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PhD Thesis Summary

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Stable sets in assignment games

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1 Background and overview of the research

Assignment games (Shapley and Shubik, 1972) are models of two-sided matching markets with transferable utilities where the aim of each player on one side is to form a profitable coalition with a player on the other side. Since only such bilateral cooperations are worthy, these games are completely defined by the matrix containing the cooperative worths of all possible pairings of players from the two sides.

Shapley and Shubik (1972) showed that the core of an assignment game is precisely the set of dual optimal solutions to the assignment optimization problem on the underlying matrix of mixed-pair profits. This implies that

1. every assignment game has a non-empty core;
2. the core can be determined without explicitly generating the entire coalitional function of the game; and
3. there are two special vertices of the core, in each of which every player from one side of the market receives her highest core-payoff while every player from the other side of the market receives her lowest core-payoff.

Besides the above fundamental results concerning the core, several important contributions dealing with other solution concepts have been published in the last decade. The classical solution concept proposed and studied by Von Neumann and Morgenstern (1944) in their monumental work has remained an intriguing exception, although Solymosi and Raghavan (2001) characterized a subclass of assignment games where the core is the unique stable set. This subclass is the assignment games which are generated from a matrix with a dominant diagonal. The existence question in the general case was settled affirmatively by Núñez and Rafels (2013), who proved that, as conjectured by Shapley (cf. Section 8.4 in Shubik (1984)), the union of the cores of certain derived subgames is always a stable set. They showed this set is the unique stable set in the principal section which contains the imputations in which every pair from the maximal value matching gets exactly their value.

In special cases we know much more then the existence of stable sets. Shapley (1959) considered the symmetric market game (glove market). He showed some nice properties of the stable sets, for example every stable set is a monotonic curve end in one endpoint of this curve every buyer gets zero payoff in the other endpoint every seller gets zero payoff.

In the fourth section we consider Harsanyi's criticism of stable sets. He argued that the definition of von Neumann–Morgenstern stable set is not good because it neglects the effect of indirect dominance. A von Neumann–Morgenstern stable set is considered stable because when a coalition forces to go out from the stable set there is always an other coalition which can go back to the stable set and that is why every deviation from the set is temporary so there is no reason to deviate from the set. Harsanyi argued it is not always true because it can happen the players in the first dominating coalition get strictly more in the payoff vector where the procedure arrived than in the original one. In this case this coalition will dominate. He proposed an alternative definition of stable set which based on a bargaining game. The stationary points (the payoff vector where no coalition want to dominate and the bargaining process will stop) of a strong Nash equilibrium should be called stable set instead of the original definition. He defined a class of games (the absolutely stable games) in which the indirect dominance is irrelevant. In this class it is obvious that the „original" definition of stable sets and the proposed new one are defines the same sets. It is easy to show that the assignment games are not absolutely stable.

In the fifth section we consider a generalization of the assignment games, the multi-sided assignment games. The definition of multi-sided assignment games and „normal" assignment games are very similar but the properties are very different. For example the core of an assignment game is non-empty Shapley and Shubik (1972), but the core of a multi-sided assignment game can be empty (Kaneko, 1982). In this class of games we don't know too much about stable sets. The existence is still an open question. The only result we know is about the stability of the core in a very special case. It is obvious that the generating matrix has a dominant diagonal is a necessary condition of the core-stability like in the normal assignment game case (Solymosi and Raghavan, 2001). But the sufficiency was known only in the smallest „real multi-sided" case, in the three-sided assignment games with two players in each side (Atay and Núñez, 2019).

2 Results of the research

Before we show the results of the thesis we have to introduce some notions and definitions.

A transferable utility cooperative game on the nonempty finite *set P of players* is defined by a *coalitional function* $w : 2^P \to \mathbb{R}$ satisfying $w(\varnothing) = 0$. The function w specifies the worth of every *coalition* $S \subseteq P$. For a payoff vector $\boldsymbol{x} \in \mathbb{R}^P$ we denote the sum of the coordinates in $S \subseteq P$ by $x(S)$

Given a game (P, w), a *payoff allocation* $x \in \mathbb{R}^P$ is called *feasible*, if $x(P) \leq w(P)$; *efficient*, if $x(P) = w(P)$; *individually rational*, if $x_i = x(\{i\}) \geq w(\{i\})$ for all $i \in P$; *coalitionally rational*, if $x(S) \geq w(S)$ for all $S \subseteq P$; where, using the standard notation, $x(S) = \sum_{i \in S} x_i$ if $S \neq \varnothing$, and $x(\varnothing) = 0$. We denote by $\mathcal{I}'(P, w)$ the *semi-imputation set* (i.e., the set of feasible and individually rational payoffs), by $\mathcal{I}(P, w)$ the *imputation set* (i.e., the set of efficient and individually rational payoffs), and by $\mathcal{C}(P, w)$ the *core* (i.e., the set of efficient and coalitionally rational payoffs) of the game (P, w). Semi-imputations which are not efficient are called strict semi-imputations.

We say that *allocation y dominates allocation x via coalition S* (notation: $y \operatorname{dom}_S x$) if $y(S) \leq w(S)$ and $y_k > x_k \ \forall\, k \in S$. We further say that *allocation y dominates allocation x* (notation: $y \operatorname{dom} x$) if there is a coalition S such that y dominates x via S. We can also define the core of a game with the dominance relation. The core of a game consist the preimputations which are not dominated by any other preimputation. Similarly to this new definition of the core we can define the core of a set $\mathcal{X}$ by the elements of $\mathcal{X}$ which are not dominated by any other element of $\mathcal{X}$.

We say a set $\mathcal{Z}$ is $\mathcal{X}$-stable if $\mathcal{Z} \subseteq \mathcal{X}$ and

- *(internal stability):* there exist no $x, y \in \mathcal{Z}$ such that $y \operatorname{dom} x$
- *(external stability):* for every $x \in \mathcal{X} \setminus \mathcal{Z}$ there exists $y \in \mathcal{Z}$ such that $y \operatorname{dom} x$.

This is a generalization of the stable set concept. The „normal” stable sets are the $\mathcal{I}$-stable sets (or $\mathcal{I}'$-stable sets).

A (nonempty) set $\mathcal{Z}$ of imputations is called a *stable set* if the following two conditions hold:

- *(internal stability):* there exist no $x, y \in \mathcal{Z}$ such that $y \operatorname{dom} x$
- *(external stability):* for every $x \in \mathcal{I} \setminus \mathcal{Z}$ there exists $y \in \mathcal{Z}$ such that $y \operatorname{dom} x$.

