i(G)}$, where e_i denotes the edge connecting to i and the node preceding it on the path connecting*

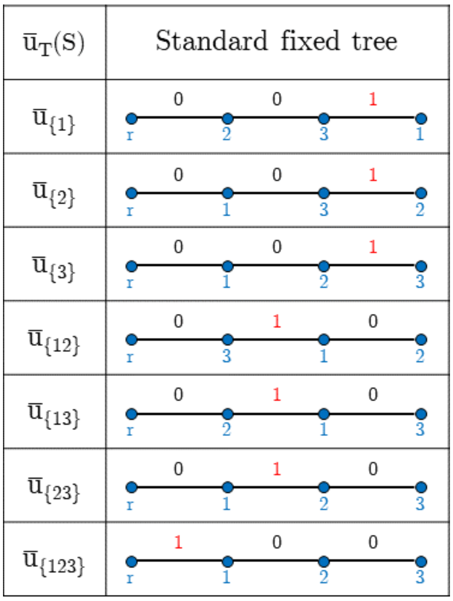

Figure 4.1: $\bar{u}_T(S)$ games represented as standard fixed trees

i to the root, while $S_i(G)$ is the set of nodes accessible from i via a directed path (i.e. the branch starting from i).

In the case of standard fixed tree games we can claim the following regarding the core (the proofs, and further representations are summarized by Koster et al., 2001).

- An allocation vector x is a core allocation if and only if $x \geq 0$ and $x(\bar{S}) \leq c(\bar{S})$, for all tree trunks $\bar{S}$.

- An allocation vector x is a core allocation if and only if $x \geq 0$ and for all edges $e = (i, j) \in E$:
$$\sum_{j \in V_e \setminus \{i\}} x_j \geq \sum_{e' \in E_e} c(e').$$

- An allocation vector x is a core allocation if and only if there exists $y^1, \ldots, y^n$, where y^j (for all $j \in 1, \ldots, n$) is a point of the unit simplex $\mathbb{R}^{S_j(G)}$, and
$$x_i = \sum_{j \in N(P_i(G))} y_i^j c(e_j), \ \forall i \in N,$$
where $P_i(G)$ denotes the set of nodes on the path connecting i to the root in the directed tree.

Based on the above points it is simple to generate a core allocation (Koster et al., 2001).

Note that in comparison to the standard definition of the core, in case of cost games the defining relations are inverted, since the core originally aims to achieve "as large as possible" payoffs, while in the case of a non-negative cost function this translates to achieving a cost "as low as possible".

Corollary 4.6 *Since fixed tree problems lead to special cost games, we know from Remark 3.25 that their cores are non-empty and that the Shapley value gives a core allocation for the respective allocation problem.*

In case of fixed tree games the Shapley value can be calculated using the following so called serial allocation rule (Littlechild and Owen, 1973):

$$x_i = \sum_{j \in P_i(G) \setminus \{\mathrm{r}\}} \frac{c(e_j)}{|S_j(G)|} .$$

This is an important result, since for certain classes of games calculating the Shapley value is computationally difficult, in this special case, i.e. in the case of fixed tree games the value can be easily calculated using the serial cost allocation rule.

Example 4.7 *In the following we demonstrate the above using the fixed tree network in Figure 4.2.*

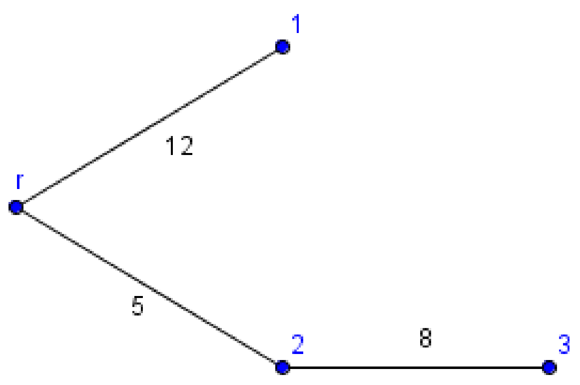

Figure 4.2: Fixed tree network in Example 4.7

The respective cost game is given by Table 4.7.

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v_c(S)$	*0*	*12*	*5*	*13*	*17*	*25*	*13*	*25*

The Shapley value of this game is $\phi(v) = (12,\ 2.5,\ 10.5)$*. This is calculated as follows. For player* 1*:* $S_1(G) = \{1\}$*,* $P_1(G) = \{r, 1\}$*, therefore* $\phi_1(v) = 12/1 = 12$*. For player* 2*:* $S_2(G) = \{2, 3\}$*,* $P_1(G) = \{r, 2\}$*, implying* $\phi_2(v) = 5/2 = 2.5$*. For player* 3*:* $S_3(G) = \{3\}$*,* $P_1(G) = \{r, 2, 3\}$*, implying* $\phi_3(v) = 5/2 + 8/1 = 10.5$*. The interpretation of the calculation is as follows. The cost of each edge is equally distributed among the players using the corresponding segment, and then for all players the partial costs for each edge are summed up.*

Note that the example describes a non-standard 3-player fixed tree, since the root does not connect to a single node. This is irrelevant with respect to the presented algorithms, but the tree can be made standard by adding after the root a new unoccupied node and an edge with value 0, if the application makes it necessary.

In the general case, similarly to the Shapley value, calculating the nucleolus is also computationally difficult. In the case of fixed tree games, however, it can be easily calculated using the painter algorithm (see for example Borm, Hamers and Hendrickx (2001) or Maschler, Potters and Reijnierse (2010)). The Shapley value can also be determined with a modified version of this algorithm.

A generalization of fixed tree games is the so called FMP-games (fixed tree games with multilocated players), where a player may occupy multiple nodes, therefore having multiple alternatives for connecting to the root node. These games are described in more detail in the paper of Hamers, Miquel, Norde and van Velzen (2006). The authors further show that the core of an FMP-game is equal to the core of the respective submodular standard fixed tree game.

In their paper Bjørndal et al. (2004) examine standard fixed tree games, where certain weights are assigned to the players (e.g. the water supply used for irrigation, etc.). As a generalization of the painter algorithm they define the so called weighted painter algorithm in both directions (starting from the residences, and starting from the root). They have shown that the two solutions provide as a result the so called weighted Shapley value, and a generalization of the nucleolus.

4.2 Applications

4.2.1 Maintenance or irrigation games

A widely used application of fixed tree games is the so called maintenance games. These describe situations in which players, a group of users connect to a certain provider (the root of the tree) through the fixed tree network. There is given a maintenance cost for all edges in the network, and the question is how to distribute "fairly" the entire network's maintenance cost (the sum of the costs on the edges) among the users.

A less widely used naming for the same fixed tree games is irrigation games, which are related to the water management problems described in Chapter 2. A group of farmers irrigate their lands using a common ditch, which connects to the main ditch at a distinct point. The costs of the network must be distributed among the farmers. Aadland and Kolpin (1998) have analyzed 25 ditches in the state of Montana, where the local farmers used two different major types of cost allocation methods, variants of the average cost and the serial cost allocations.

Moreover, Aadland and Kolpin (2004) also studied the environmental and geographical conditions that influenced the cost allocation principle chosen in the case of different ditches.

4.2.2 River sharing and river cleaning problems

From a certain standpoint, we can also categorize the games modeling river sharing and river cleaning problems as fixed tree games as well. Basically, the model is a fixed tree game where the tree consists of a single path. These trees are called chains.

Let there be given a river, and along the river players that may be states, cities, enterprises, and so forth. Their positions along the river defines a natural ordering among the players in the direction of the river's flow. In this case $i < j$ means that player i is further upstream the river (closer to the spring) than player j. Let there be given a perfectly distributable good, money, and the water quantity acquirable from the river, which is valued by the players according to a utility function. In the case of river sharing problems in an international environment, from a certain standpoint each state has complete control over the water from its segment of the river. For downstream users the quality and quantity of water let on by the state is of concern, and conversely, how upstream users are managing the river water is of concern to the state. These questions can be regulated by international treaties, modeling these agreements is beyond the scope of our present discussion of fixed tree games. In the paper of Ambec and Sprumont (2002) the position of the users (states) along the river defines the quantity of water they have control over, and the welfare they can therefore achieve. Ambec and Ehlers (2008) studied how a river can be distributed efficiently among the connected states. They have shown that cooperation provides a profit for the participants, and have given the method for the allocation of the profit.

In the case of river cleaning problems, the initial structure is similar. There is given a river, the states (enterprises, factories, etc.) along the river, and the amount of pollution emitted by the agents. There are given cleanup costs for each segment of the river as well, therefore the question is how to distribute these costs among the group. Since the pollution of those further upstream influences the pollution and cleanup costs further downstream as well, we get a fixed tree structure with a single path.

Ni and Wang (2007) analyzed the problem of the allocation of cleanup costs from two different aspects. Two international doctrines exist, absolute territorial sovereignty, and unlimited territorial integrity. According to the first, the state has full sovereignty over the river segment within its borders, while the second states that no state has the right to change natural circumstances so that it's harmful to other neighboring states. Considering these two doctrines, they analyzed the available methods for the distribution of river cleanup costs, and the properties thereof. They have shown that in both cases there exists an allocation method that is equal to the Shapley value in the corresponding cooperative game. Based on this Gómez-Rúa (2013) studied how the cleanup cost may be distributed taking into consideration certain environmental taxes. The article discusses the expected properties that are prescribed by states in real situations in the taxation strategies, and how these can be implemented for concrete models. Furthermore, the article describes the properties useful for the characterization properties of certain allocation methods, shows that one of the allocation rules is equal to the weighted Shapley value of the associated game.

Ni and Wang (2007) initially studied river sharing problems where the river has a single spring. This has been generalized by Dong, Ni and Wang (2012) for the case of rivers with multiple springs. They have presented the so called polluted river problem and three different allocation solutions: “local responsibility sharing” (LRS), “upstream equal sharing” (UES), and “downstream equal sharing” (DES). They have provided the axiomatization of these allocations and have shown that they are equal to the Shapley value of the associated cooperative games, respectively. Based on these results van den Brink, He and Huang (2018) have shown that the UES and DES allocations are equal to the so called conjunctive permission value of permission structure games, presented by van den Brink and Gilles (1996). In these games there is defined a hierarchy, and the players must get approval from some supervisors to form a coalition. The polluted river problem can be associated with games of this type. Thereby, van den Brink et al. (2018) have presented new axiomatizations for UES and DES allocations, utilizing axioms related to the conjunctive permission value. Furthermore, they have proposed and axiomatized a new allocation, leveraging the “alternative disjunctive permission value”, which is equal to the Shapley value of another corresponding restricted game. The paper demonstrates the power of the Shapley value, highlighting its useful properties that ensure that it is applicable in practice.

In present thesis we do not analyze concepts in international law regarding river sharing, a recent overview of the topic and the related models are provided by Béal, Ghintran, Rémila and Solal (2013) and Kóczy (2018). Upstream responsibility (UR) games, discussed in Chapter 6 provide a model different from models LRS and UES describing the concepts of ATS and UTI, respectively. In the case of UR games, a more generalized definition is possible regarding which edges a user is directly or indirectly responsible for, therefore we will consider river-sharing problems from a more abstract viewpoint.

Khmelnitskaya (2010) discusses problems where the river sharing problem can be represented by a graph comprising a root or a sink. In the latter case the direction of the graph is the opposite of in the case where the graph comprises a root, in other words, the river unifies flows from multiple springs (from their respective the river deltas) in a single point, the sink. Furthermore, the paper discusses the “tree- and sink-value” and their characterizations. It is shown that these are natural extensions of the so called lower and upper equivalent solutions of van den Brink, van der Laan and Vasil’ev (2007) on chains.

Ansink and Weikard (2012) also consider river sharing problems for cases when a linear ordering can be given among the users. Leveraging this ordering they trace back the original problem to a series of two-player river sharing problems that each of them is mathematically equivalent to a bankruptcy problem. The class of serial cost allocation rules they present provides a solution to the original river sharing problem. This approach also gives an extension of bankruptcy games.

Chapter 5

Airport and irrigation games

In this chapter, similarly to the previous one, we will consider cost-sharing problems given by rooted trees. We assign transferable utility (TU) cooperative games (henceforth games) to these cost-sharing problems. The induced games are the *irrigation games* discussed in the previous chapter. Let us recall the problem. Consider an irrigation ditch joined to the stream by a head gate and a group of users who use this ditch to irrigate their own farms. The objective is sharing the functional and maintenance costs of the ditch among the users.

The irrigation ditch can be represented by a rooted tree. The root is the head gate, nodes denote users, and the edges represent the segments of the ditch between users. Employing this representation Littlechild and Owen (1973) have shown that the contribution vector (the solution for the cost-sharing problem) given by the "sequential equal contributions rule" (henceforth SEC rule, or Baker-Thompson rule; Baker (1965), Thompson (1971)) is equivalent to the Shapley value (Shapley, 1953). According to this rule, for all segments their respective costs must be distributed evenly among those using the given segment, and for all users the costs of the segments they are using must be summed up. This sum is the cost the user must cover.

Previously, in Chapter 2 we have described an empirical and axiomatic analysis of a real cost-sharing problem, an irrigation ditch located in a south-central Montana community (Aadland and Kolpin, 1998). A similar definition is presented in Kayi (2007), using c_i for the whole cost joining user i to the head gate, instead of the cost of the segment immediately preceding i.

When considering special rooted trees with no branches (i.e. chains), we arrive at the well-known class of airport games (Littlechild and Thompson, 1977), therefore this class is the proper subset of the class of irrigation games. Thomson (2007) gives an overview on the results for airport games. In the literature, up to now, two axiomatizations of the Shapley value are given for airport games, those of Shapley (1953) and Young (1985b). Dubey (1982) shows that Shapley's characterization is valid on the class of airport games, and Moulin and Shenker (1992) prove that Young's axiomatization is also valid on this subclass of games.

It is well-known that the validity of a solution concept may vary from subclass to subclass, e.g. Shapley's axiomatization is valid on the class of monotone games but not on the class of strictly monotone games. Therefore, we must consider each subclass of games separately.

In this chapter we introduce irrigation games and characterize their class. We show that the class of irrigation games is a non-convex cone which is a proper subset of the finite convex cone spanned by the duals of the unanimity games, therefore every irrigation game is concave. Furthermore, as a corollary we show that the class of airport games has the same characteristics as that of irrigation games.

In addition to the previously listed results, we extend the results of Dubey (1982) and Moulin and Shenker (1992) to the class of irrigation games. Furthermore, we "translate" the axioms used in the cost-sharing literature (see e.g. Thomson, 2007) to the language of transferable utility cooperative games, and show that the results of Dubey (1982) and Moulin and Shenker (1992) can be deduced directly from Shapley (1953)'s and Young (1985b)'s results. Namely, we present two new variants of Shapley (1953)'s and Young (1985b)'s results, and we provide Dubey (1982)'s, Moulin and Shenker (1992)'s and our characterizations as direct corollaries of the two new variants.

In our characterization results we relate the TU games terminologies to the cost sharing terminologies, therefore we bridge between the two fields.