Assignment games:

The player set is $P = M \cup N$ with $M \cap N = \emptyset$, players $i \in M = \{1, \ldots, m\}$ are called sellers, and players $j \in N = \{1', \ldots, n'\}$ are called buyers. The coalitional function $w = w_A$ is generated from the $m \times n$ nonnegative matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

consisting of the profits that pairs of a seller and a buyer can make. We define

$$w_A(S) = \max_{\sigma \in \Pi(S \cap M, S \cap N)} \sum_{i=1}^{m} a_{i\sigma(i)}$$

Where $\Pi(X, Y)$ denotes the value of the maximal matching between sets X and Y.

In assignment game w_A if domination occurs among semi-imputations it also occurs via coalitions $\{i, j'\}$ with $a_{ij} > 0$. We shall simply write $(\boldsymbol{u}; \boldsymbol{v}) \operatorname{dom}_{ij} (\boldsymbol{u}'; \boldsymbol{v}')$ if $u_i + v_j \leq a_{ij}$ and $u_i > u'_i$, $v_j > v'_j$. To emphasize the special role of the sellers and buyers, we shall write the payoff allocations as $(\boldsymbol{u}; \boldsymbol{v}) \in \mathbb{R}^m \times \mathbb{R}^n$.

2.1 Section 1: Stable sets and assignment games

- We give a new proof to the theorem of Solymosi and Raghavan (2001) which says that the core of an assignment game is stable if and only if the matrix of the assignment game has a dominant diagonal.
  The key of the proof is Lemma 2.2.4. This lemma says that if $(\boldsymbol{x}; \boldsymbol{y}) \in \mathcal{C}$ and $x_i + y_j = a_{ij}$ then the vector $(\boldsymbol{x}^{(\tau)}; \boldsymbol{y}^{(\tau)})$ is also an element of $\mathcal{C}$ for every $\tau \in \mathbb{R}$ where $x_k^{(\tau)} = \operatorname{med}(0; x_k + \tau; a_{kk})$ and $y_k^{(\tau)} = \operatorname{med}(0; x_k - \tau; a_{kk})$.

- We give a new proof for the stability of the set proposed by Shapley (Shubik, 1984). Núñez and Rafels (2013) have already proved the stability of this set but we think our proof is easier and the characterization in the next section is the generalization of this proof.

2.2 Section 2.: Characterization

In this section is the main theorem of the first part. We give a new characterization of the stable sets in assignment games.

Theorem 2.2.1 *A set $\mathcal{V} \subseteq \mathcal{I}$ is stable in an assignment game if and only if it*

1. *is internally stable,*

2. *is connected,*

3. *contains an imputation with* 0 *payoff to all buyers and an imputation with* 0 *payoff to all sellers,*

4. *contains the core of the semi-imputations in the rectangular set spanned by any two points of* $\mathcal{V}$.

The necessity of these properties was proved by Shapley (1959) for glove markets (assignment games with $a_{ij} = 1$ for all $i \in M$ and $j \in N$). The proof of the necessity in the general case is similar to the proof of Shapley. Before the proof we need some preparation. Suppose that $\mathcal{V}$ is a subset of the set of imputations which satisfies the four conditions in 2.2.1 Theorem. We denote the coordinatewise maximum of the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ by $\vee$ and the minimum by $\wedge$. Observe that if $(\boldsymbol{x};\boldsymbol{y})$ dominates $(\boldsymbol{u}^1 \vee \boldsymbol{u}^2; \boldsymbol{v}^1 \wedge \boldsymbol{v}^2)$, then it also dominates $(\boldsymbol{u}^1;\boldsymbol{v}^1)$ or $(\boldsymbol{u}^2;\boldsymbol{v}^2)$. The set $\mathcal{X} \subseteq \mathcal{I}'$ is said to be a lattice if for every $(\boldsymbol{u}^1;\boldsymbol{v}^1), (\boldsymbol{u}^2;\boldsymbol{v}^2) \in \mathcal{X}$ the payoff vectors $(\boldsymbol{u}^1 \vee \boldsymbol{u}^2; \boldsymbol{v}^1 \wedge \boldsymbol{v}^2), (\boldsymbol{u}^1 \wedge \boldsymbol{u}^2; \boldsymbol{v}^1 \vee \boldsymbol{v}^2)$ are also in $\mathcal{X}$. Shapley and Shubik (1972) showed that the core of an assignment game is a lattice and Shapley (1959) showed that this also holds for stable sets in glove markets. This property is also true in assignment games. To see this suppose that for some stable set $\mathcal{V}$ the vector $(\boldsymbol{u}^1 \vee \boldsymbol{u}^2; \boldsymbol{v}^1 \wedge \boldsymbol{v}^2)$ is not in $\mathcal{V}$. If it is a semi-imputation it is dominated by an element of $\mathcal{V}$. In this case this vector also dominates $(\boldsymbol{u}^1;\boldsymbol{v}^1)$ or $(\boldsymbol{u}^2;\boldsymbol{v}^2)$ in contradiction with the internal stability of $\mathcal{V}$. If it is not a semi-imputation then $(\boldsymbol{u}^1 \wedge \boldsymbol{u}^2; \boldsymbol{v}^1 \vee \boldsymbol{v}^2)$ is a strict semi-imputation and since $\mathcal{V} \subseteq \mathcal{I}$ we have $(\boldsymbol{u}^1 \wedge \boldsymbol{u}^2; \boldsymbol{v}^1 \vee \boldsymbol{v}^2) \notin \mathcal{V}$ which leads to the same contradiction. See also in Núñez and Rafels (2013).

With the lattice property of the set $\mathcal{V}$ we can easily see the necessity of the third condition: since $\mathcal{V}$ is a closed lattice, there is a vector $(\underline{\boldsymbol{u}};\overline{\boldsymbol{v}}) \in \mathcal{V}$ which gives the minimal payoffs to the sellers and the maximal payoffs to the buyers. If $\underline{\boldsymbol{u}} \neq \mathbf{0}$, then $(\mathbf{0};\overline{\boldsymbol{v}})$ is a strict semi-imputation which is not dominated by $\mathcal{V}$ because no buyers can get more in $\mathcal{V}$ which contradicticts the external stability of $\mathcal{V}$.