We also note that the Shapley value (which is equivalent to the SEC rule, see Claim 5.23) is stable for irrigation games, i.e. it is always in the core (Shapley, 1955; Gillies, 1959). This result is a simple consequence of the Ichiishi-Shapley theorem (Shapley, 1971; Ichiishi, 1981) and that every irrigation game is concave.

Up to our knowledge these are the first results in the literature which provide a precise characterization of the class of irrigation games, and extend Shapley's and Young's axiomatizations of the Shapley value to this class of games. We conclude that applying the Shapley value to cost-tree problems is theoretically well-founded, therefore, since the Shapley value behaves well from the viewpoint of computational complexity theory (Megiddo, 1978), the Shapley value is a desirable tool for solving cost-tree problems. The results of present chapter have been published in Márkus, Pintér and Radványi (2011).

As discussed previously in Section 4.1, Granot et al. (1996) introduce the notion of standard fixed tree games. Irrigation games are equivalent to standard fixed tree games, except for that in irrigation games the tree may vary, while in the approach of Granot et al. (1996) it is fixed. Koster et al. (2001) study the core of fixed tree games. Ni and Wang (2007) characterize the rules meeting properties additivity and independence of irrelevant costs on the class of standard fixed tree games. Fragnelli and Marina (2010) characterize the SEC rule for airport games.

Ambec and Ehlers (2008) examined how to share a river efficiently among states connected to the river. They have shown that cooperation exerts positive externalities on the benefit of a coalition and explored how to distribute this benefit among the members of the coalition. By Ambec and Sprumont (2002) the location of an agent (i.e. state) along the river determines the quantity of water the agent controls, and the welfare it can thereby secure for itself. The authors call the corresponding cooperative game *consecutive game* and prove that the game is convex, therefore the Shapley value is in the core (see Shapley (1971) and Ichiishi (1981))

A further problem is presented regarding the allocation of costs and benefits from regional cooperation by Dinar and Yaron (1986), by defining *regional games*. Cooperative game-theoretic allocations are applied, like the core, the nucleolus, the Shapley value and the generalized Shapley value; and are compared with an allocation based on marginal cost pricing. Dinar, Ratner and Yaron (1992) analyze a similar problem in the TU and the NTU settings (in the NTU case the core of the related game is non-convex, therefore the Nash-Harsányi solution is applied).

In this chapter we consider only Shapley (1953)'s and Young (1985b)'s axiomatizations. The validity of further axiomatizations of the Shapley value, see e.g. van den Brink (2001) and Chun (1991) among others, on the classes of airport games and irrigation games, is intended to be the topic of future research.

5.1 Introduction to airport and irrigation games

Let there be given a graph $G = (V, E)$, a cost function $c : E \to \mathbb{R}_+$ and a cost tree (G, c). One possible interpretation to the presented problem, as seen in Chapter 2 is the following. An irrigation ditch is joined to the stream by a head gate, and the users (the nodes of the graph except for the root) irrigate their farms using this ditch. The functional and maintenance costs of the ditch are given by c, and it is paid for by the users (more generally, the nodes might be departments of a company, persons, etc.). For any $e \in A$, $e = \overline{ij}$, c_e denotes the cost of connecting player j to player i.

In this section we build on the duals of unanimity games. As discussed previously, given N, the *unanimity game* corresponding to coalition T for all $T \in 2^N \setminus \{\emptyset\}$ and $S \subseteq N$ is the following:

$$u_T(S) = \begin{cases} 1, & \text{if } T \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the *dual of the unanimity game* for all $T \in 2^N \setminus \{\emptyset\}$ and $S \subseteq N$ is:

$$\bar{u}_T(S) = \begin{cases} 1, & \text{if } T \cap S \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, all unanimity games are convex, and their duals are concave games. Moreover, we know that the set of unanimity games $\{u_T | \emptyset \neq T \subseteq N\}$ gives a basis of $\mathcal{G}^N$ (see e.g. Peleg and Sudhölter, 2007, pp. 153., Lemma 8.1.4).

Henceforth, we assume that there are at least two players for all cost tree games, i.e. $|V| \geq 3$ and $|N| \geq 2$. In the following we define the notion of an irrigation game. Let (G, c) be a cost tree, and N the set of players (the set of nodes, except for the root). Let us consider the nonempty coalition $S \subseteq N$, then the cost of connecting members of coalition S to the root is equal to the cost of the minimal, rooted subtree that covers members of S. This minimal spanning tree corresponding to coalition S is called the *trunk*, denoted by $\bar{S}$. For all cost trees a corresponding *irrigation game* can be formally defined as follows.

Definition 5.1 (Irrigation game) *For all cost trees (G, c) and player set $N = V \setminus \{r\}$, and coalition S let*

$$v_{(G,c)}(S) = \sum_{e \in \bar{S}} c_e \ ,$$

where the value of the empty sum is 0. The games given above by v are called irrigation games, their class on the set of players N is denoted by $\mathcal{G}_I^N$. Furthermore, let $\mathcal{G}_G$ denote the subclass of irrigation games that is induced by cost tree problems defined on rooted tree G.

Since tree G also defines the set of players, instead of the notation $\mathcal{G}_G^N$ for set of players N, we will use the shorter form $\mathcal{G}_G$. These notations imply that

$$\mathcal{G}_I^N = \bigcup_{\substack{G(V,E) \\ N = V \setminus \{r\}}} \mathcal{G}_G .$$

Example 5.2 demonstrates the above definition.

Example 5.2 *Let us consider the cost tree in Figure 5.1. The rooted tree $G = (V, E)$ can be described as follows: $V = \{r, 1, 2, 3\}$, $E = \{\overline{r1}, \overline{r2}, \overline{23}\}$. The definition of cost function c is $c(\overline{r1}) = 12$, $c(\overline{r2}) = 5$, and $c(\overline{23}) = 8$.*

Therefore, the irrigation game is defined as $v_{(G,c)} = (0, 12, 5, 13, 17, 25, 13, 25)$, implying $v_{(G,c)}(\emptyset) = 0$, $v_{(G,c)}(\{1\}) = 12$, $v_{(G,c)}(\{2\}) = 5$, $v_{(G,c)}(\{3\}) = 13$, $v_{(G,c)}(\{1,2\}) = 17$, $v_{(G,c)}(\{1,3\}) = 25$, $v_{(G,c)}(\{2,3\}) = 13$, and $v_{(G,c)}(N) = 25$.

The notion of airport games was introduced by Littlechild and Thompson (1977). An airport problem can be illustrated by the following example. Let there be given an airport with one runway, and k different types of planes. For each type of planes i a cost c_i is determined representing the maintenance cost of the runway required for i. For example if i stands for small planes, then the maintenance cost

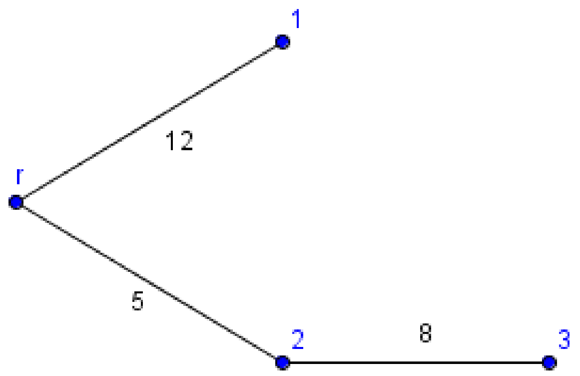

Figure 5.1: Cost tree (G, c) of irrigation problem in Example 5.2

of a runway for them is denoted by c_i. If j is the category of big planes, then $c_i < c_j$, since big planes need a longer runway. Consequently, on player set N a partition is given: $N = N_1 \uplus \dots \uplus N_k$, where N_i denotes the number of planes of type i, and each type i determines a maintenance cost c_i, such that $c_1 < \ldots < c_k$. Considering a coalition of players (planes) S, the maintenance cost of coalition S is the maximum of the members' maintenance costs. In other words, the cost of coalition S is the maintenance cost of the biggest plane of coalition S.
In the following we present two equivalent definitions of airport games.

Definition 5.3 (Airport games I.) *For an airport problem let $N = N_1 \uplus \dots \uplus N_k$ be the set of players, and let there be given $c \in \mathbb{R}^k_+$, such that $c_1 < \ldots < c_k \in \mathbb{R}_+$. Then the airport game $v_{(N,c)} \in \mathcal{G}^N$ can be defined as $v_{(N,c)}(\emptyset) = 0$, and for all non-empty coalitions $S \subseteq N$:*

$$v_{(N,c)}(S) = \max_{i: N_i \cap S \neq \emptyset} c_i \ .$$

An alternative definition is as follows.

Definition 5.4 (Airport games II.) *For an airport problem let $N = N_1 \uplus \dots \uplus N_k$ be the set of players, and $c = c_1 < \ldots < c_k \in \mathbb{R}_+$. Let $G = (V, E)$ be a chain such that $V = N \cup \{r\}$, and $E = \{\overline{r1}, \overline{12}, \ldots, \overline{(|N|-1)|N|}\}$, $N_1 = \{1, \ldots, |N_1|\}, \ldots, N_k = \{|N| - |N_k| + 1, \ldots, |N|\}$. Furthermore, for all $\overline{ij} \in E$ let $c(\overline{ij}) = c_{N(j)} - c_{N(i)}$, where $N(i) = \{N^* \in \{N_1, \ldots, N_k\} : i \in N^*\}$.*

For a cost tree (G, c), an airport game $v_{(N,c)} \in \mathcal{G}^N$ can be defined as follows. Let $N = V \setminus \{r\}$ be the set of players, then for each coalition S (the empty sum *is* 0)

$$v_{(N,c)}(S) = \sum_{e \in \bar{S}} c_e \ .$$

Clearly, both definitions above define the same games, therefore let the class of airport games with player set N be denoted by $\mathcal{G}_A^N$. Furthermore, let $\mathcal{G}_G$ denote the subclass of airport games induced by airport problems on chain G. Note that the notation $\mathcal{G}_G$ is consistent with the notation introduced in Definition 5.1, since if G is a chain, then $\mathcal{G}_G \subseteq \mathcal{G}_A^N$, otherwise, if G is not a chain, then $\mathcal{G}_G \setminus \mathcal{G}_A^N \neq \emptyset$. Since not every rooted tree is a chain, $\mathcal{G}_A^N \subset \mathcal{G}_I^N$.

Example 5.5 *Consider airport problem* (N, c') *by Definition 5.3, and Figure 5.5 corresponding to the problem, where* $N = \{\{1\} \uplus \{2,3\}\}$, $c'_{N(1)} = 5$, *and* $c'_{N(2)} = c'_{N(3)} = 8$ *(*$N(2) = N(3)$*). Next, let us consider Definition 5.4 and the cost tree in Figure 5.5, where rooted tree* $G = (V, E)$ *is defined as* $V = \{r, 1, 2, 3\}$, $E = \{\overline{r1}, \overline{12}, \overline{23}\}$, *and for cost function c:* $c(\overline{r1}) = 5$, $c(\overline{12}) = 3$ *and* $c(\overline{23}) = 0$.

Figure 5.2: Cost tree (G, c) of airport problem in Example 5.5

Then the induced airport game is as follows: $v_{(G,c)} = (0, 5, 8, 8, 8, 8, 8, 8)$, *i.e.* $v_{(G,c)}(\emptyset) = 0$, $v_{(G,c)}(\{1\}) = 5$, $v_{(G,c)}(\{2\}) = v_{(G,c)}(\{3\}) = v_{(G,c)}(\{1,2\}) = v_{(G,c)}(\{1,3\}) = v_{(G,c)}(\{2,3\}) = v_{(G,c)}(N) = 8$.

In the following we will characterize the class of airport and irrigation games. First we note that for all rooted trees G: $\mathcal{G}_G$ is a cone, therefore by definition for all $\alpha \geq 0$: $\alpha \mathcal{G}_G \subseteq \mathcal{G}_G$. Since the union of cones is also a cone, $\mathcal{G}_A^N$ and $\mathcal{G}_I^N$ are also cones. Henceforth, let Cone $\{v_i\}_{i \in N}$ denote the convex (i.e. closed to convex combination) cone spanned by given v_i games.

Lemma 5.6 *For all rooted trees G:* $\mathcal{G}_G$ *is a cone, therefore* $\mathcal{G}_A^N$ *and* $\mathcal{G}_I^N$ *are also cones.*

In the following lemma we will show that the duals of unanimity games are airport games.

Lemma **5.7** *For an arbitrary coalition $\emptyset \neq T \subseteq N$, for which $T = S_i(G)$, $i \in N$, there exists chain G, such that $\bar{u}_T \in \mathcal{G}_G$. Therefore $\{\bar{u}_T\}_{T \in 2^N \setminus \{\emptyset\}} \subset \mathcal{G}_A^N \subset \mathcal{G}_I^N$.*

Proof: For all $i \in N$, $N = (N \setminus S_i(G)) \uplus S_i(G)$ let $c_1 = 0$ and $c_2 = 1$, implying that the cost of members of coalition $N \setminus S_i(G)$ is equal to 0, and the cost of members of coalition $S_i(G)$ is equal to 1 (see Definition 5.3). Then for the induced airport game $v_{(G,c)} = \bar{u}_{S_i(G)}$.

On the other hand, clearly, there exists an airport game which is not the dual of any unanimity game (for example an airport game containing two non-equal positive costs). ∎

Let us consider Figure 5.3 demonstrating the representation of dual unanimity games as airport games on player set $N = \{1, 2, 3\}$.

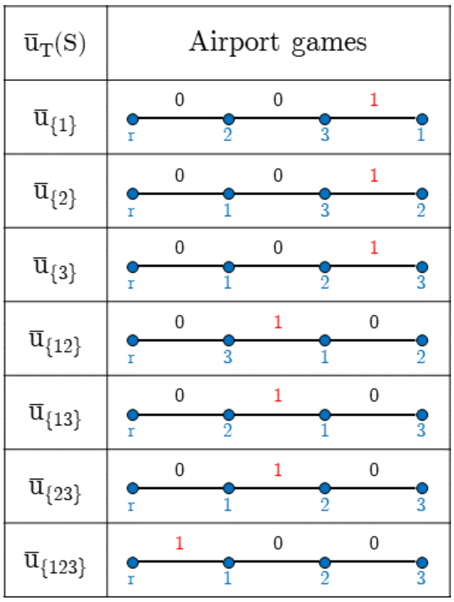

Figure 5.3: Duals of unanimity games as airport games

It is important to note the relationships between the classes of airport and irrigation games, and the convex cone spanned by the duals of unanimity games.

Lemma **5.8** *For all rooted trees G: $\mathcal{G}_G \subset \text{Cone}\,\{\bar{u}_{S_i(G)}\}_{i \in N}$. Therefore, $\mathcal{G}_A^N \subset \mathcal{G}_I^N \subset \text{Cone}\,\{\bar{u}_T\}_{T \in 2^N \setminus \{\emptyset\}}$.*

Proof: First we show that $\mathcal{G}_G \subset \text{Cone}\ \{\bar{u}_{S_i(G)}\}_{i \in N}$.

Let $v \in \mathcal{G}_G$ be an irrigation game. Since $G = (V, E)$ is a rooted tree, for all $i \in N$: $|\{j \in V : \overline{ji} \in E\}| = 1$. Therefore, a notation can be introduced for the player preceding player i, let $i_- = \{j \in V : \overline{ji} \in E\}$. Then for all $i \in N$ let $\alpha_{S_i(G)} = c_{\overline{i_- i}}$.

Finally, it can be easily shown the $v = \sum_{i \in N} \alpha_{S_i(G)} \bar{u}_{S_i(G)}$.

Secondly, we show that $\text{Cone}\ \{\bar{u}_{S_i(G)}\}_{i \in N} \setminus \mathcal{G}_G \neq \emptyset$. Let $N = \{1, 2\}$, then $\sum_{T \in 2^N \setminus \{\emptyset\}} \bar{u}_T \notin \mathcal{G}_G$. Consequently, $(2, 2, 3)$ is not an irrigation game, since in the case of two players and the value of coalitions with one member is 2, then we may get two different trees. If the tree consists of a single chain, where the cost of the last edge is 0, then the value of the grand coalition should be 2, while if there are two edges pointing outwards from the root with cost 2 for each, then the value of the grand coalition should be 4. ■

We demonstrate the above results with the Example 5.9.

Example 5.9 *Let us consider the irrigation game given in Example 5.2 with the cost tree given in Figure 5.4.*

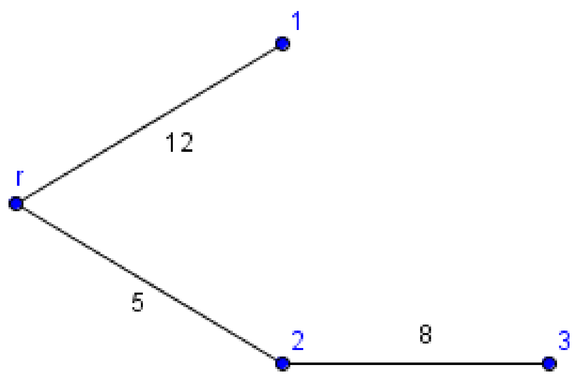

Figure 5.4: Cost tree (G, c) of irrigation problems in Examples 5.2 and 5.9

Consequently, $S_1(G) = \{1\}$, $S_2(G) = \{2, 3\}$ *and* $S_3(G) = \{3\}$. *Moreover,* $\alpha_{S_1(G)} = 12$, $\alpha_{S_2(G)} = 5$ *and* $\alpha_{S_3(G)} = 8$. *Finally,* $v_{(G,c)} = 12\bar{u}_{\{1\}} + 5\bar{u}_{\{2,3\}} + 8\bar{u}_{\{3\}} = \sum_{i \in N} \alpha_{S_i(G)} \bar{u}_{S_i(G)}$.

In the following we discuss further corollaries of Lemmata 5.7 and 5.8. First we prove that for rooted tree G, even if set $\mathcal{G}_G$ is convex, the set of airport and irrigation games are not convex.

Lemma **5.10** *Neither $\mathcal{G}_A^N$ nor $\mathcal{G}_I^N$ is convex.*

Proof: Let $N = \{1, 2\}$. According to Lemma 5.7 we know that $\{\bar{u}_T\}_{T \in 2^N \setminus \{\emptyset\}} \subseteq \mathcal{G}_A^N$. However, $\sum_{T \in 2^N \setminus \{\emptyset\}} \frac{1}{3} \bar{u}_T \notin \mathcal{G}_I^N$, consequently $(2/3, 2/3, 1)$ is not an irrigation game. ■

The following corollary is essential for the axiomatization of the Shapley value given by Young (1985b) from the standpoint of airport and irrigation games. It is well-known that the duals of unanimity games are linearly independent vectors. Based on Lemma 5.8 for all rooted trees G and game $v \in \mathcal{G}_G$: $v = \sum_{i \in N} \alpha_{S_i(G)} \bar{u}_{S_i(G)}$, where weights $\alpha_{S_i(G)}$ are well-defined, i.e. unambiguously determined. The following Lemma 5.11 claims that for all games $v \in \mathcal{G}_G$, if the weight of any of the basis vectors (duals of unanimity games) is decreased to 0 (erased), then we get a game corresponding to $\mathcal{G}_G$.

Lemma **5.11** *For all rooted trees G and $v = \sum_{i \in N} \alpha_{S_i(G)} \bar{u}_{S_i(G)} \in \mathcal{G}_G$, and for all $i^* \in N$: $\sum_{i \in N \setminus \{i^*\}} \alpha_{S_i(G)} \bar{u}_{S_i(G)} \in \mathcal{G}_G$. Then for all airport games $v = \sum_{T \in 2^N \setminus \{\emptyset\}} \alpha_T \bar{u}_T$ and coalition $T^* \in 2^N \setminus \{\emptyset\}$: $\sum_{T \in 2^N \setminus \{\emptyset, T^*\}} \alpha_T \bar{u}_T \in \mathcal{G}_A^N$, and for all irrigation games $v = \sum_{T \in 2^N \setminus \{\emptyset\}} \alpha_T \bar{u}_T$ and $T^* \in 2^N \setminus \{\emptyset\}$: $\sum_{T \in 2^N \setminus \{\emptyset, T^*\}} \alpha_T \bar{u}_T \in \mathcal{G}_I^N$.*

Proof: Let $v = \sum_{i \in N} \alpha_{S_i(G)} \bar{u}_{S_i(G)}$ and $i^* \in N$. Then cost function c' is defined as follows. For all $e \in E$ (see proof of Lemma 5.8)

$$c'_e = \begin{cases} 0, & \text{if } e = \overline{i^*\_i^*}, \\ c_e & \text{otherwise.} \end{cases}$$

Then $\sum_{i \in N \setminus \{i^*\}} \alpha_{S_i(G)} \bar{u}_{S_i(G)} = v_{(G,c')}$, i.e. $\sum_{i \in N \setminus \{i^*\}} \alpha_{S_i(G)} \bar{u}_{S_i(G)} \in \mathcal{G}_G$. ■

The next example illustrates the above results.

Example 5.12 *Let us consider the irrigation game in Example 5.2, and player 2. Then*

$$c'_e = \begin{cases} 12, & \textit{if } e = \overline{r1}, \\ 0, & \textit{if } e = \overline{r2}, \\ 8, & \textit{if } e = \overline{23}. \end{cases}$$

Moreover, $\sum_{i \in N \setminus \{i^*\}} \alpha_{S_i(G)} \bar{u}_{S_i(G)} = 12\bar{u}_{\{1\}} + 8\bar{u}_{\{3\}} = v_{(G,c')}$ is an irrigation game.

Finally, a simple observation.

Lemma **5.13** *All irrigation games are concave.*

Proof: Duals of unanimity games are concave, therefore the claim follows from Lemma 5.8. ■

Our results can be summarized as follows.

Corollary **5.14** *In the case of a fixed set of players the class of airport games is a union of finitely many convex cones, but the class itself is not convex. Moreover, the class of airport games is a proper subset of the class of irrigation games. The class of irrigation games is also a union of finitely many convex cones, but is not convex, either. Furthermore, the class of irrigation games is a proper subset of the finite convex cone spanned by the duals of the unanimity games, therefore every irrigation game is concave, and consequently every airport game is concave too.*

5.2 Solutions for irrigation games

In this section we present solutions for irrigation games. In the following we discuss two previously described solutions, the Shapley value (Shapley, 1953), and the core (Shapley, 1955; Gillies, 1959).

Let $v \in \mathcal{G}^N$ and

$$p^i_{Sh}(S) = \begin{cases} \dfrac{|S|!(|(N \setminus S)| - 1)!}{|N|!}, & \text{if } i \notin S, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\phi_i(v)$, the Shapley value of player i in game v is the p^i_{Sh}-weighted expected value of all v'_i. In other words:

$$\phi_i(v) = \sum_{S \subseteq N} v'_i(S)\; p^i_{Sh}(S)\ . \tag{5.1}$$

Henceforth let ϕ denote the Shapley value.

It follows from the definition that the Shapley value is a single-valued solution, i.e. a value.

As a reminder we define the previously discussed core. Let $v \in \mathcal{G}^N_I$ be an irrigation game. Then the core of v is

$$C(v) = \left\{ x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N), \text{ and for all } S \subseteq N : \sum_{i \in S} x_i \leq v(S) \right\} .$$

The core consists of the stable allocations of the value of the grand coalition, i.e. any allocation in the core is such that the allocated cost is the total cost ($\sum_{i \in N} x_i = v(N)$) and no coalition has incentive to deviate from the allocation. Taking irrigation games as an example, the core allocation describes the distributions which do not force any group of users to pay a higher cost than the total cost of the segments of the ditch they are using. In Chapter 2 this property was expressed by the subsidy-free axiom (Axiom 2.9) for rooted subtree coalitions. The following theorem describes the relationship between the subsidy-free property and the core for the case of irrigation games.

Definition 5.15 *Value ψ in game class $A \subseteq \mathcal{G}^N$ is* core compatible*, if for all $v \in A$ it holds that $\psi(v) \in C(v)$.*

That is, core compatibility holds, if the cost allocation rule given for an irrigation problem provides a solution that is a core allocation in the corresponding irrigation game.

Theorem 5.16 *A cost allocation rule ξ on the cost tree (G, c) is subsidy-free, if and only if the value generated by the cost allocation rule ξ on the irrigation game $v_{(G,c)}$ induced by the cost tree is core compatible.*

Proof: For irrigation games it holds that $v_{(G,c)}(S) = v_{(G,c)}(\overline{S})$ and $\sum\limits_{i \in S} \xi_i \leq \sum\limits_{i \in \overline{S}} \xi_i$. For core allocations it holds that $\sum\limits_{i \in S} \xi_i \leq v_{(G,c)}(S)$. Since

$$\sum_{i \in S} \xi_i \leq \sum_{i \in \overline{S}} \xi_i \leq v_{(G,c)}(\overline{S}) = v_{(G,c)}(S),$$

it is sufficient to examine cases where $S = \overline{S}$. Therefore the allocations in the core are those for which

$$\sum_{i \in \overline{S}} \xi_i \leq v_{(G,c)}(\overline{S}) = \sum_{i \in \overline{S}} c_i.$$

Comparing the beginning and end of the inequality, we get the subsidy-free property. ■

Among the cost allocation methods described in Chapter 2 the serial and restricted average cost allocations were subsidy-free, and Aadland and Kolpin (1998) have shown that the serial allocation is equivalent to the Shapley value. However, it is more difficult to formulate which further aspects can be chosen and is worth choosing from the core. There are subjective considerations depending on the situation that may influence decision makers regarding which core allocation to choose (which are the "best" alternatives for rational decision makers). Therefore, in case of a specific distribution it is worth examining in detail, which properties a given distribution satisfies, and which properties characterize the distribution within the game class.

In the following definition we discuss properties that we will use to characterize single-valued solutions. Some of these have already been described in Section 3.3, now we will discuss specifically those that we are going to use in this section. As a reminder, the marginal contribution of player i to coalition S in game v is $v_i'(S) = v(S \cup \{i\}) - v(S)$ for all $S \subseteq N$. Furthermore, $i, j \in N$ are equivalent in game v, i.e. $i \sim^v j$ if $v_i'(S) = v_j'(S)$ for all $S \subseteq N \setminus \{i, j\}$.

Definition 5.17 *A single-valued solution ψ on $A \subseteq \mathcal{G}^N$ is / satisfies*

- *Pareto optimal (PO), if for all $v \in A$, $\sum\limits_{i \in N} \psi_i(v) = v(N)$,*
- *null-player property (NP), if for all $v \in A$, $i \in N$, $v_i' = 0$ implies $\psi_i(v) = 0$,*
- *equal treatment property (ETP), if for all $v \in A$, $i, j \in N$, $i \sim^v j$ implies $\psi_i(v) = \psi_j(v)$,*
- *additive (ADD), if for all $v, w \in A$ such that $v + w \in A$, $\psi(v + w) = \psi(v) + \psi(w)$,*
- *marginal (M), if for all $v, w \in A$, $i \in N$, $v_i' = w_i'$ implies $\psi_i(v) = \psi_i(w)$.*

A brief interpretations of the above axioms is as follows. Firstly, another commonly used name of axiom *PO* is *Efficiency*. This axiom requires that the total cost must be shared among the players. Axiom *NP* requires that if the player's marginal contribution to all coalitions is zero, i.e. the player has no influence on the costs in the given situation, then the player's share (value) must be zero.

Axiom ETP requires that if two players have the same effect on the change of costs in the given situation, then their share must be equal when distributing the total cost. Going back to our example this means that if two users are equivalent with respect to their irrigation costs, then their cost shares must be equal as well.

A solution meets axiom ADD, if for any two games, we get the same result by adding up the games first and then evaluating the share of players, as if we evaluated the players first and then added up their shares. Finally, axiom M requires that if a given player in two games has the same marginal contributions to the coalitions, then the player must be evaluated equally in the two games. Let us consider the following observation.

Claim 5.18 *Let $A, B \subseteq \mathcal{G}^N$. If a set of axioms S characterizes value ψ on class of games both A and B, and ψ satisfies the axioms in S on set $A \cup B$ then it characterizes the value on class $A \cup B$ as well.*

In the following we examine two characterizations of the Shapley value on the class of airport and irrigation games. The first one is the original axiomatization by Shapley (Shapley, 1953).

Theorem 5.19 *For all rooted trees G a value ψ is PO, NP, ETP and ADD on $\mathcal{G}_G$ if and only if $\psi = \phi$, i.e. the value is the Shapley value. In other words, a value ψ is PO, NP, ETP and ADD on the class of airport and irrigation games if and only if $\psi = \phi$.*

Proof: $\Rightarrow$: It is known that the Shapley value is PO, NP, ETP and ADD, see for example Peleg and Sudhölter (2007).

$\Leftarrow$: Based on Lemmata 5.6 and 5.7, ψ is defined on the cone spanned by $\{\bar{u}_{S_i(G)}\}_{i \in N}$.

Let us consider player $i^* \in N$. Then for all $\alpha \geq 0$ and players $i, j \in S_{i^*}(G)$: $i \sim^{\alpha \bar{u}_{S_{i^*}(G)}} j$, and for all players $i \notin S_{i^*}(G)$: $i \in NP(\alpha \bar{u}_{S_{i^*}(G)})$.

Then from property NP it follows that for all players $i \notin S_{i^*}(G)$ it holds that $\psi_i(\alpha \bar{u}_{S_{i^*}(G)}) = 0$. Moreover, according to axiom ETP for all players $i, j \in S_{i^*}(G)$: $\psi_i(\alpha \bar{u}_{S_{i^*}(G)}) = \psi_j(\alpha \bar{u}_{S_{i^*}(G)})$. Finally, axiom PO implies that $\sum_{i \in N} \psi_i(\alpha \bar{u}_{S_{i^*}(G)}) = \alpha$.