Since $\operatorname{med}(x,y,z) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ where $\operatorname{med}(x,y,z)$ denotes the median of x, y and z, we have that the median of every three elements of $\mathcal{V}$ is also in $\mathcal{V}$. Observe that if $(\boldsymbol{x};\boldsymbol{y})$ is between $(\boldsymbol{u}^1;\boldsymbol{v}^1)$ and $(\boldsymbol{u}^2;\boldsymbol{v}^2)$ (which means $(\boldsymbol{x};\boldsymbol{y}) = \operatorname{med}((\boldsymbol{u}^1;\boldsymbol{v}^1), (\boldsymbol{x};\boldsymbol{y}), (\boldsymbol{u}^2;\boldsymbol{v}^2))$, $(\boldsymbol{u}^3;\boldsymbol{v}^3) \operatorname{dom}_{ij} (\boldsymbol{x};\boldsymbol{y})$ and $(\boldsymbol{u}^1;\boldsymbol{v}^1), (\boldsymbol{u}^2;\boldsymbol{v}^2), (\boldsymbol{u}^3;\boldsymbol{v}^3)$ don't dominate each other, then
$\operatorname{med}((\boldsymbol{u}^1;\boldsymbol{v}^1), (\boldsymbol{u}^2;\boldsymbol{v}^2), (\boldsymbol{u}^3;\boldsymbol{v}^3)) \operatorname{dom}_{ij} (\boldsymbol{x};\boldsymbol{y})$.

If we use this observation for a vector $(\boldsymbol{x};\boldsymbol{y}) \notin \mathcal{V}$ which is between two elements $(\boldsymbol{u}^1;\boldsymbol{v}^1)$ and $(\boldsymbol{u}^2;\boldsymbol{v}^2)$ of $\mathcal{V}$, we have more than the external stability of $\mathcal{V}$: we get an element of $\mathcal{V}$ which dominates $(\boldsymbol{x};\boldsymbol{y})$ and this vector is between $(\boldsymbol{u}^1;\boldsymbol{v}^1)$ and $(\boldsymbol{u}^2;\boldsymbol{v}^2)$.

From this property we get immediately the necessity of the fourth condition. We can also get the second one: we show that between every two points of $\mathcal{V}$ there is also a third point. Let $(\boldsymbol{u}^1;\boldsymbol{v}^1)$ and $(\boldsymbol{u}^2;\boldsymbol{v}^2)$ be two elements of $\mathcal{V}$. If the average of these two points is in $\mathcal{V}$ then we have a third point between $(\boldsymbol{u}^1;\boldsymbol{v}^1)$ and $(\boldsymbol{u}^2;\boldsymbol{v}^2)$. If the average is not in $\mathcal{V}$ then there is a vector $(\boldsymbol{u}^3;\boldsymbol{v}^3)\in\mathcal{V}$ which is between $(\boldsymbol{u}^1;\boldsymbol{v}^1)$ and $(\boldsymbol{u}^2;\boldsymbol{v}^2)$ and this vector dominates $(\boldsymbol{x};\boldsymbol{y})$. With the closedness of $\mathcal{V}$ we can prove following Shapley (1959) that every stable set is connected. To prove the sufficiency of these properties we need a couple of lemmas:

Lemma 2.2.1 *Every set $\mathcal{V}$ satisfying the four properties in theorem 2.2.1 is a lattice.*

PROOF.
Let $(\boldsymbol{u}^1;\boldsymbol{v}^1),(\boldsymbol{u}^2;\boldsymbol{v}^2)$ be two elements of $\mathcal{V}$. Observe that the vectors $(\boldsymbol{u}^1\vee\boldsymbol{u}^2;\boldsymbol{v}^1\wedge\boldsymbol{v}^2)$ and $(\boldsymbol{u}^1\wedge\boldsymbol{u}^2;\boldsymbol{v}^1\vee\boldsymbol{v}^2)$ are not dominated by any vectors between $(\boldsymbol{u}^1;\boldsymbol{v}^1)$ and $(\boldsymbol{u}^2;\boldsymbol{v}^2)$. Because of the fourth condition if $(\boldsymbol{u}^1\vee\boldsymbol{u}^2;\boldsymbol{v}^1\wedge\boldsymbol{v}^2)$ or $(\boldsymbol{u}^1\wedge\boldsymbol{u}^2;\boldsymbol{v}^1\vee\boldsymbol{v}^2)$ is an imputation then it is also an element of $\mathcal{V}$. If $(\boldsymbol{u}^1\vee\boldsymbol{u}^2;\boldsymbol{v}^1\wedge\boldsymbol{v}^2)$ or $(\boldsymbol{u}^1\vee\boldsymbol{u}^2;\boldsymbol{v}^1\wedge\boldsymbol{v}^2)$ is a strict semi-imputation, then by the fourth condition it is an element of $\mathcal{V}$ which contradicts the condition $\mathcal{V}\subseteq\mathcal{I}$. If $(\boldsymbol{u}^1\vee\boldsymbol{u}^2;\boldsymbol{v}^1\wedge\boldsymbol{v}^2)$ or $(\boldsymbol{u}^1\wedge\boldsymbol{u}^2;\boldsymbol{v}^1\vee\boldsymbol{v}^2)$ is not a semi-imputation, then the other one is a strict semi-imputation which leads to a contradiction. □

Lemma 2.2.2 *Every two points of $\mathcal{V}$ is connected with a coordinatewise monotonic curve in $\mathcal{V}$.*

PROOF.