Consequently, we know that $\psi(\alpha\bar{u}_{S_{i^*}(G)})$ is well-defined (uniquely determined), and since the Shapley value is PO, NP and ETP, it follows that $\psi(\alpha\bar{u}_{S_{i^*}(G)}) = \phi(\alpha\bar{u}_{S_{i^*}(G)})$.

Furthermore, we know that games $\{u_T\}_{T\in 2^N\setminus\emptyset}$ give a basis of $\mathcal{G}^N$, and the same holds for games $\{\bar{u}_T\}_{T\in 2^N\setminus\emptyset}$. Let $v \in \mathcal{G}_G$ be an irrigation game. Then it follows from Lemma 5.8 that

$$v = \sum_{i\in N} \alpha_{S_i(G)}\bar{u}_{S_i(G)} \ ,$$

where for all $i \in N$: $\alpha_{S_i(G)} \geq 0$. Therefore, since the Shapley value is ADD, and for all $i \in N$, $\alpha_{S_i(G)} \geq 0$: $\psi(\alpha_{S_i(G)}\bar{u}_{S_i(G)}) = \phi(\alpha_{S_i(G)}\bar{u}_{S_i(G)})$, i.e. $\psi(v) = \phi(v)$.

Finally, Claim 5.18 can be applied. ■

In the proof of Theorem 5.19 we have applied a modified version of Shapley's original proof. In his proof Shapley uses the basis given by unanimity games $\mathcal{G}^N$. In the previous proof we considered the duals of the unanimity games as a basis and used Claim 5.18 and Lemmata 5.6, 5.7, 5.8. It is worth noting that (as we discuss in the next section) for the class of airport games Theorem 5.19 was also proven by Dubey (1982), therefore in this sense our result is also an alternative proof for Dubey (1982)'s result.

In the following we consider Young's axiomatization of the Shapley value (Young, 1985b). This was the first axiomatization of the Shapley value not involving axiom ADD.

Theorem 5.20 *For any rooted tree G, a single-valued solution ψ on $\mathcal{G}_G$ is PO, ETP and M if and only if $\psi = \phi$, i.e. it is the Shapley value. Therefore, a single-valued solution ψ on the class of airport games is PO, ETP and M if and only if $\psi = \phi$, and a single-valued solution ψ on the class of irrigation games is PO, ETP and M if and only if $\psi = \phi$.*

Proof: $\Rightarrow$: We know that the Shapley value meets axioms PO, ETP and M, see e.g. Peleg and Sudhölter (2007).

$\Leftarrow$: The proof is by induction, similarly to that of Young's. For any irrigation game $v \in \mathcal{G}_G$, let $\mathcal{B}(v) = |\{\alpha_{S_i(G)} > 0 \mid v = \sum_{i\in N} \alpha_{S_i(G)}\bar{u}_{S_i(G)}\}|$. Clearly, $\mathcal{B}(\cdot)$ is well-defined.

If $\mathcal{B}(v) = 0$ (i.e. $v \equiv 0$), then axioms PO and ETP imply that $\psi(v) = \phi(v)$.

Let us assume that for any game $v \in \mathcal{G}_G$ for which $\mathcal{B}(v) \leq n$ it holds that $\psi(v) = \phi(v)$. Furthermore, let $v = \sum_{i \in N} \alpha_{S_i(G)} \bar{u}_{S_i(G)} \in \mathcal{G}_G$ such that $\mathcal{B}(v) = n+1$. Then one of the following two points holds.

1. There exists player $i^* \in N$ such that there exists $i \in N$ that it holds that $\alpha_{S_i(G)} \neq 0$ and $i^* \notin S_i(G)$. Then Lemmata 5.8 and 5.11 imply that $\sum_{j \in N \setminus \{i\}} \alpha_{S_j(G)} \bar{u}_{S_j(G)} \in \mathcal{G}_G$, and for the marginal contributions it holds that

$$\left(\sum_{j \in N \setminus \{i\}} \alpha_{S_j(G)} \bar{u}_{S_j(G)} \right)'_{i^*} = v'_{i^*} \ ,$$

   therefore, from axiom M

$$\psi_{i^*}(v) = \psi_{i^*} \left(\sum_{j \in N \setminus \{i\}} \alpha_{S_j(G)} \bar{u}_{S_j(G)} \right) ,$$

   i.e. $\psi_{i^*}(v)$ is well-defined (uniquely determined) and due to the induction hypothesis, $\psi_{i^*}(v) = \phi_{i^*}(v)$.

2. There exist players $i^*, j^* \in N$ such that for all $i \in N$: $\alpha_{S_i(G)} \neq 0$ it holds that $i^*, j^* \in S_i(G)$. Then $i^* \sim^v j^*$, therefore, axiom ETP implies that $\psi_{i^*}(v) = \psi_{j^*}(v)$.

   Axiom PO implies that $\sum_{i \in N} \psi_i(v) = v(N)$. Therefore, $\psi(v)$ is well-defined (uniquely determined), consequently, since the Shapley value meets PO, ETP and M, due to the induction hypothesis, $\psi(v) = \phi(v)$.

Finally, we can apply Claim 5.18. ■

In the above proof we apply the idea of Young's proof, therefore we do not need the alternative approaches provided by Moulin (1988) and Pintér (2015). Our approach is valid, because Lemma 5.11 ensures that when the induction step is applied, we do not leave the considered class of games. It is also worth noting that (as we also discuss in the following) for the class of airport games Theorem 5.20 has also been proven by Moulin and Shenker (1992), therefore in this sense our result is also an alternative proof for Moulin and Shenker (1992)'s result.

Finally, Lemma 5.13 and the well-known results of Shapley (1971) and Ichiishi (1981) imply the following corollary.

Corollary 5.21 *For any irrigation game v, $\phi(v) \in C(v)$, i.e. the Shapley value is in the core. Moreover, since every airport game is an irrigation game, for any airport game v: $\phi(v) \in C(v)$.*

The above corollary shows that the Shapley value is stable on both classes of games we have considered.

5.3 Cost sharing results

In this section we reformulate our results using the classic cost sharing terminology. In order to unify the different terminologies used in the literature, we exclusively use Thomson (2007)'s notions. First we introduce the notion of a *rule*, which is in practice analogous to the notion of a cost-sharing rule introduced by Aadland and Kolpin (1998) (see Definition 2.2). Let us consider the class of allocation problems defined on cost trees, i.e. the set of cost trees. A *rule* is a mapping which assigns a cost allocation to a cost tree allocation problem, providing a method by which the cost is allocated among the players. Note that the rule is a single-valued mapping. The analogy between the rule and the solution is clear, the only significant difference is that while the solution is a set-valued mapping, the rule is single-valued.

Let us introduce the rule known in the literature as *sequential equal contributions rule.*

Definition 5.22 (SEC rule) *For all cost trees (G, c) and for all players i the distribution according to the* SEC rule *is given as follows:*

$$\xi_i^{SEC}(G, c) = \sum_{j \in P_i(G) \setminus \{r\}} \frac{c_{\overline{j_- j}}}{|S_j(G)|} .$$

In the case of airport games, where graph G is a chain, the SEC rule can be given as follows, for any player i:

$$\xi_i^{SEC}(G,c) = \frac{c_1}{n} + \cdots + \frac{c_{\overline{i\_i}}}{n-i+1} ,$$

where the players are ordered according to the their positions in the chain, i.e. player i is at the ith position of the chain.

Littlechild and Owen (1973) have shown that the SEC rule and the Shapley value are equivalent on the class of irrigation games.

Claim 5.23 (Littlechild and Owen, 1973) *For any cost-tree (G, c) it holds that $\xi(G, c) = \phi(v_{(G,c)})$, where $v_{(G,c)}$ is the irrigation game generated by cost tree (G, c). In other words, for cost-tree allocation problems the SEC rule and the Shapley value are equivalent.*

In the following we consider certain properties of *rules* (see Thomson, 2007).

Definition 5.24 *Let $G = (V, A)$ be a rooted tree. Rule χ defined on the set of cost trees denoted by G satisfies*

- *non-negativity, if for each cost function c, $\chi(G, c) \geq 0$,*
- *cost boundedness, if for each cost function c, $\chi(G, c) \leq \left(\sum_{e \in A_{P_i(G)}} c_e \right)_{i \in N}$,*
- *efficiency, if for each cost function c, $\sum_{i \in N} \chi_i(G, c) = \sum_{e \in A} c_e$,*
- *equal treatment of equals, if for each cost function c and pair of players $i, j \in N$, $\sum_{e \in A_{P_i(G)}} c_e = \sum_{e \in A_{P_j(G)}} c_e$ implies $\chi_i(G, c) = \chi_j(G, c)$,*
- *conditional cost additivity, if for any pair of cost functions c, c', $\chi(G, c + c') = \chi(G, c) + \chi(G, c')$,*
- *independence of at-least-as-large costs, if for any pair of cost functions c, c' and player $i \in N$ such that for each $j \in P_i(G)$, $\sum_{e \in A_{P_j(G)}} c_e = \sum_{e \in A_{P_j(G)}} c'_e$, $\chi_i(G, c) = \chi_i(G, c')$.*

The interpretations of the above properties of rules defined above are as follows (Thomson, 2007). Non-negativity claims that for each problem the rule must only give a non-negative cost allocation vector as result. Cost boundedness requires that the individual costs must be an upper bound for the elements of the cost allocation vector. Efficiency describes that coordinates of the cost allocation vector must add up to the maximal cost. Equal treatment of equals states that players with equal individual costs must pay equal amounts. Conditional cost additivity requires that if two cost trees are summed up (the tree is fixed), then the cost allocation vector defined by the sum must be the sum of the cost allocation vectors of the individual problems. Finally, independence of at-least-as-large costs means that the sum payed by a player must be independent of the costs of the segments he does not use.

These properties are similar to the axioms we defined in Definition 5.17. The following proposition formalizes the similarity.

Claim **5.25** *Let G be a rooted tree, χ be defined on cost trees (G, c), solution ψ be defined on $\mathcal{G}_G$ such that $\chi(G, c) = \psi(v_{(G,c)})$ for any cost function c. Then, if χ satisfies*

- *non-negativity and cost boundedness, then ψ is NP,*
- *efficiency, then ψ is PO,*
- *equal treatment of equals, then ψ is ETP,*
- *conditional cost additivity, then ψ is ADD,*
- *independence of at-least-as-large costs, then ψ is M.*

Proof: NN and $CB \Rightarrow NP$: Clearly, player i is NP, if and only if $\sum_{e \in A_{P_i(G)}} c_e = 0$. Then NN implies that $\chi_i(G, c) \geq 0$, and from CB: $\chi_i(G, c) \leq 0$. Consequently, $\chi_i(G, c) = 0$, therefore $\psi(v_{(G,c)}) = 0$.

$E \Rightarrow PO$: From the definition of irrigation games (Definition 5.1): $\sum_{e \in A} c_e = v_{(G,c)}(N)$, therefore $\sum_{i \in N} \psi_i(v_{(G,c)}) = \sum_{i \in N} \chi_i(G, c) = \sum_{e \in A} c_e = v_{(G,c)}(N)$.

$ETE \Rightarrow ETP$: Clearly, if $i \sim^{v_{(G,c)}} j$ for $i, j \in N$, then $\sum_{e \in A_{P_i(G)}} c_e = \sum_{e \in A_{P_j(G)}} c_e$, so $\chi_i(G, c) = \chi_j(G, c)$. Therefore, if $i \sim^{v_{(G,c)}} j$ for $i, j \in N$, then $\psi_i(v_{(G,c)}) = \psi_j(v_{(G,c)})$.

$CCA \Rightarrow ADD$: $\psi(v_{(G,c)} + v_{(G,c')}) = \psi(v_{(G,c+c')}) = \chi(G, c + c') = \chi((G, c) + (G, c')) = \chi(G, c) + \chi(G, c') = \psi(v_{(G,c)}) + \psi(v_{(G,c')})$.

$IALC \Rightarrow M$: It is easy to check that if for cost trees (G,c), (G,c') and player $i \in N$ it holds that $(v_{(G,c)})'_i = (v_{(G,c')})'_i$, then for each $j \in P_i(G)$: $\sum_{e \in A_{P_j(G)}} c_e = \sum_{e \in A_{P_j(G)}} c'_e$, therefore $\chi_i(G,c) = \chi_i(G,c')$. Consequently, $(v_{(G,c)})'_i = (v_{(G,c')})'_i$ implies $\psi_i(v_{(G,c)}) = \psi_i(v_{(G,c')})$. ■

It is worth noting that all but the efficiency point are tight, i.e. efficiency and the Pareto-optimal property are equivalent, in the other cases the implication holds in only one way. Therefore the cost-sharing axioms are stronger than the game theoretical axioms.

The above results and Theorem 5.19 imply Dubey (1982)'s result as a direct corollary.

Theorem 5.26 (Dubey, 1982) *Rule χ on airport the class of airport games satisfies non-negativity, cost boundedness, efficiency, equal treatment of equals and conditional cost additivity, if and only if $\chi = \xi$, i.e. χ is the SEC rule.*

Proof: $\Rightarrow$: With a simple calculation it can be shown that the SEC rule satisfies properties NN, CB, E, ETE, and CCA (see e.g.Thomson, 2007).

$\Leftarrow$: Claim 5.25 implies that we can apply Theorem 5.19 and thereby get the Shapley value. Then Claim 5.23 implies that the Shapley value and the SEC rule coincide. ■

Moulin and Shenker (1992)'s result can be deduced from the results above and Theorem 5.20, similarly to Dubey (1982)'s result.

Theorem 5.27 (Moulin and Shenker, 1992) *Rule χ on the class of airport problems satisfies efficiency, equal treatment of equals and independence of at-least-as-large costs, if and only if $\chi = \xi$, i.e. χ is the SEC rule.*

Proof: $\Rightarrow$: It can be shown with a simple calculation that the SEC rule satisfies the E, ETE and $IALC$ properties (see e.g. Thomson, 2007).

$\Leftarrow$: Based on Claim 5.25, Theorem 5.19 can be applied and thereby get the Shapley value. Then Claim 5.23 implies that the Shapley value and the SEC rule coincide. ■

Finally, in the following two theorems we extend the results of Dubey (1982) and Moulin and Shenker (1992) to any cost tree allocation problem. The proofs of these results is the same as those of the previous two theorems.

Theorem **5.28** *Rule* χ *on cost-tree problems satisfies non-negativity, cost boundedness, efficiency, equal treatment of equals and conditional cost additivity, if and only if* $\chi = \xi$*, i.e.* χ *is the SEC rule.*

Theorem **5.29** *Rule* χ *on cost-tree problems satisfies efficiency, equal treatment of equals and independence of at-least-as-large costs, if and only if* $\chi = \xi$*, i.e.* χ *is the SEC rule.*

Chapter 6

Upstream responsibility

In this chapter we consider further cost sharing problems given by rooted trees, called *cost-tree problems*, but we are considering different applications from those so far. We will consider energy supply chains with a motivated dominant leader, who has the power to determine the responsibilities of suppliers for both direct and indirect emissions. The induced games are called *upstream responsibility games* (Gopalakrishnan, Granot, Granot, Sosic and Cui, 2017), and henceforth we will refer to it as *UR game*.

For an example let us consider a supply chain where we look at the responsibility allocation of greenhouse gas (GHG) emission among the firms in the chain. One of the main questions is how to share the costs related to the emission among the firms. The supply chain and the related firms (or any other actors) can be represented by a rooted tree.

The root of the tree represents the end product produced by the supply chain. The root is connected to only one node which is the leader of the chain. Each further node represents one firm, and the edges of the rooted tree represent the manufacturing process among the firms with the related emissions. Our goal is to share the responsibility of the emission while embodying the principle of upstream emission responsibility.

In this chapter we utilize the TU game model of Gopalakrishnan et al. (2017), called GHG Responsibility-Emissions and Environment (GREEN) game. The Shapley value is used as an allocation method by Gopalakrishnan et al., who also consider some pollution-related properties that an emission allocation rule should meet, and provide several axiomatizations as well.

One of the extensions of airport games is the well-known class of standard fixed-tree games. One of the applications of these games is the irrigation problems, considered by Aadland and Kolpin (1998) and also by us (Márkus, Pintér and Radványi, 2011). However, UR games are a different generalization from irrigation games. Since the validity of a specific solution concept may change from subclass to subclass, it is important to examine the classes of games individually.

However, we note that even though as stated above UR games are a different generalization of airport games from irrigation games, the elements of the two classes are "structurally" similar. Both classes of games are defined on rooted trees, the important difference is the "direction" of the graphs. While in neither case must the graph be directed, the motivation, the real-life problem behind the graph defines an "ordering" among the players, i.e. the nodes of the graph. In the case of irrigation games, the most important aspect is the number of users for a segment of the ditch, i.e. for each node the following users ($S_i(G)$) play an important role. Contrarily, in the case of UR games, the according to the interpretation we move from the leaves towards the root, for any node the most important aspect will be the set of preceding nodes ($P_i(G)$), i.e. the set of nodes on the path from the leaf to the root. Our results presented for irrigation and UR games will be similar due to the structure of the defining graphs. However, the different motivations generate different classes of games not only from a mathematical, but also from an application standpoint, and carrying out an analysis is relevant in both cases.

In the following, we consider upstream responsibility games and characterize their class. We show that the class is a non-convex cone which is a proper subset of the finite convex cone spanned by the duals of the unanimity games, therefore every upstream responsibility game is concave. Furthermore, as a corollary we show that Shapley's and Young's axiomatizations are valid on the class of upstream responsibility games. We also note that the Shapley value is stable for this class as well, i.e. it is always in the core. This result is a simple corollary of the Ichiishi-Shapley theorem (Shapley, 1971; Ichiishi, 1981) and that every upstream responsibility game is concave. It is also an implication of our characterization that the Shapley value can be computed efficiently on this class of games (this result is analogous to that of Megiddo (1978) for fixed-tree games). The results on the class of UR games presented in the following have been published in working paper Radványi (2018).

6.1 Upstream responsibility games

In this section we will leverage the notations and notions introduced in Section 4.1 and Chapter 5. In this case, a rooted graph $G = (V, E)$ models the manufacturing process throughout a supply chain. The nodes of set V represent the entities and firms cooperating in the supply chain, and the distinctive node $r \in V$ represents the end product. Edge $e \in E$ directed from player i towards the root node denotes the activities and contributions of player i to the manufacturing of the end product. The directed edge pointing from i to j is denoted by $\vec{ij}$. We assume that there is a single node from which an edge is directed to the root. This node is denoted by 1 and represents the end user.

A partial ordering on the nodes of the tree will be needed. As a reminder, $i \leq j$, if the unique path from j to the root passes through i. For all $i \in V$: $S_i(G) = \{j \in V : j \geq i\}$ is the set comprising i, and the users following i. Similarly, for all $i \in V$: $P_i(G) = \{j \in V : j \leq i\}$ is the set comprising i, and the users preceding i. Moreover, for any $V' \subseteq V$, let $(P_{V'}(G), E_{V'})$ be the sub-rooted-tree of (V, E), where $P_{V'}(G) = \cup_{i \in V'} P_i(G)$ and $E_{V'} = \{\overline{ij} \in E : i, j \in P_{V'}(G)\}$.

Let there be given a cost function $c : E \to \mathbb{R}_+$ and a cost tree (G, c). In the above interpretation for supply chains the nodes $V = N \cup \{\mathrm{r}\}$ are the participants of the manufacturing process, and the end product. Node 1 denotes the end user. For all edges $e \in E$ there is given an associated activity, and a cost denoted by $c(e)$ corresponding to the emission of the activity. Let e_i be the unique edge that is directed towards the root from node i. Then $c(e_i)$ represents the direct emission cost corresponding to e_i, i.e. to user i. Furthermore, i may be responsible for other emissions in the supply chain. For all nodes i let $\mathcal{E}_i$ denote the set of edges whose corresponding emission is the direct or indirect responsibility of i.

In the following we assume that the game consists of at least two players, i.e. $|V| \geq 3$ and $|N| \geq 2$.

In the following we introduce the *upstream responsibility games* (UR game - URG; Gopalakrishnan et al., 2017). Let (G, c) be the cost tree representing the supply chain, and let N be the set of the members of the chain, henceforth players (nodes of the graph, except for the root). Let c_j denote the pollution associated with edge j (e.g. expressed as monetary cost). Let $v(\{i\})$ be the total emission cost for which player i is directly, or indirectly responsible, therefore $v(\{i\}) = c(\mathcal{E}_i) \equiv \sum_{j \in \mathcal{E}_i} c_j$. For all $S \subseteq N$ let $\mathcal{E}_S$ be the set of all edges for which their corresponding emissions are the direct or indirect responsibilities of players in S. In other words, $\mathcal{E}_S = \cup_{i \in S} \mathcal{E}_i$, and the degree of pollution that is the direct or indirect responsibility of coalition S is $v(S) = c(\mathcal{E}_S) \equiv \sum_{j \in \mathcal{E}_S} c_j$. There may be several answers to the question of which edges i is directly or indirectly responsible for, and it is usually specific to the application which edges of the graph are the elements of set $\mathcal{E}_i$. In present thesis we only examine cases where $\mathcal{E}_i$ contains edges that belong to the rooted subtree branching out from i (for these i is indirectly responsible), and the edge leading from i towards the root (for this i is directly responsible).

Let $\mathcal{G}_{UR}^N$ denote the class of UR games corresponding to the set of players N. Let $\mathcal{G}_G$ be the subclass of UR games associated with rooted cost tree G.

Definition 6.1 (UR game) *For all cost trees (G, c), player set $N = V \setminus \{r\}$, and coalition S let the UR game be defined as follows.*

$$v_{(G,c)}(S) = \sum_{j \in \mathcal{E}_S} c_j \ ,$$

where the value of the empty set is 0.

The following example demonstrates the above definition.

Example 6.2 *Let us consider the cost tree in Figure 6.1, and let the rooted tree $G = (V, E)$ be defined as follows: $V = \{r, 1, 2, 3\}$, $E = \{\vec{1r}, \vec{21}, \vec{31}\}$. Therefore, the degree of the pollution i.e. the cost arising from the emission is given for each edge by vector $c = (2, 4, 1)$, i.e. $c_{\vec{1r}} = 2$, $c_{\vec{21}} = 4$, and $c_{\vec{31}} = 1$. Individual players are responsible for the following edges: $\mathcal{E}_1 = \{\vec{1r}, \vec{21}, \vec{31}\}$, $\mathcal{E}_2 = \{\vec{21}\}$, $\mathcal{E}_3 = \{\vec{31}\}$.*

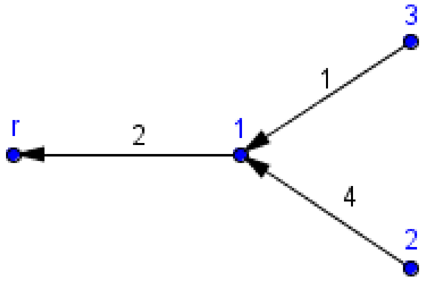

Figure 6.1: Cost tree (G, c) of UR game in Example 6.2

Then the UR game is as follows. $v_{(G,c)} = (0, 7, 4, 1, 7, 7, 5, 7)$*, i.e.* $v_{(G,c)}(\emptyset) = 0$, $v_{(G,c)}(\{1\}) = 7$, $v_{(G,c)}(\{2\}) = 4$, $v_{(G,c)}(\{3\}) = 1$, $v_{(G,c)}(\{1, 2\}) = 7$, $v_{(G,c)}(\{1, 3\}) = 7$, $v_{(G,c)}(\{2, 3\}) = 5$, *and* $v_{(G,c)}(N) = 7$.

Referring back to the airport games discussed in the previous chapter (their class denoted by $\mathcal{G}_A^N$), the following claim can be made similarly to the case of irrigation games. For a subclass of airport games defined on chain G, denoted by $\mathcal{G}_G$, it holds, that $\mathcal{G}_G \subseteq \mathcal{G}_A$, where $\mathcal{G}_A = \cup_N \mathcal{G}_A^N$ $(0 < |N| < \infty)$. In other cases, when G is not a chain, $\mathcal{G}_G \setminus \mathcal{G}_A \neq \emptyset$. Since not every rooted tree is a chain, $\mathcal{G}_A^N \subset \mathcal{G}_{UR}^N$.

Therefore, airport games are not only irrigation games, but also UR games as well. Let us illustrate this with the example given in Figure 6.2.

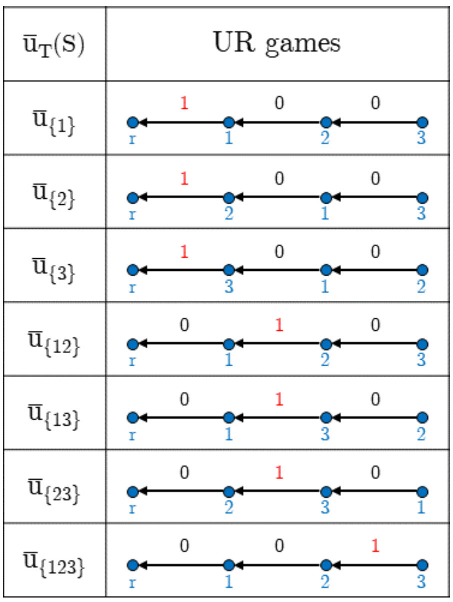

Figure 6.2: Duals of unanimity games as UR games

Next we characterize the class of upstream responsibility games. First, note that for any rooted tree G: $\mathcal{G}_G$ is a cone, that is for any $\alpha \geq 0$: $\alpha \mathcal{G}_G \subseteq \mathcal{G}_G$. Since union of cones is also a cone, both $\mathcal{G}_A^N$ and $\mathcal{G}_{UR}^N$ are cones.

Lemma 6.3 *For all rooted trees G: $\mathcal{G}_G$ is a cone, therefore $\mathcal{G}_A^N$ and $\mathcal{G}_{UR}^N$ are also cones.*

In the following lemma we show that the duals of unanimity games are airport and UR games.

Lemma 6.4 *For any chain G, $T \subseteq N$ such that $T = P_i(G)$, $i \in N$, $\bar{u}_T \in \mathcal{G}_G$ it holds that $\{\bar{u}_T\}_{T \in 2^N \setminus \{\emptyset\}} \subset \mathcal{G}_A^N \subset \mathcal{G}_{UR}^N$.*

Proof: For any $i \in N$ and $N = (N \setminus P_i(G)) \uplus P_i(G)$, let $c_1 = 0$ and $c_2 = 1$, that is, the cost of the members of coalition $N \setminus P_i(G)$ is 0, and the cost of the members of coalition $P_i(G)$ is 1 (see Definition 5.3). Then for the generated airport game $v_{(G,c)} = \bar{u}_{P_i(G)}$.

On the other hand, it is clear that there exists an airport game which is not the dual of any unanimity game. ■

Given player set $N = \{1, 2, 3\}$, let us now consider Figure 6.3 for the representation of dual unanimity games as airport and UR games, comparing it to the case of irrigation games.

It is important to note the relationship between the classes of airport and UR games, and the convex cone spanned by the duals of unanimity games.

Lemma 6.5 *For all rooted trees G: $\mathcal{G}_G \subset$ Cone $\{\bar{u}_{P_i(G)}\}_{i \in N}$. Therefore, $\mathcal{G}_A \subset \mathcal{G}_{UR}^N \subset$ Cone $\{\bar{u}_T\}_{T \in 2^N \setminus \{\emptyset\}}$.*

Proof: First we show that $\mathcal{G}_G \subset$ Cone $\{\bar{u}_{P_i(G)}\}_{i \in N}$.

Let $v \in \mathcal{G}_G$ be an upstream responsibility game. Since $G = (V, E)$ is a rooted tree, for all $i \in N$: $|\{j \in V : \overline{ji} \in E\}| = 1$, i.e. the player preceding player i is unique, let this player be denoted by $i_- = \{j \in V : \overline{ji} \in E\}$. Then for all $i \in N$ let $\alpha_{P_i(G)} = c_{\overline{i_- i}}$. Finally, it is easy to show that $v = \sum_{i \in N} \alpha_{P_i(G)} \bar{u}_{P_i(G)}$.