Let $(\boldsymbol{u}^0;\boldsymbol{v}^0)$ and $(\boldsymbol{u}^1;\boldsymbol{v}^1)$ be two elements of $\mathcal{V}$. We can assume that $\boldsymbol{u}^0\le\boldsymbol{u}^1$ and $\boldsymbol{v}^0\ge\boldsymbol{v}^1$ because we showed in lemma 2.2.1 that $(\boldsymbol{u}^0\vee\boldsymbol{u}^1;\boldsymbol{v}^0\wedge\boldsymbol{v}^1)\in\mathcal{V}$ and if there is a monotone curve between $(\boldsymbol{u}^0;\boldsymbol{v}^0)$ and $(\boldsymbol{u}^0\vee\boldsymbol{u}^1;\boldsymbol{v}^0\wedge\boldsymbol{v}^1)$ and another one between $(\boldsymbol{u}^0\vee\boldsymbol{u}^1;\boldsymbol{v}^0\wedge\boldsymbol{v}^1)$ and $(\boldsymbol{u}^1;\boldsymbol{v}^1)$ and we connect these two curves together we get a monotone curve between $(\boldsymbol{u}^0;\boldsymbol{v}^0)$ and $(\boldsymbol{u}^1;\boldsymbol{v}^1)$.
Since $\mathcal{V}$ is connected there is a continuous curve $(\boldsymbol{u}^t;\boldsymbol{v}^t)_{t\in[0;1]}\subseteq\mathcal{V}$ between $(\boldsymbol{u}^0;\boldsymbol{v}^0)$ and $(\boldsymbol{u}^1;\boldsymbol{v}^1)$. Let $\boldsymbol{u}'^t=\operatorname{med}(\boldsymbol{u}^0,\boldsymbol{u}^t,\boldsymbol{u}^1)$ and $\boldsymbol{v}'^t=\operatorname{med}(\boldsymbol{v}^0,\boldsymbol{v}^t,\boldsymbol{v}^1)$. Since $\mathcal{V}$ is a lattice $(\boldsymbol{u}'^t;\boldsymbol{v}'^t)_{t\in[0;1]}\subseteq\mathcal{V}$. Let $\boldsymbol{u}''^t=\min_{s\le t}\boldsymbol{u}'^s$ and $\boldsymbol{v}''^t=\max_{s\le t}\boldsymbol{v}'^s$. Obviously the curve $(\boldsymbol{u}''^t;\boldsymbol{v}''^t)_{t\in[0;1]}$ is monotone, $(\boldsymbol{u}''^0;\boldsymbol{v}''^0)=(\boldsymbol{u}^0;\boldsymbol{v}^0)$, $(\boldsymbol{u}''^1;\boldsymbol{v}''^1)=(\boldsymbol{u}^1;\boldsymbol{v}^1)$ and since $\mathcal{V}$ is a lattice $(\boldsymbol{u}''^t;\boldsymbol{v}''^t)_{t\in[0;1]}\subseteq\mathcal{V}$. □

With this lemma we can prove a condition which is stronger then the internal stability.

Corollary 2.2.3 *Let $(\boldsymbol{x};\boldsymbol{y}),(\boldsymbol{u};\boldsymbol{v})\in\mathcal{V}$ such that $x_i>u_i$ and $y_j>v_j$ for some $i\in M$ and $j'\in N$ then $u_i+v_j\ge a_{ij}$ (the internal stability states only $x_i+y_j>a_{ij}$*

Proof.

Suppose that $u_i + v_j < a_{ij}$. Let $s, t \in \mathbb{R}$ such that $s + t \leq a_{ij}, u_i < s < x_i$ and $v_j < t < y_j$. $(\boldsymbol{u} \vee \boldsymbol{x}, \boldsymbol{v} \wedge \boldsymbol{y}), (\boldsymbol{u} \wedge \boldsymbol{x}, \boldsymbol{v} \vee \boldsymbol{y}) \in \mathcal{V}$ because $\mathcal{V}$ is a lattice. There is a vector $(\boldsymbol{x}^1, \boldsymbol{y}^1) \in \mathcal{V}$ in the monotonic curve connecting $(\boldsymbol{u}; \boldsymbol{v})$ and $(\boldsymbol{u} \vee \boldsymbol{x}, \boldsymbol{v} \wedge \boldsymbol{y})$, and a point $(\boldsymbol{x}^2, \boldsymbol{y}^2) \in \mathcal{V}$ in the monotonic curve connecting $(\boldsymbol{x}; \boldsymbol{y})$ and $(\boldsymbol{u} \wedge \boldsymbol{x}, \boldsymbol{v} \vee \boldsymbol{y})$ such that $x_i^1 = s = x_i^2$. Note that $y_j^1 = v_j$ and $y_j^2 = y_j$. There is a vector $(\boldsymbol{x}^3, \boldsymbol{y}^3) \in \mathcal{V}$ in the monotonic curve connecting $(\boldsymbol{x}^1, \boldsymbol{y}^1)$ and $(\boldsymbol{x}^2, \boldsymbol{y}^2)$ such that $y_j^3 = t$. For this vector $x_i^3 = s$ means that $(\boldsymbol{x}^3; \boldsymbol{y}^3) \operatorname{dom}_{ij} (\boldsymbol{u}; \boldsymbol{v})$ which contracicts the internal stability. □

Lemma 2.2.4 *Every set $\mathcal{V}$ satisfying the four properties in Theorem 2.2.1 is closed.*

Proof.

Let $(\boldsymbol{u}^i; \boldsymbol{v}^i)_{i\in\mathbb{N}} \subseteq \mathcal{V}$ and let $(\boldsymbol{u}; \boldsymbol{v})$ be the limit of this sequence. Since each $(\boldsymbol{u}^i; \boldsymbol{v}^i)$ is in $\mathcal{V} \subseteq \mathcal{I}$ we get $(\boldsymbol{u}; \boldsymbol{v}) \in \mathcal{I}$. By the second condition, there are elements $(\overline{\boldsymbol{u}}; \boldsymbol{0})$ and $(\boldsymbol{0}; \overline{\boldsymbol{v}})$ in $\mathcal{V}$. As $\mathcal{V}$ is a lattice, every element of $\mathcal{V}$ is between these two vectors. Since each $(\boldsymbol{u}^i; \boldsymbol{v}^i)$ is between $(\overline{\boldsymbol{u}}; \boldsymbol{0})$ and $(\boldsymbol{0}; \overline{\boldsymbol{v}})$ we get that $(\boldsymbol{u}; \boldsymbol{v})$ is also between them.

Now suppose that $(\boldsymbol{u}; \boldsymbol{v}) \notin \mathcal{V}$. Then $(\boldsymbol{u}; \boldsymbol{v})$ is between $(\overline{\boldsymbol{u}}; \boldsymbol{0})$ and $(\boldsymbol{0}; \overline{\boldsymbol{v}})$, thus there is a mixed pair $\{i; j'\}$ which can dominate $(\boldsymbol{u}; \boldsymbol{v})$ with a vector between $(\overline{\boldsymbol{u}}; \boldsymbol{0})$ and $(\boldsymbol{0}; \overline{\boldsymbol{v}})$. Because of lemma 2.2.2, there is a vector $(\boldsymbol{x}; \boldsymbol{y}) \in \mathcal{V}$ between $(\overline{\boldsymbol{u}}; \boldsymbol{0})$ and $(\boldsymbol{0}; \overline{\boldsymbol{v}})$ such that $x_i - y_j = u_i - v_j$. If $x_i > u_i$ and $y_j > v_j$ then $\exists k : x_i > u_i^k, y_j > v_j^k$ and $u_i^k + v_j^k < a_{ij}$ in contradiction with 2.2.3 corollary.

Now we can assume that $x_i \leq u_i$ and $y_j \leq v_j$. Let $(\boldsymbol{x}^1; \boldsymbol{y}^1) = (\boldsymbol{u} \wedge \boldsymbol{x}; \boldsymbol{v} \vee \boldsymbol{y})$, $(\boldsymbol{x}^2; \boldsymbol{y}^2) = (\boldsymbol{u} \vee \boldsymbol{x}; \boldsymbol{v} \wedge \boldsymbol{y})$. There are two cases:

- $(\boldsymbol{x}^1; \boldsymbol{y}^1)$ or $(\boldsymbol{x}^2; \boldsymbol{y}^2)$ is a semi-imputation but is not in $\mathcal{V}$: assume that $(\boldsymbol{x}^1; \boldsymbol{y}^1)$ is this vector. By lemma 2.2.1, $(\boldsymbol{u}'^i; \boldsymbol{v}'^i) = (\boldsymbol{u}^i \wedge \boldsymbol{x}; \boldsymbol{v}^i \vee \boldsymbol{y}) \in \mathcal{V}\ \forall i \in \mathbb{N}$ and $\lim(\boldsymbol{u}'^i; \boldsymbol{v}'^i) = (\boldsymbol{x}^1; \boldsymbol{y}^1) = (\boldsymbol{u}'; \boldsymbol{v}')$. Thus, $(\boldsymbol{x}^1; \boldsymbol{y}^1)$ is between $(\boldsymbol{0}; \overline{\boldsymbol{v}})$ and $(\boldsymbol{x}; \boldsymbol{y})$ but between these points there is no vector which dominates $(\boldsymbol{x}^1; \boldsymbol{y}^1)$ via the mixed-pair $\{i; j'\}$.

- both $(\boldsymbol{x}^1; \boldsymbol{y}^1), (\boldsymbol{x}^2; \boldsymbol{y}^2) \in \mathcal{V}$: since lemma 2.2.1, $(\boldsymbol{u}'^i; \boldsymbol{v}'^i) = \operatorname{med}((\boldsymbol{x}^1; \boldsymbol{y}^1); (\boldsymbol{u}^i; \boldsymbol{v}^i); (\boldsymbol{x}^2; \boldsymbol{y}^2)) \in \mathcal{V}\ \forall i \in \mathbb{N}$ and $\lim(\boldsymbol{u}'^i; \boldsymbol{v}'^i) = (\boldsymbol{u}; \boldsymbol{v}) = (\boldsymbol{u}'; \boldsymbol{v}')$. Thus, $(\boldsymbol{u}; \boldsymbol{v})$ is between $(\boldsymbol{x}^1; \boldsymbol{y}^1)$ and $(\boldsymbol{x}^2; \boldsymbol{y}^2)$ but between these points there is no vector which dominates $(\boldsymbol{u}; \boldsymbol{v})$ via the mixed-pair $\{i; j'\}$.

In both cases we got two points from $\mathcal{V}$ and a sequence $(\boldsymbol{u}'^i; \boldsymbol{v}'^i) \subseteq \mathcal{V}$ between them such that the limit of this sequence is outside of the set $\mathcal{V}$ and this limit is not dominated by any vector in the rectangular set spanned by the two points of $\mathcal{V}$. Now change $(\overline{\boldsymbol{u}}; \boldsymbol{0})$ and $(\boldsymbol{0}; \overline{\boldsymbol{v}})$ to these two points, $(\boldsymbol{u}; \boldsymbol{v})$ to $(\boldsymbol{u}'; \boldsymbol{v}')$ and the sequence $(\boldsymbol{u}^i; \boldsymbol{v}^i)$ to $(\boldsymbol{u}'^i; \boldsymbol{v}'^i)$. If we do this step again we can exclude another possible dominating mixed-pair. After a finite number of steps we exclude all mixed-pairs and we get two points of $\mathcal{V}$ and a third outside of $\mathcal{V}$ between them which is not dominated by any vector of the rectangular set spanned by the two vectors of $\mathcal{V}$ in contradiction with the fourth property. □

Now we can prove the sufficiency of the four conditions. The proof will be very similar to the proof of lemma 2.2.4.

The internal stability of $\mathcal{V}$ is our first condition thus we only need to prove the external stability of $\mathcal{V}$. Let $(\boldsymbol{u};\boldsymbol{v})$ be a semi-imputation outside of $\mathcal{V}$. We can assume that $(\boldsymbol{u};\boldsymbol{v})$ is between $(\boldsymbol{0};\overline{\boldsymbol{v}})$ and $(\overline{\boldsymbol{u}};\boldsymbol{0})$. To see this suppose that this claim does not hold and let $(\boldsymbol{u}';\boldsymbol{v}') = \operatorname{med}((\boldsymbol{0};\overline{\boldsymbol{v}});(\boldsymbol{u};\boldsymbol{v});(\overline{\boldsymbol{u}};\boldsymbol{0}))$. This vector is also a semi-imputation outside of $\mathcal{V}$ and if this is dominated by a vector from $\mathcal{V}$, this vector also dominates $(\boldsymbol{u};\boldsymbol{v})$.