Second we show that Cone $\{\bar{u}_{P_i(G)}\}_{i \in N} \setminus \mathcal{G}_G \neq \emptyset$. Let $N = \{1, 2\}$, then $\sum_{T \in 2^N \setminus \{\emptyset\}} \bar{u}_T \notin \mathcal{G}_G$, i.e. $(2, 2, 3)$ is not an upstream responsibility game. Namely,

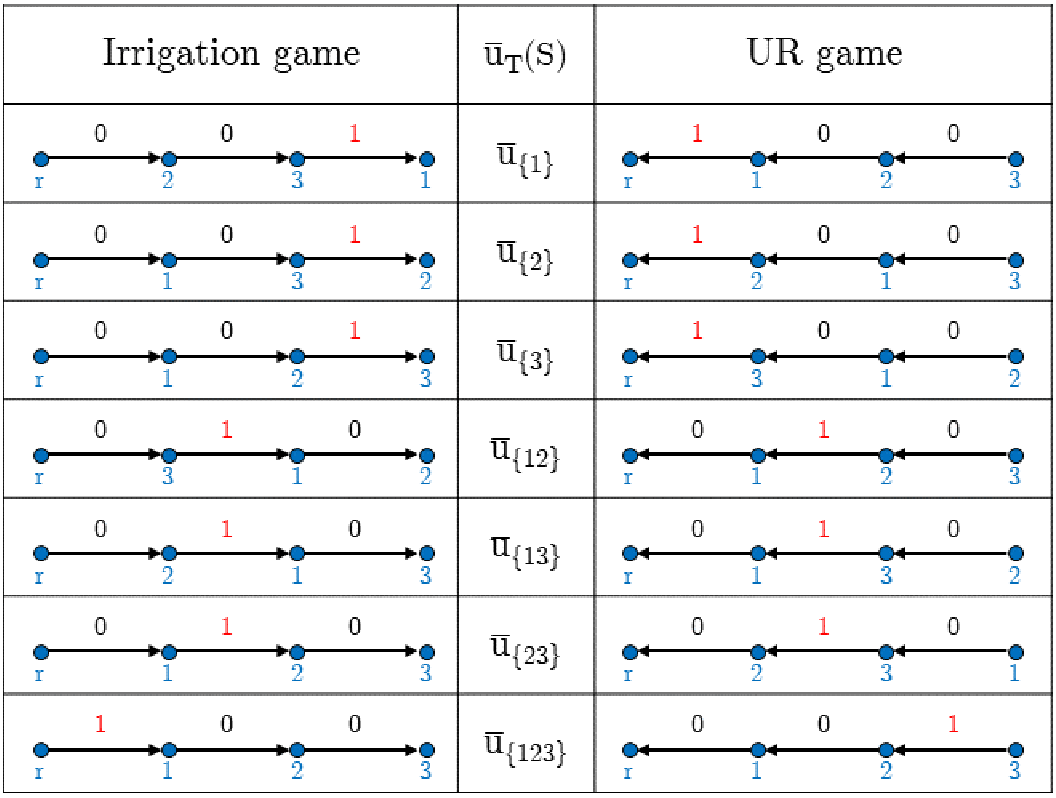

Figure 6.3: Duals of unanimity games as irrigation and UR games

in case of two players in two separate coalitions whose values are 2 each, we may get two different trees. If the tree consists of a single chain where the cost of the first edge is 0, and the cost of the second is 2, then the value of the grand coalition should be 2. Conversely, if there are two edges directed into the root with costs 2, then the value of the grand coalition should be 4. ■

Let us demonstrate the above results with the following example.

Example 6.6 *Let us consider the UR game in Example 6.2. Then* $P_1(G) = \{1\}$, $P_2(G) = \{1,2\}$, *and* $P_3(G) = \{1,3\}$. *Moreover,* $\alpha_{P_1(G)} = 2$, $\alpha_{P_2(G)} = 4$, *and* $\alpha_{P_3(G)} = 1$. *Finally, we get* $v_{(G,c)} = 2\bar{u}_{\{1\}} + 4\bar{u}_{\{1,2\}} + \bar{u}_{\{1,3\}} = \sum_{i \in N} \alpha_{P_i(G)} \bar{u}_{P_i(G)}$.

Next we discuss further corollaries of Lemmata 6.4 and 6.5. First we show that even if for any rooted tree G it holds that $\mathcal{G}_G$ is a convex set, the classes of airport games and upstream responsibility games are not convex.

Lemma 6.7 *Neither* $\mathcal{G}_A^N$ *nor* $\mathcal{G}_{UR}^N$ *is convex.*

Proof: Let $N = \{1,2\}$. Lemma 6.3 implies that $\{\bar{u}_T\}_{T \in 2^N \setminus \{\emptyset\}} \subseteq \mathcal{G}_A^N$, however, $\sum_{T \in 2^N \setminus \{\emptyset\}} \frac{1}{3} \bar{u}_T \notin \mathcal{G}_{UR}^N$, that is, game $(2/3, 2/3, 1)$ is not a UR game. ■

The following corollary has a key role in Young's axiomatization of the Shapley value on the classes of airport and upstream responsibility games. It is well-known that the duals of the unanimity games are linearly independent vectors. From Lemma 6.5 we know that for any rooted tree G and $v \in \mathcal{G}_G$: $v = \sum_{i\in N} \alpha_{P_i(G)}\bar{u}_{P_i(G)}$, where weights $\alpha_{P_i(G)}$ are well-defined, i.e. uniquely determined. The following lemma states that for any game $v \in \mathcal{G}_G$, if we decrease the weight of any of the basis vectors (the duals of the unanimity games) to 0, then we get a game corresponding to $\mathcal{G}_G$.

Lemma 6.8 *For any rooted tree G and $v = \sum_{i\in N} \alpha_{P_i(G)}\bar{u}_{P_i(G)} \in \mathcal{G}_G$, and for each $i^* \in N$ it holds that $\sum_{i\in N\setminus\{i^*\}} \alpha_{P_i(G)}\bar{u}_{P_i(G)} \in \mathcal{G}_G$. Therefore, for any airport game $v = \sum_{T\in 2^N\setminus\{\emptyset\}} \alpha_T\bar{u}_T$ and $T^* \in 2^N \setminus \{\emptyset\}$ it holds that $\sum_{T\in 2^N\setminus\{\emptyset,T^*\}} \alpha_T\bar{u}_T \in \mathcal{G}_A^N$, and for any upstream responsibility game $v = \sum_{T\in 2^N\setminus\{\emptyset\}} \alpha_T\bar{u}_T$ and $T^* \in 2^N \setminus \{\emptyset\}$ it holds that $\sum_{T\in 2^N\setminus\{\emptyset,T^*\}} \alpha_T\bar{u}_T \in \mathcal{G}_{UR}^N$.*

Proof: Let $v = \sum_{i\in N} \alpha_{P_i(G)}\bar{u}_{P_i(G)}$ and $i^* \in N$. Then let the cost function c' be defined as follows. For any $e \in E$ (see the proof of Lemma 6.5),

$$c'_e = \begin{cases} 0, & \text{if } e = \overline{i^*\_i^*}, \\ c_e & \text{otherwise.} \end{cases}$$

Then game $\sum_{i\in N\setminus\{i^*\}} \alpha_{P_i(G)}\bar{u}_{P_i(G)} = v_{(G,c')}$ i.e. $\sum_{i\in N\setminus\{i^*\}} \alpha_{P_i(G)}\bar{u}_{P_i(G)} \in \mathcal{G}_G$. ■
The following example is an illustration of the above result.

Example 6.9 *Let us consider the upstream responsibility game of Example 6.2, and player 2. Then*

$$c'_e = \begin{cases} 2, & \textit{if } e = \vec{1r}, \\ 0, & \textit{if } e = \vec{21}, \\ 1, & \textit{if } e = \vec{31}. \end{cases}$$

Furthermore, $\sum_{i\in N\setminus\{i^*\}} \alpha_{P_i(G)}\bar{u}_{P_i(G)} = 2\bar{u}_{\{1\}} + \bar{u}_{\{13\}} = v_{(G,c')}$ is an upstream responsibility game.

Finally, a simple observation.

Lemma 6.10 *All UR games are concave.*

Proof: Duals of unanimity games are concave, therefore the proposition is implied by Lemma 6.5. ∎

Let us summarize our results in the following.

Corollary 6.11 *For a fixed player set the class of airport games is a union of finitely many convex cones, but the class is not convex. Moreover, the class of airport games is a proper subset of the class of upstream responsibility games. The class of upstream responsibility games is also a union of finitely many convex cones, but is not convex either. Finally, the class of UR games is a proper subset of the finite convex cone spanned by the duals of the unanimity games, therefore every upstream responsibility game is concave, so every airport game is concave too.*

6.2 Solutions for UR games

In this section we present solution proposals for UR games. We are building on the definitions introduced in Chapter 5 regarding the Shapley value and its axiomatizations.

We consider two characterizations of the Shapley value on the class of upstream responsibility games. The first one is Shapley's original axiomatization (Shapley, 1953).

Theorem 6.12 *For all rooted trees G a value ψ is PO, NP, ETP and ADD on $\mathcal{G}_G$, if and only if $\psi = \phi$, i.e. the value is the Shapley value. In other words, value ψ on the class of UR games is PO, NP, ETP and ADD, if and only if $\psi = \phi$.*

Proof: $\Rightarrow$: We know that the Shapley value is PO, NP, ETP and ADD, see for example Peleg and Sudhölter (2007).

$\Leftarrow$: Based on Lemmata 6.3 and 6.4, ψ is defined on the cone spanned by $\{\bar{u}_{P_i(G)}\}_{i \in N}$.

Let us consider player $i^* \in N$. Then for all $\alpha \geq 0$ and players $i, j \in P_{i^*}(G)$: $i \sim^{\alpha \bar{u}_{P_{i^*}(G)}} j$, and for all players $i \notin P_{i^*}(G)$: $i \in NP(\alpha \bar{u}_{P_{i^*}(G)})$.

Then property NP implies that for all players $i \notin P_{i^*}(G)$: $\psi_i(\alpha \bar{u}_{P_{i^*}(G)}) = 0$. Moreover, according to axiom ETP, for all players $i, j \in P_{i^*}(G)$: $\psi_i(\alpha \bar{u}_{P_{i^*}(G)}) = \psi_j(\alpha \bar{u}_{P_{i^*}(G)})$. Finally, axiom PO implies that $\sum_{i \in N} \psi_i(\alpha \bar{u}_{P_{i^*}(G)}) = \alpha$.

Consequently, $\psi(\alpha \bar{u}_{P_{i^*}(G)})$ is well-defined (uniquely determined), and since the Shapley value is PO, NP and ETP, it follows that $\psi(\alpha \bar{u}_{P_{i^*}(G)}) = \phi(\alpha \bar{u}_{P_{i^*}(G)})$.

Furthermore, we know that games $\{u_T\}_{T \in 2^N \setminus \emptyset}$ give a basis of $\mathcal{G}^N$, and so do games $\{\bar{u}_T\}_{T \in 2^N \setminus \emptyset}$ as well.

Let $v \in \mathcal{G}_G$ be an upstream responsibility game. Then Lemma 6.5. implies that

$$v = \sum_{i \in N} \alpha_{P_i(G)} \bar{u}_{P_i(G)} ,$$

where for all $i \in N$: $\alpha_{P_i(G)} \geq 0$. Therefore, since the Shapley value is ADD, and for all $i \in N$, $\alpha_{P_i(G)} \geq 0$ it holds that $\psi(\alpha_{P_i(G)} \bar{u}_{P_i(G)}) = \phi(\alpha_{P_i(G)} \bar{u}_{P_i(G)})$, it follows that $\psi(v) = \phi(v)$.

Finally, Claim 5.18 can be applied. ■

Next we consider Young's axiomatization of the Shapley value. As a reminder, this was the first axiomatization of the Shapley value not involving the axiom of additivity.

Theorem 6.13 *For any rooted tree G, a single-valued solution ψ on $\mathcal{G}_G$ is PO, ETP and M if and only if $\psi = \phi$, i.e. it is the Shapley value. Therefore, a value ψ on the class of UR games is PO, ETP and M if and only if $\psi = \phi$.*

Proof: $\Rightarrow$: We know that the Shapley value meets axioms PO, ETP and M, see e.g. Peleg and Sudhölter (2007).

$\Leftarrow$: The proof is done by induction, similarly to that of Young's. For all UR games $v \in \mathcal{G}_G$ let $\mathcal{B}(v) = |\{\alpha_{P_i(G)}| > 0 \mid v = \sum_{i \in N} \alpha_{P_i(G)} \bar{u}_{P_i(G)}\}$. Clearly, $\mathcal{B}(\cdot)$ is well-defined.

If $\mathcal{B}(v) = 0$ (i.e. $v \equiv 0$), then axioms PO and ETP imply that $\psi(v) = \phi(v)$.

Let us assume that for all games $v \in \mathcal{G}_G$ for which $\mathcal{B}(v) \leq n$ it holds that $\psi(v) = \phi(v)$. Moreover, let $v = \sum_{i \in N} \alpha_{P_i(G)} \bar{u}_{P_i(G)} \in \mathcal{G}_G$ such that $\mathcal{B}(v) = n + 1$.

Then one of the following two points holds.

1. There exists player $i^* \in N$ such that there exists $i \in N$ that it holds that $\alpha_{P_i(G)} \neq 0$ and $i^* \notin P_i(G)$. Then Lemmata 6.5 and 6.8 imply that $\sum_{j \in N \setminus \{i\}} \alpha_{P_j(G)} \bar{u}_{P_j(G)} \in \mathcal{G}_G$, and for the marginal contributions it holds that

$$\left(\sum_{j \in N \setminus \{i\}} \alpha_{P_j(G)} \bar{u}_{P_j(G)} \right)'_{i^*} = v'_{i^*} \ ,$$

therefore, from axiom M

$$\psi_{i^*}(v) = \psi_{i^*} \left(\sum_{j \in N \setminus \{i\}} \alpha_{P_j(G)} \bar{u}_{P_j(G)} \right) \ ,$$

i.e. $\psi_{i^*}(v)$ is well-defined (uniquely determined) and due to the induction hypothesis, $\psi_{i^*}(v) = \phi_{i^*}(v)$.

2. There exist players $i^*, j^* \in N$ such that for all $i \in N$: $\alpha_{P_i(G)} \neq 0$ it holds that $i^*, j^* \in P_i(G)$. Then $i^* \sim^v j^*$, therefore, axiom ETP implies that $\psi_{i^*}(v) = \psi_{j^*}(v)$.

   Axiom PO implies that $\sum_{i \in N} \psi_i(v) = v(N)$. Therefore, $\psi(v)$ is well-defined (uniquely determined), consequently, since the Shapley value meets PO, ETP and M, due to the induction hypothesis, $\psi(v) = \phi(v)$.

Finally, Claim 5.18 can be applied. ■

Based on Lemma 6.10 and the well-known results of Shapley (1971) and Ichiishi (1981) we can formulate the following corollary.

Corollary **6.14** *For all UR games v it holds that $\phi(v) \in C(v)$ i.e. the Shapley value is in the core.*

The above corollary shows that on the considered class of games the Shapley value is stable.

Finally, as a direct corollary of our characterization of the class of upstream responsibility games, Theorem 6.5, we get the following result.

Corollary **6.15** *The Shapley value on the class of the upstream responsibility games can be calculated in polynomial time.*

Proof: Let us consider an arbitrary UR game, $v \in \mathcal{G}_{UR}^N$. Then

$$v = \sum_{i \in N} \alpha_{P_i(G)} \bar{u}_{P_i(G)}.$$

Moreover,

$$\phi_j(\bar{u}_{P_i(G)}) = \begin{cases} \frac{1}{|P_i(G)|}, & \text{if } j \in P_i(G), \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, since $\alpha_{P_i(G)} = c_{\overline{i_- i}}$, for all $j \in N$ we get

$$\phi_j(v) = \sum_{j \in P_i(G)} \frac{c_{\overline{i_- i}}}{|P_i(G)|}.$$

■

The result is analogous to that of Megiddo (1978)'s, who proved that for fixed tree games the Shapley value can be calculated in $O(n)$ time.

Chapter 7

Shortest path games

In this chapter we consider the class of *shortest path games*. There are given some agents, a good, and a network. The agents own the nodes of the network and they want to transport the good from certain nodes of the network to others. The transportation cost depends on the chosen path within the network, and the successful transportation of a good generates profit. The problem is not only choosing the shortest path (a path with minimum cost, i.e. with maximum profit), but we also have to divide the profit among the players.