By the fourth condition, there is at least one mixed pair which can dominate $(\boldsymbol{u};\boldsymbol{v})$ between $(\boldsymbol{0};\overline{\boldsymbol{v}})$ and $(\overline{\boldsymbol{u}};\boldsymbol{0})$. The proof is similar to the proof of the closedness of $\mathcal{V}$. There are two cases:

1. There exists a mixed pair $\{i;j'\}$ such that $(\boldsymbol{u};\boldsymbol{v})$ can be dominated via this coalition between $(\boldsymbol{0};\overline{\boldsymbol{v}})$ and $(\overline{\boldsymbol{u}};\boldsymbol{0})$ and there is a vector $(\boldsymbol{x};\boldsymbol{y}) \in \mathcal{V}$ between $(\boldsymbol{0};\overline{\boldsymbol{v}})$ and $(\overline{\boldsymbol{u}};\boldsymbol{0})$ such that $x_i > u_i$ and $y_i > v_j$
2. For each mixed pair $\{i;j'\}$ such that $(\boldsymbol{u};\boldsymbol{v})$ can be dominated via this coalition between $(\boldsymbol{0};\overline{\boldsymbol{v}})$ and $(\overline{\boldsymbol{u}};\boldsymbol{0})$ there is no vector $(\boldsymbol{x};\boldsymbol{y}) \in \mathcal{V}$ between $(\boldsymbol{0};\overline{\boldsymbol{v}})$ and $(\overline{\boldsymbol{u}};\boldsymbol{0})$ with $x_i > u_i$ and $y_i > v_j$.

In the second case we can do the same as in the proof of the closedness of $\mathcal{V}$ because by the internal stability of $\mathcal{V}$ if $(\boldsymbol{u}';\boldsymbol{v}')$ is dominated by a vector from $\mathcal{V}$ the vector $(\boldsymbol{u};\boldsymbol{v})$ is also dominated via the same coalition.

In the first case if $x_i + y_j \le a_{ij}$ then $(\boldsymbol{x};\boldsymbol{y})$ dominates $(\boldsymbol{u};\boldsymbol{v})$. Let $x_i + y_j > a_{ij}$. Because of the connectedness of $\mathcal{V}$ we can assume that $u_i - v_j = x_i - y_j$. Let $s, t \in \mathbb{R}$ such that $s + t = a_{ij}$ and $s - t = u_i - v_j = x_i - y_j$. By lemma 2.2.2, there are two vectors $(\boldsymbol{x}^1;\boldsymbol{y}^1), (\boldsymbol{x}^2;\boldsymbol{y}^2) \in \mathcal{V}$ such that $(\boldsymbol{x}^1;\boldsymbol{y}^1)$ is between $(\boldsymbol{0};\overline{\boldsymbol{v}})$ and $(\boldsymbol{x};\boldsymbol{y})$, $(\boldsymbol{x}^2;\boldsymbol{y}^2)$ is between $(\boldsymbol{x};\boldsymbol{y})$ and $(\overline{\boldsymbol{u}};\boldsymbol{0})$, $x^1_i = s$ and $y^2_j = t$. Let $(\boldsymbol{x}^3;\boldsymbol{y}^3) = (\boldsymbol{x}^1 \vee \boldsymbol{u};\boldsymbol{y}^1 \wedge \boldsymbol{v})$ and $(\boldsymbol{x}^4;\boldsymbol{y}^4) = (\boldsymbol{x}^2 \wedge \boldsymbol{x}^3;\boldsymbol{y}^2 \vee \boldsymbol{y}^3) = \operatorname{med}((\boldsymbol{x}^1;\boldsymbol{y}^1),(\boldsymbol{u};\boldsymbol{v}),(\boldsymbol{x}^2;\boldsymbol{y}^2))$. Since $x^4_i = s$ and $y^4_j = t$, the vector $(\boldsymbol{x}^4;\boldsymbol{y}^4)$ dominates $(\boldsymbol{u};\boldsymbol{v})$. If it is in $\mathcal{V}$, we have proved that $\mathcal{V}$ dominates $(\boldsymbol{u};\boldsymbol{v})$. If $(\boldsymbol{x}^4;\boldsymbol{y}^4) \notin \mathcal{V}$ then there are two cases:

1. If $(\boldsymbol{x}^4;\boldsymbol{y}^4)$ is a semi-imputation, then it is enough to show that $\mathcal{V}$ dominates $(\boldsymbol{x}^4;\boldsymbol{y}^4)$ because if a vector from $\mathcal{V}$ dominates $\operatorname{med}((\boldsymbol{x}^1;\boldsymbol{y}^1),(\boldsymbol{u};\boldsymbol{v}),(\boldsymbol{x}^2;\boldsymbol{y}^2))$, then it also dom-

inates one of $(\boldsymbol{x}^1;\boldsymbol{y}^1)$, $(\boldsymbol{u};\boldsymbol{v})$, $(\boldsymbol{x}^2;\boldsymbol{y}^2)$. Because of the internal stability of $\mathcal{V}$, we get that this vector dominates $(\boldsymbol{u};\boldsymbol{v})$. Thus $(\boldsymbol{x}^4;\boldsymbol{y}^4)$ is between $(\boldsymbol{x}^1;\boldsymbol{y}^1)$ and $(\boldsymbol{x}^2;\boldsymbol{y}^2)$ and between these vectors the coalition $\{i;j'\}$ can't dominate anything. Thus we excluded one coalition.

2. If $(\boldsymbol{x}^4;\boldsymbol{y}^4)$ is not a semi-imputation, then $(\boldsymbol{u}\wedge\boldsymbol{x}^1;\boldsymbol{v}\vee\boldsymbol{y}^1)$ or $(\boldsymbol{u}\vee\boldsymbol{x}^2;\boldsymbol{v}\wedge\boldsymbol{y}^2)$ is a strict semi-imputation (because if $(\boldsymbol{x}^3;\boldsymbol{y}^3)$ is a semi-imutation, then $(\boldsymbol{x}^3\vee\boldsymbol{x}^2;\boldsymbol{y}^3\wedge\boldsymbol{y}^2)=(\boldsymbol{u}\vee\boldsymbol{x}^2;\boldsymbol{v}\wedge\boldsymbol{y}^2)$ or $(\boldsymbol{x}^4;\boldsymbol{y}^4)$ is a semi-imputation and if $(\boldsymbol{x}^3;\boldsymbol{y}^3)$ is not a semi-imputation, then $(\boldsymbol{u}\wedge\boldsymbol{x}^1;\boldsymbol{v}\vee\boldsymbol{y}^1)$ is a semi-imputation). Let this vector be $(\boldsymbol{x}^5;\boldsymbol{y}^5)$. If $(\boldsymbol{x}^5;\boldsymbol{y}^5)$ is dominated by $\mathcal{V}$ then $(\boldsymbol{u};\boldsymbol{v})$ is also dominated thus it is enough to show that $\mathcal{V}$ dominates $(\boldsymbol{x}^5;\boldsymbol{y}^5)$.

Now we can do the same, once again with $(\boldsymbol{x}^5;\boldsymbol{y}^5)$ instead of $(\boldsymbol{u};\boldsymbol{v})$, and $(\boldsymbol{x}^1;\boldsymbol{y}^1)$ instead of $(\overline{\boldsymbol{u}};\boldsymbol{0})$ or $(\boldsymbol{x}^2;\boldsymbol{y}^2)$ instead of $(\boldsymbol{0};\overline{\boldsymbol{v}})$. But now $(\boldsymbol{x}^5;\boldsymbol{y}^5)$ is a strict semi imputation, and because of the closedness of $\mathcal{V}$ there exists $\epsilon>0$ for all $(\boldsymbol{x};\boldsymbol{y})\in\mathcal{V}$ such that $x_i>x_i^5$ and $y_j>y_j^5$ satisfying $x_i+y_j>a_{ij}+\epsilon$ If we do the same the coalition $\{i;j'\}$ get more than in $(\boldsymbol{x}^5;\boldsymbol{y}^5)$ with at least $\epsilon/2$ thus after a finite number of repetition we get a vector $(\boldsymbol{x}^k;\boldsymbol{y}^k)\notin\mathcal{V}$ such that $x_i^k+y_j^k\geq a_{ij}$. If $(\boldsymbol{x}^k;\boldsymbol{y}^k)$ is dominated by $\mathcal{V}$ then $(\boldsymbol{u};\boldsymbol{v})$ is also dominated via the same coalition. Thus after a finite number of steps we can exclude one coalition.

□

- Based on the above characterization we can give a simpler proof than in Núñez and Rafels (2013) to the conjecture by Shapley is stable. It is easy to prove the four property in 2.2.1 is true for this set.

- Núñez and Rafels (2013) proved that this set is stable and it is the unique stable set in the principal section. Using the characterization we can get a stronger result for uniqueness. This set is the unique stable set which contains the buyeroptimal and the selleroptimal points of the principal section.

- The characterization works for a bigger class of games also. If we have an assignment game and we fix the value of the one player coalitions and the grand coalition and increase some other the characterization of stable sets in 2.2.1 works.

- If we replace the stable set to $\mathcal{X}$-stable set in 2.2.1 where $\mathcal{X}\subseteq\mathcal{I}*$ is a closed connected lattice the characterization also works.

- With the characterization it is easy to prove that if we know a curve between the buyeroptimal and the selleroptimalpoint of the stable set it determines the stable set. If we have two set containing the same curve then the intersection of this set is also stable.

- Every stable set is $\mathcal{I}*$-stable and every $\mathcal{I}*$-stable set is stable.

- Let $A, A' \in \mathcal{R}^{m\times n}$ such that $A \leq A'$ and $w_A(P) = w_{A'}(P)$. If $\mathcal{V}$ is stable in the assignment game belonging to the matrix A and $\mathcal{V}'$ is $\mathcal{V}$-stable in the assignment game belongs to the matrix A', then $\mathcal{V}'$ is stable (not only $\mathcal{V}$-stable) in the game belonging to A'.

- It can be easily checked that if A and A' differ in only one element the core of V in the game belonging to A' is always V-stable (and also stable) in the game belonging to A'.

- With the last two observations we can construct stable sets, and give an other proof to the theorem of Núñez and Rafels (2013): if A is a diagonal matrix then the principal section is obviously stable. In the first step we increase one element of the matrix A and take the core of the original stable set in the new game. This set is stable in the new game. Then we increase another element of the matrix and so on.

- We also proved that if the core of an assignment game is not stable then there is infinite many stable set: it can be easily checked for games with two buyers and two sellers. Similarly we can show it to assignment games with a matrix which has only one non zero element not in their diagonal. Begin the construction in the previous point from this infinite many stable set the procedure ends in infinite many stable sets.

2.3 Section 3: 1-seller case

In this section we give an other characterization of stable set if there is only one seller. We also show that the union of all stable sets can be described as the union of convex polytopes all of whose vertices are marginal contribution payoff vectors. You can find the proofs in Bednay (2014). In this section let the generating matrix $A = [a_1, a_2, \dots, a_n]$ with $a_1 \geq a_2 \geq \dots \geq a_n (\geq a_{n+1} = 0)$.
Let $\mathcal{X} = \{(u, \boldsymbol{v}) \in \mathcal{I} : \quad \forall\ 1 \leq i \leq n\ v_i = 0\ \text{ or }\ u + v_i \leq a_i\}$ and
$\mathcal{U} = \{(u, \boldsymbol{v}) \in \mathcal{I},\ \forall\ k : \sum_{j=k}^{n} v_j \leq |a_k - u|_+\}$.
Let M_i be the set of marginal payoff vectors where the seller gets exactly a_i.
The results of this section:

- Let w_A be a one-seller assignment game. Then $\mathcal{Z} \subseteq \mathcal{I}$ is a stable set in w_A, if and only if $\mathcal{Z}$ is a $[0; a_1]$-continuous, monotone curve in $\mathcal{X}$.
- Let w_A be a one-seller assignment game. Then $\mathcal{Z} \subseteq \mathcal{I}$ is a stable set in w_A, if and only if $\mathcal{Z}$ is a $[0; a_1]$-continuous, monotone curve in $\mathcal{U}$.
- $\mathcal{U}$ is the union of stable sets.
- $\mathcal{U} = \bigcup_{i=1}^{n} \operatorname{conv}(M_i \cup M_{i+1})$.
- In one-seller assignment games the Shapley value is only in very special case in $\mathcal{U}$.

2.4 Section 4: Bargaining equilibrium

In this section using the characterization of stable sets in the previous sections we showed that in assignment games the original definition of stable sets and the definition based on the bargaining game proposed by Harsanyi (1974) are the same.

2.5 Section 5: Multi-sided assignment games

In this section we showed that the core of a multi-sided assignment game (not only in the $2+2+2$ case like Atay and Núñez (2019)) is stable if and only if the generating (poly)matrix has a dominant diagonal (like in assignment games (Solymosi and Raghavan, 2001)).
The key of the proof is the following lemma:
Let A be a polymatrix with dominant diagonal and $\boldsymbol{x} \in \mathcal{I}\backslash\mathcal{C}$. If $a_{i_1 i_2 \ldots i_r}$ is a maximal element of A and the core inequality belonging to $a_{i_1 i_2 \ldots i_r}$ doesn't hold then the core dominates $\boldsymbol{x}$ via the mixed r-tuple $(i_1, i_2, \ldots, i_r)$.