Fragnelli, García-Jurado and Méndez-Naya (2000) introduce the notion of shortest path games and they prove that the class of such games coincides with the well-known class of monotone games.

In this chapter we consider further characterizations of the Shapley value: Shapley (1953)'s, Young (1985b)'s, Chun (1989)'s, and van den Brink (2001)'s axiomatizations. We analyze their validity on the class of shortest path games, and conclude that all aforementioned axiomatizations are valid on the class.

Our results are different from Fragnelli et al. (2000) in two aspects. Firstly, Fragnelli et al. (2000) give a new axiomatization of the Shapley value, conversely, we consider four well-known characterizations. Secondly, Fragnelli et al. (2000)'s axioms are based on the graph behind the problem, in this chapter we consider game-theoretic axioms only. Namely, while Fragnelli et al. (2000) consider a fixed-graph problem, we consider all shortest path problems, and examine them from the viewpoint of an abstract decision maker who focuses rather on the abstract problem, instead of the concrete situations.

The following results have been published in Pintér and Radványi (2013).

7.1 Notions and notations

As a reminder, let $v \in \mathcal{G}^N$, $i \in N$, and for all $S \subseteq N$ let $v_i'(S) = v(S \cup \{i\}) - v(S)$ the marginal contribution of player i to coalition S in game v. Furthermore, $i, j \in N$ are equivalent in game v, i.e. $i \sim^v j$ if for all $S \subseteq N \setminus \{i, j\}$: $v_i'(S) = v_j'(S)$. Function ψ is a value on set $A \subseteq \Gamma^N = \bigcup_{T \subseteq N,\ T \neq \emptyset} \mathcal{G}^T$, if $\forall T \subseteq N$, $T \neq \emptyset$ it holds that $\psi|_{\mathcal{G}^T \cap A} : \mathcal{G}^T \cap A \to \mathbb{R}^T$. In the following definition we present the axioms used for the characterization of a value (some of them have been introduced before, we provide the entire list as a reminder).

Definition 7.1 *On set $A \subseteq \mathcal{G}^N$ value ψ is / satisfies*

- *Pareto optimal, or efficient (PO), if for all players $v \in A$: $\sum_{i \in N} \psi_i(v) = v(N)$,*
- *null-player property (NP), if for all games $v \in A$, and players $i \in N$: $v_i' = 0$ implies $\psi_i(v) = 0$,*
- *equal treatment property (ETP), if for all games $v \in A$, and players $i, j \in N$: $i \sim^v j$ implies $\psi_i(v) = \psi_j(v)$,*
- *additive (ADD), if for all games $v, w \in A$ such that $v + w \in A$: $\psi(v + w) = \psi(v) + \psi(w)$,*
- *fairness property (FP), if for all games $v, w \in A$ and players $i, j \in N$ for which $v + w \in A$ and $i \sim^w j$: $\psi_i(v + w) - \psi_i(v) = \psi_j(v + w) - \psi_j(v)$,*
- *marginal (M), if for all games $v, w \in Az$ and player $i \in N$: $v_i' = w_i'$ implies $\psi_i(v) = \psi_i(w)$.*
- *coalitional strategic equivalence (CSE), if for all games $v \in A$, player $i \in N$, coalition $T \subseteq N$, and $\alpha > 0$: $i \notin T$ and $v + \alpha u_T \in A$ implies $\psi_i(v) = \psi_i(v + \alpha u_T)$.*

For a brief interpretation of the above axioms let us consider the following situation. There is given a network of towns and a set of companies. Let only one company be based in each town, in this case we say that the company *owns* the city. There is given a good (e.g. a raw material or a finished product) that some of the towns are producing (called *sources*) and some other towns are consuming (called *sinks*). Henceforth, we refer to a series of towns as *path*, and we say a path is *owned* by a group of companies if and only if all towns of the path are owned by one of these companies. A group of companies is able to transport the good from a source to a sink if there exists a path connecting the source to the sink which is owned by the same group of companies. The delivery of the good from source to sink results in a fixed value benefit and a cost depending on the chosen transportation path. The goal is the transportation of the good through a path which provides the maximal profit, provided that the path has minimal cost.

Given the interpretation above, let us consider the axioms introduced earlier. The axiom PO requires that the entire profit stemming from the transportation is distributed among the companies, players. In our example Axiom NP states that if a company plays no role in the delivery, then the company's share of the profit must be zero.

Property ETP is responsible for that if two companies are equivalent with respect to the transportation profit of the good, then their shares from the total profit must be equal.

A solution meets axiom ADD if for any two games the results are equal if we add up the games first and evaluate the players later, or if we evaluate the players first and add up their evaluations later. Let us modify our example so that we consider the same network in two consecutive years. In this case ADD requires that if we evaluate the profit of a company for these two years, then the share must be equal to the sum of the shares of the company in the two years separately.

For describing axiom FP, let us assume that we add up two games such that in the second game two players are equivalent. The axiom requires that after summing up the two games, the difference between the players' summed evaluations and their evaluations in the first game must be equal. Let us consider this axiom in the context of our example, and let us consider the profit of a group of companies in two consecutive years and assume that there exists two companies in the group with equal profit changes in the second year. Then the summed profit change after the second year compared to the first year must be equal for the two companies. It is worth noting that the origin of this axiom goes back to Myerson (1977).

Axiom M requires that if a given player in two games produces the same marginal contributions then that player must be evaluated equally in those games. Therefore, in our example if we consider profits for two consecutive years and there is given a company achieving the same profit change from the transportation (e.g. it raises the profit with the same amount) in both years, then the shares in the profit of the company must be equal in the two years.

CSE can be interpreted as follows. Let us assume that some companies (coalition T) are together responsible for the change (increase) in the profit of the transportation. Then a CSE solution evaluates the companies such that the shares of the companies do not change if they are not responsible for the profit increase.

It is worth noting that Chun (1989)'s original definition of CSE is different from ours. CSE was defined as "ψ satisfies coalitional strategic equivalence (CSE), if for each $v \in A$, $i \in N$, $T \subseteq N$, $\alpha \in \mathbb{R}$: $i \notin T$ and $v + \alpha u_T \in A$ imply $\psi_i(v) = \psi_i(v + \alpha u_T)$." However if for some $\alpha < 0$: $v + \alpha u_T \in A$ then by $w = v + \alpha u_T$ we get "$i \notin T$ and $w + \beta u_T \in A$ imply $\psi_i(w) = \psi_i(w + \beta u_T)$", where $\beta = -\alpha > 0$. Therefore the two CSE definitions are equivalent.

The following lemma formulates some well-known relations among the above axioms.

Lemma 7.2 *Let us consider the following points.*

1. *If value ψ is ETP and ADD, then it is FP.*

2. *If value ψ is M, then it is CSE.*

Proof: It is easy to prove, see point 1. in van den Brink, 2001, Claim 2.3. point (i) on pp. 311. ■

Finally, we present a well-known result that we will use later.

Claim 7.3 *The Shapley value is PO, NP, ETP, ADD, FP, M, and CSE.*

7.2 Introduction to shortest path games

Shortest path problems describe situations where the nodes of the network are owned by agents whose goal is to transport a good from source nodes to sink nodes at a minimum cost (on the shortest path). The minimal cost (or the generated profit) must be allocated among agents that contributed to the transportation. The games corresponding to this situation are called shortest path games. In our definition of this game class we rely on Fragnelli et al. (2000).

Definition 7.4 *A shortest path problem Σ is a tuple (V, A, L, S, T), where*

- *(V, A) is an acyclic digraph, where V is a finite set of nodes. In this section, as opposed to previous ones, elements of V will be denoted by x_i for clarity, since v will denote the corresponding TU game. Therefore, $V = \{x_i : i = 1, \dots, |V|\}$. The set of nodes A is a subset of $V \times V$ such that every $a = (x_1, x_2) \in A$ satisfies that $x_1 \neq x_2$. For each $a = (x_1, x_2) \in A$ we say that x_1 and x_2 are the endpoints of a.*

- *L is a map assigning to each edge $a \in A$ a non-negative real number $L(a)$. $L(a)$ represents the length of a.*

- *S and T are non-empty and disjoint subsets of V. The elements of S and T are called sources and sinks, respectively.*

A path P in Σ connecting two nodes x_0 and x_p is a collection of nodes (with no recurrent nodes) $\{x_0, \dots, x_p\}$ where $(x_{i-1}, x_i) \in A$, $i = 1, \dots, p$. $L(P)$, the *length of path* P is the sum $\sum_{i=1}^{p} L(x_{i-1}, x_i)$. We note that by *path* we mean a *path connecting a source to a sink*. A path P is a *shortest path* if there exists no other path P' with $L(P') < L(P)$. In a shortest path problem we search for such shortest paths.

Let us now introduce the related cooperative game. There is given a shortest path problem Σ, where nodes are owned by a finite set of players N according to a map $o : 2^V \to 2^N$, where $o(\{x\}) = \{i\}$ denotes that player i is the owner of node x. For each path P: $o(P)$ gives the set of players who own the nodes of P. We assume that the transportation of a good from a source to a sink produces an income g, and the cost of the transportation is determined by the length of the used path. A path P is owned by a coalition $S \subseteq N$, if $o(P) \subseteq S$, and we assume that a coalition S can only transport a good through own paths.

Definition 7.5 *A* shortest path cooperative situation σ *is a tuple* (Σ, N, o, g). *We can identify* σ *with the corresponding cooperative TU game* v_σ *given by, for each* $S \subseteq N$*:*

$$v_\sigma(S) = \begin{cases} g - L_S, & \text{if } S \text{ owns a path in } \Sigma \text{ and } L_S < g, \\ 0 & \text{otherwise,} \end{cases}$$

where L_S *is the length of the shortest path owned by* S*.*

Definition 7.6 *A* shortest path game v_σ *is a game associated with a shortest path cooperative situation* σ*. Let* SPG *denote the class of shortest path games.*

Let us consider the following example.

Example 7.7 *Let* $N = \{1, 2\}$ *be the set of players, the graph in Figure 7.1 represents the shortest path cooperative situation,* s_1*,* s_2 *are the sources,* t_1*,* t_2 *are the sink nodes. The numbers on the edges identify their costs or lengths, and* $g = 7$*. Player* 1 *owns nodes* s_1*,* x_1*, and* t_1*, Player* 2 *owns nodes* s_2*,* x_2*, and* t_2*. Table 7.1 gives the induced shortest path game.*

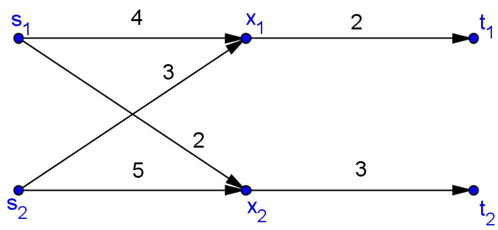

Figure 7.1: Graph of shortest path cooperative situation in Example 7.7

Let us consider Fragnelli et al. (2000)'s result on the relationship between the classes of shortest path games and monotone games.

Definition 7.8 *A game* $v \in \mathcal{G}^N$ *is monotone, if* $\forall S, T \in N$*, and* $S \subseteq T$ *it holds that* $v(S) \leq v(T)$*.*

Theorem 7.9 (Fragnelli et al., 2000) $SPG = MO$*, where* MO *denotes the class of monotone games. In other words, the class of shortest path games is equal to the class of monotone games.*

S	Shortest path owned by S	$L(S)$	$v(S)$
$\{1\}$	$\{s_1, x_1, t_1\}$	6	1
$\{2\}$	$\{s_2, x_2, t_2\}$	8	0
$\{1, 2\}$	$\{s_1, x_2, t_2\} \sim \{s_2, x_1, t_1\}$	5	2

Table 7.1: Shortest path game induced by Example 7.7

7.3 Characterization results

In this section we organize our results into thematic subsections.

7.3.1 The potential

In this subsection we characterize the Shapley value on the class of monotone games leveraging the potential (Hart and Mas-Colell, 1989).

Definition 7.10 *Let there be given a $v \in \mathcal{G}^N$ and $T \subseteq N$: $T \neq \emptyset$. Then subgame $v^T \in \mathcal{G}^T$ corresponding to coalition T in game v can be defined as follows: for all $S \subseteq T$ let $v^T(S) = v(S)$.*

Clearly, defining v^T is only required on subsets of T.

Definition 7.11 *Let $A \subseteq \Gamma^N$, and $P : A \to \mathbb{R}$ a mapping. For all $v \in \mathcal{G}^T \cap A$ and player $i \in T$, where $|T| = 1$ or $v^{T \setminus \{i\}} \in A$*

$$P'_i(v) = \begin{cases} P(v), & \text{if } |T| = 1, \\ P(v) - P(v^{T \setminus \{i\}}) & \text{otherwise.} \end{cases} \tag{7.1}$$

Furthermore, if for all games $v \in \mathcal{G}^T \cap A$ such that $|T| = 1$ or for all players $i \in T$: $v^{T \setminus \{i\}} \in A$

$$\sum_{i \in T} P'_i(v) = v(T) \ ,$$

then P is the potential *on set A.*

Definition 7.12 *A set $A \subseteq \Gamma^N$ is* subgame-closed *if for all coalitions $T \subseteq N$ for which $|T| > 1$, game $v \in \mathcal{G}^T \cap A$ and player $i \in T$: $v^{T \setminus \{i\}} \in A$.*

The concept of subgame is meaningful only if the original game has at least two players. Therefore, in the above definition we require that for each player i: $v^{T\setminus\{i\}}$ be in the set under consideration only if there are at least two players in T.

Theorem 7.13 *Let $A \subseteq \Gamma^N$ be a subgame-closed set of games. Then function P on A is a potential, if and only if for each game $v \in \mathcal{G}^T \cap A$ and player $i \in T$ it holds that $P_i'(v) = \phi_i(v)$.*

Proof: See e.g. Theorem 8.4.4. on pp. 162 in Peleg and Sudhölter (2007). ∎

In the following we focus on the class of monotone games.

Corollary 7.14 *A function P is a potential on the class of monotone games, if and only if for all monotone games $v \in \mathcal{G}^T$ and player $i \in T$ it holds that $P_i'(v) = \phi_i(v)$, i.e. P_i' is the Shapley value ($i \in N$).*

Proof: It is easy to verify that the class of monotone games is a subgame-closed set of games. Therefore we can apply Theorem 7.13. ∎

7.3.2 Shapley's characterization

In this subsection we consider Shapley (1953)'s original characterization. The next theorem fits into the sequence of increasingly enhanced results of Shapley (1953), Dubey (1982), Peleg and Sudhölter (2007).

Theorem 7.15 *Let there be given $A \subseteq \mathcal{G}^N$ such that the* Cone $\{u_T\}_{T\subseteq N,\ T\neq\emptyset} \subseteq A$*. Then value ψ on A is PO, NP, ETP and ADD, if and only if $\psi = \phi$.*

Proof: $\Rightarrow$: See Claim 7.3.