In the $2+2+2$ case we have also an other proof. In this case the structure of the core is similar to the „normal" assignment games. A generalization of the lemma used in the proof of the theorem by Solymosi and Raghavan (2001) in section 1 holds. We a also showed that the $2+2+2$ case is the only non-trivial case where it works.

3 References of the author

3.1 In English

Bednay D, Bozóki S (2013) Constructions for nontransitive dice sets, *Proceedings of the 8th Japanese-Hungarian Symposium on Discrete Mathematics and Its Applications (Veszprém, Hungary, 2013)*: 15-23.

Bednay D (2014) Stable sets in one-seller assignment games, *Annals of Operations Research* 222: 143 - 152.

Bednay D, Moskalenko A, Tasnadi A (2017) Does avoiding bad voting rules result in good ones?, *Operations Research Letters* 45: 448-451.

Bednay D, Moskalenko A, Tasnadi A (2019) Dictatorship versus manipulability, *Mathematical Social Sciences* 101: 72-76.

3.2 In Hungarian

Bednay D (2012) Alkuegyensúlyok és stabil halmazok, *Egyensúly és optimum. Tanulmányok Forgó Ferenc 70. születésnapjára. Budapest, Aula Kiadó 2012*: 3-12.

References

Atay, A. (2016). *Essays on multi-sided assignment markets*. Ph. D. thesis, Universitat de Barcelona. www.tdx.cat/bitstream/handle/10803/402148/ATAY_PhD_THESIS.pdf Accessed on 7 September 2019.

Atay, A. (2017). An alternative proof of the characterization of core stability for the assignment game. *Operations Research Letters 45*(3), 217–219.

Atay, A. and M. Núñez (2019). A note on the relationship between the core and stable sets in three-sided markets. *Mathematical Social Sciences 98*.

Bednay, D. (2009). Stabil halmazok hozzárendelési játékokban. Master's thesis, Budapesti Corvinus Egyetem. http://szd.lib.uni-corvinus.hu/1981/1/bednay_dezso.pdf Accessed on 2019-08-30.

Bednay, D. (2012). *Egyensúly és optimum. Tanulmányok Forgó Ferenc 70. születésnapjára*, Chapter Alkuegyensúlyok és stabil halmazok, pp. 3–12. Budapest: Aula Kiadó.

Bednay, D. (2014). Stable sets in one-seller assignment games. *Annals of Operations Research 222*, 143–152.

Bondareva, O. N., T. Kulakovskaya, and N. Naumova (1979). Solution of arbitrary cooperative four person games. *Vestnik Leningradskogo Universiteta, Matematika, Mekhanika, Astronomija 1979*.

Diamantoudi, E. and L. Xue (2006). Lucas Counter Example Revisited. Departmental Working Papers 2005-09, McGill University, Department of Economics.

Ehlers, L. (2007). Von Neumann–Morgenstern stable sets in matching problems. *Journal of Economic Theory 134*, 537–547.

Gale, D. and L. S. Shapley (1962). College admissions and the stability of marriage. *The American Mathematical Monthly 69*(1), 9–15.

Greenberg, J. (1990). *The Theory of Social Situations: An Alternative Game-Theoretic Approach*. Cambridge University Press.

Harsanyi, J. C. (1974). An equilibrium-point interpretation of stable sets and a proposed alternative definition. *Management Science 20*(11), 1472–1495.

Kaneko, Mamoru andWooders, M. H. (1982). Cores of partitioning games. *Mathematical Social Sciences 3*(4), 313 – 327.

Lucas, W. F. (1968a). A game with no solution. *Bull. Amer. Math. Soc. 74*(2), 237–239.

Lucas, W. F. (1968b). On solutions for n-person games.

Lucas, W. F. (1971). Some recent developments in n-person game theory. *SIAM Review 13*(4), 491–523.

Lucas, W. F. (1992). Chapter 17 von Neumann–Morgenstern stable sets. *Handbook of Game Theory With Economic Applications 1*, 543–590.

Nash, J. F. (1953). Two-person cooperative games. *Econometrica 21*(1), 128–140.

Nash, J. F. (1958). *Games and Decisions: Introduction and Critical Survey*, Volume 121.

Núñez, M. and C. Rafels (2013). Von Neumann–Morgenstern solutions in the assignment market. *Journal of Economic Theory 148*(3), 1282–1291.

Peleg, B. (1986). A proof that the core of an ordinal convex game is a von neumann-morgenstern solution. *Mathematical Social Sciences 11*(1), 83–87.

Schotter, A. (1974). Auctioning Bohm–Bawerk's horses. *International Journal of Game Theory 3*, 195–215.

Shapley, L. S. (1959). *Contributions to the Theory of Games (AM-40), Volume IV*. Princeton University Press.

Shapley, L. S. (1971). Cores of convex games. *International Journal of Game Theory 1*(1), 11–26.

Shapley, L. S. and M. Shubik (1969). On market games. *Journal of Economic Theory 1*(1), 9–25.

Shapley, L. S. and M. Shubik (1972). The assignment game i: The core. *International Journal of Game Theory 1*, 111–130.

Shubik, M. (1984). *A Game Theoretical Approach to Political Economy*. Cambridge, MA: MIT Press.

Solymosi, T. (2007). *Kooperatív játékok (elektronikus jegyzet)*. http://gametheory.uni-corvinus.hu/Solymosi-KooperativJatekok-2007maj31.pdf Accessed 2019-08-28.

Solymosi, T. and T. Raghavan (2001). Assignment games with stable core. *International Journal of Game Theory 30*, 177–185.

Suk-Young, C. M. (1994). Farsighted coalitional stability. *Journal of Economic Theory 63*(2), 299–325.

Tejada, O. (2011). *A theoretical study on assignment markets, power indices and taxation.* Ph. D. thesis, Universitat de Barcelona.

Von Neumann, J. and O. Morgenstern (1944). *Theory of games and economic behavior.* Princeton University Press Princeton.

Wako, J. (2010). A polynomial-time algorithm to find von Neumann–Morgenstern stable matchings in marriage games. *Algorithmica 58*, 188–220.

Weber, R. J. (1977). Probabilistic values for games. Cowles Foundation Discussion Papers 471R, Cowles Foundation for Research in Economics, Yale University.