$\Leftarrow$: Let $v \in A$ be a game and ψ a value on A, that is PO, NP, ETP and ADD. If $v = 0$, then NP implies that $\psi(v) = \phi(v)$, therefore without the loss of generality we can assume that $v \neq 0$.

We know that there exist weights $\{\alpha_T\}_{T\subseteq N,\ T\neq\emptyset} \subseteq \mathbb{R}$ such that

$$v = \sum_{T\subseteq N,\ T\neq\emptyset} \alpha_T u_T \ .$$

Let $Neg = \{T : \alpha_T < 0\}$. Then

$$\left(-\sum_{T\in Neg} \alpha_T u_T\right) \in A$$

and

$$\left(\sum_{T\in 2^N\setminus(Neg\cup\{\emptyset\})} \alpha_T u_T\right) \in A \ .$$

Furthermore,

$$v + \left(-\sum_{T\in Neg} \alpha_T u_T\right) = \sum_{T\in 2^N\setminus(Neg\cup\{\emptyset\})} \alpha_T u_T \ .$$

Since for all unanimity games u_T, and $\alpha \geq 0$ axioms PO, NP and ETP imply that $\psi(\alpha u_T) = \phi(\alpha u_T)$, and since axiom ADD holds:

$$\psi\left(-\sum_{T\in Neg} \alpha_T u_T\right) = \phi\left(-\sum_{T\in Neg} \alpha_T v_T\right)$$

and

$$\psi\left(\sum_{T\in 2^N\setminus(Neg\cup\{\emptyset\})} \alpha_T u_T\right) = \phi\left(\sum_{T\in 2^N\setminus(Neg\cup\{\emptyset\})} \alpha_T u_T\right) \ .$$

Then from Claim 7.3, and axiom ADD imply that $\psi(v) = \phi(v)$. ■

Based on Theorem 7.15 we can conclude the following for the class of monotone games.

Corollary **7.16** *A value ψ is PO, NP, ETP and ADD on the class of monotone games, if and only if $\psi = \phi$, i.e. it is the Shapley value.*

Proof: The class of monotone games comprises the convex cone $\{u_T\}_{T\subseteq N,\ T\neq\emptyset}$ spanned by unanimity games, therefore we can apply Theorem 7.15. ■

7.3.3 The van den Brink axiomatization

In this subsection we discuss van den Brink (2001)'s characterization of the Shapley value on the class of monotone games.

The following lemma extends Claim 2.4 of van den Brink (2001) (point (ii) on pp. 311) in the sense that it is sufficient to consider the value on a subset that comprises the 0-game.

Lemma **7.17** *Let there be given $A \subseteq \mathcal{G}^N$ such that $0 \in A$, and the value ψ on A, such that axioms NP and FP are satisfied. Then ψ satisfies axiom ETP.*

Proof: Let $v \in A$ be such that $i \sim^v j$, and $w = 0$, then axiom NP implies that $\psi(0) = 0$. Since ψ meets FP,

$$\psi_i(v + w) - \psi_i(w) = \psi_j(v + w) - \psi_j(w) \ ,$$

consequently, $\psi_i(v + w) = \psi_j(v + w)$. Applying FP again we get

$$\psi_i(v + w) - \psi_i(v) = \psi_j(v + w) - \psi_j(v) \ .$$

Then equation $\psi_i(v + w) = \psi_j(v + w)$ implies that $\psi_i(v) = \psi_j(v)$. ■
The following claim is the key result of this subsection.

Claim **7.18** *Let ψ be a value on the convex cone spanned by unanimity games, i.e. the set* Cone $\{u_T\}_{T \subseteq N,\ T \neq \emptyset}$ *such that PO, NP and FP are all satisfied. Then ψ meets ADD, as well.*

Proof: Firstly, we show that ψ is well-defined on the set Cone $\{u_T\}_{T \subseteq N,\ T \neq \emptyset}$.

Let $v \in$ Cone $\{u_T\}_{T \subseteq N,\ T \neq \emptyset}$ be a monotone game, i.e. $v = \sum_{T \subseteq N,\ T \neq \emptyset} \alpha_T u_T$, and let $I(v) = \{T \mid \alpha_T > 0\}$. The proof is done using induction on $|I(v)|$.

$|I(v)| \leq 1$: Based on axiom NP and Lemma 7.17, $\psi(v)$ is uniquely determined.

Let us assume for $1 \leq k < |I(v)|$ that for all $A \subseteq I(v)$ such that $|A| \leq k$ the value $\psi(\sum_{T \in A} \alpha_T u_T)$ is well-defined. Let $C \subseteq I(v)$ such that $|C| = k + 1$, and $z = \sum_{T \in C} \alpha_T u_T$.

Case 1: There exist $u_T, u_S \in C$ such that there exist $i^*, j^* \in N$, such that $i^* \sim^{u_T} j^*$, but $i^* \nsim^{u_S} j^*$. In this case, based on FP and $z - \alpha_T u_T, z - \alpha_S u_S$ $\in$ Cone $\{u_T\}_{T \subseteq N,\ T \neq \emptyset}$ it holds that for all players $i \in N \backslash \{i^*\}$ such that $i \sim^{\alpha_S u_S} i^*$:

$$\psi_{i^*}(z) - \psi_{i^*}(z - \alpha_S u_S) = \psi_i(z) - \psi_i(z - \alpha_S u_S) \ , \qquad (7.2)$$

and for all players $j \in N \setminus \{j^*\}$ such that $j \sim^{\alpha_S u_S} j^*$:

$$\psi_{j^*}(z) - \psi_{j^*}(z - \alpha_S u_S) = \psi_j(z) - \psi_j(z - \alpha_S u_S) \ , \tag{7.3}$$

and

$$\psi_{i^*}(z) - \psi_{i^*}(z - \alpha_T u_T) = \psi_{j^*}(z) - \psi_{j^*}(z - \alpha_T u_T) \ . \tag{7.4}$$

Furthermore, PO implies that

$$\sum_{i \in N} \psi_i(z) = z(N) \ . \tag{7.5}$$

Assuming the induction hypothesis holds, the linear system of equations (7.2), (7.3), (7.4), (7.5), consisting of $|N|$ variables ($\psi_i(z)$, $i \in N$), and $|N|$ equations, has a unique solution. Therefore $\psi(z)$ is well-defined.

Case 2: $z = \alpha_T u_T + \alpha_S u_S$ such that $S = N \setminus T$. Then $z = m(u_T + u_S) + (\alpha_T - m)u_T + (\alpha_S - m)u_T$, where $m = \min\{\alpha_T, \alpha_S\}$. Without the loss of generality we can assume that $m = \alpha_T$. Since $i \sim^{m(u_T + u_S)} j$, $\psi((\alpha_S - m)u_S)$ is well-defined (based on the induction hypothesis), and PO implies that $\psi(z)$ is well-defined as well.

To summarize, ψ is well-defined on the set Cone $\{u_T\}_{T \subseteq N,\ T \neq \emptyset}$. Then from Claim 7.3 it follows that ψ is ADD on the set Cone $\{u_T\}_{T \subseteq N,\ T \neq \emptyset}$. ■

The following theorem is the main result of this subsection, extending van den Brink (2001)'s Theorem 2.5 (pp. 311–315) for the case of monotone games.

Theorem 7.19 *A value ψ is PO, NP and FP on the class of monotone games, if and only if $\psi = \phi$, i.e. it is the Shapley value.*

Proof: $\Rightarrow$: See Claim 7.3.

$\Leftarrow$: Theorem 7.15 and Claim 7.18 imply that on the set Cone $\{u_T\}_{T \subseteq N,\ T \neq \emptyset}$ it holds that $\psi = \phi$. Let $v = \sum_{T \subseteq N,\ T \neq \emptyset} \alpha_T u_T$ be a monotone game, and $w = (\alpha + 1) \sum_{T \subseteq N,\ T \neq \emptyset} u_T$, where $\alpha = \max\{-\min_T \alpha_T, 0\}$.

Then $v + w \in$ Cone $\{u_T\}_{T \subseteq N,\ T \neq \emptyset}$, for all players $i, j \in N$, $i \sim^w j$, therefore from axioms PO and FP it follows that $\psi(v)$ is well-defined. Finally, we can apply Claim 7.3. ■

7.3.4 Chun's and Young's approaches

In this subsection Chun (1989)'s and Young (1985b)'s approaches are discussed. In the case of Young's axiomatization we only refer to an external result, in the case of Chun (1989)'s axiomatization we connect it to Young's characterization. The following result is from the paper of Pintér (2015).

Claim 7.20 *A value ψ is PO, ETP and M on the class of monotone games, if and only if $\psi = \phi$, i.e. it is the Shapley value.*

In the game theory literature there is confusion about the relationship between Chun's and Young's characterizations. According to van den Brink (2007), CSE is equivalent to M. However, this argument is not true, e.g. on the class of assignment games this does not hold (see Pintér (2014), pp. 92, Example 13.26). Unfortunately, the class of monotone games does not bring to surface the difference between axioms M and CSE. The next lemma formulates this statement.

Lemma 7.21 *On the class of monotone games M and CSE are equivalent.*

Proof: $CSE \Rightarrow M$: Let v, w be monotone games, and let player $i \in N$ be such that $v'_i = w'_i$. It can be easily proven that $(v-w)'_i = 0$, $v-w = \sum_{T\subseteq N,\ T\neq\emptyset} \alpha_T u_T$, and for all $T \subseteq N$, $T \neq \emptyset$, if $i \in T$, then $\alpha_T = 0$. Therefore, $v = w + \sum_{T\subseteq N\setminus\{i\}, T\neq\emptyset} \alpha_T u_T$.

Let $T^+ = \{T \subseteq N \mid \alpha_T > 0\}$. Then for all monotone games z, $\alpha > 0$, and unanimity game u_T: $z + \alpha u_T$ is a monotone game, therefore, we get that $w + \sum_{T\in T^+} \alpha_T u_T$ is a monotone game as well, and $w'_i = (w + \sum_{T\in T^+} \alpha_T u_T)'_i$. Axiom CSE implies that $\psi_i(w) = \psi_i(w + \sum_{T\in T^+} \alpha_T u_T)$.

Furthermore, since all z are monotone games, for $\alpha > 0$, and unanimity game u_T, $z + \alpha u_T$ is a monotone game as well, we get that $v + \sum_{T\notin T^+} -\alpha_T u_T$ is also a monotone game, and $v'_i = (v + \sum_{T\notin T^+} -\alpha_T u_T)'_i$. Moreover, CSE implies that $\psi_i(v) = \psi_i(v + \sum_{T\notin T^+} -\alpha_T u_T)$.

Then $w + \sum_{T\in T^+} \alpha_T u_T = v + \sum_{T\notin T^+} -\alpha_T u_T$, therefore

$$\psi_i(w) = \psi_i\left(w + \sum_{T\in T^+} \alpha_T u_T\right) = \psi_i\left(v + \sum_{T\notin T^+} -\alpha_T u_T\right) = \psi_i(v)\ .$$

$M \Rightarrow CSE$: See Lemma 7.2. ■

Based on the above we can formulate the following corollary.

Corollary **7.22** *A value* ψ *is PO, ETP and CSE on the class of monotone games, if and only if* $\psi = \phi$*, i.e. it is the Shapley value.*

Chapter 8

Conclusion

In our thesis we examined economic situations modeled with rooted trees and directed, acyclic graphs. In the presented problems the collaboration of economic agents (players) incurred costs or created a profit, and we have sought answers to the question of "fairly" distributing this common cost or profit. We have formulated properties and axioms describing our expectations of a "fair" allocation. We have utilized cooperative game theoretical methods for modeling.

After the introduction, in Chapter 2 we analyzed a real-life problem and its possible solutions. These solution proposals, namely the average cost-sharing rule, the serial cost sharing rule, and the restricted average cost-sharing rule have been introduced by Aadland and Kolpin (2004). We have also presented two further water management problems that arose during the planning of the economic development of Tennessee Valley, and discussed solution proposals for them as well (Straffin and Heaney, 1981). We analyzed if these allocations satisfied the properties we associated with the notion of "fairness".

In Chapter 3 we introduced the fundamental notions and concepts of cooperative game theory. We defined the core (Shapley, 1955; Gillies, 1959) and the Shapley value (Shapley, 1953), that play an important role in finding a "fair" allocation.

In Chapter 4 we presented the class of fixed-tree game and relevant applications from the domain of water management.

In Chapter 5 we discussed the classes of airport and irrigation games, and the characterizations of these classes. We extended the results of Dubey (1982) and Moulin and Shenker (1992) on axiomatization of the Shapley value on the class of airport games to the class of irrigation games. We have "translated" the axioms used in cost allocation literature to the axioms corresponding to TU games, thereby providing two new versions of the results of Shapley (1953) and Young (1985b).

In Chapter 6 we introduced the upstream responsibility games and characterized the game class. We have shown that Shapley's and Young's characterizations are valid on this class as well.

In Chapter 7 we discussed shortest path games and have shown that this game class is equal to the class of monotone games. We have shown that further axiomatizations of the Shapley value, namely Shapley (1953)'s, Young (1985b)'s, Chun (1989)'s, and van den Brink (2001)'s characterizations are valid on the class of shortest path games.

The thesis presents multiple topics open for further research, in the following we highlight some of them. Many are new topics and questions that are currently conjectures and interesting ideas, but for some we have already got partial results. These results are not yet mature enough to be parts of this thesis, but we still find it useful to show the possible directions of our future research.

The restricted average cost-share rule presented in Chapter 2 may be a subject to further research. Theorem 2.21 proves the rule's existence and that it is unique, but the recursive construction used in the proof is complex for large networks, since the number of rooted subtrees that must be considered is growing exponentially with the number of nodes. One conjecture that could be a subject for further research is that during the recursion it is sufficient to consider subtrees whose complements are a single branch, since their number is equal to the number of agents. Therefore, the restricted average cost sharing rule on rooted trees could be computed in polynomial time. Another interesting question is whether the axiomatization of Aadland and Kolpin (1998) (Theorem 2.25) can be extended to tree structures with an appropriate extension of the axiom of reciprocity.

In connection with UR games presented in Chapter 6 it is an open question what relationship exists between the axioms related to pollution discussed by Gopalakrishnan et al. (2017) and the classic axioms of game theory, and what can be said in this context about the known characterizations of the Shapley value. We have published existing related results in Pintér and Radványi (2019a). In one of the possible generalizations of UR games the game does not depend on the tree graph, only on the responsibility matrix, which describes the edges a player is responsible for. Then the structure of the graph is not limited to the case of trees. We have studied the characterization of such an extension of UR games and published our results in Pintér and Radványi (2019b), presenting characterizations on unanimity games and their duals as well. Based on these results we have also studied further axiomatization possibilities of the Shapley value.

Another relevant area for future research is the relationship between the axioms of the networks of shortest path games (Chapter 7) and the axioms of game theory. It is also an open question, which characterizations of the Shapley value presented in the thesis remain valid if the base network is fixed, and only the weights of the edges are altered. This could shed light on the connection between already known results and Fragnelli et al. (2000)'s characterization using properties of networks.

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